Il est enfin à remarquer que l'on peut transformer K en un continu *jordanien* (image continue de l'intervalle).

Désignons à ce but par J l'ensemble formé d'une suite infinie de segments verticaux issus des points rationnels de l'intervalle 01, la longueur de ces segments tendant vers 0[1]).

Ajoutons au contour du carré C_1 l'ensemble J ainsi que les trois ensembles qui lui sont symétriques par rapport soit au centre du carré, soit à l'une ou l'autre diagonale de ce carré. Puis, ajoutons au contour de chaque carré $C_{k_1 \ldots k_n}$ une figure analogue (convenablement diminuée). Le continu ainsi obtenu est jordanien et répond au problème.

[1]) Cf. Janiszewski, Thèse, Paris 1911, p. 18.

The Double-Elliptic Case of the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry [1]).

By

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Introduction.

In his paper *Ueber die Grundlagen der Geometrie*, Hilbert [2]) formulates a set of axioms concerning a group of motions which is sufficient to necessitate that this group should be simply isomorphic with either the Euclidean or the Bolyai-Lobatschefskian group of rigid motions in a plane. He assumes, however, that the set of points which undergoes the transformation is a number manifold. In his paper *On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry*, R. L. Moore [3]) gives a treatment in which this assumption is not made in advance, but in which there is a simultaneous analysis of the group of transformations and of the space which undergoes this transformation. In this paper we shall give a similar analysis for the Double-Elliptic case. After a group of preliminary theorems we shall prove that every motion distinct from the identity leaves fixed exactly two points, which we shall call poles. We shall then introduce the notions of great circles, intervals, congruence of intervals, of triangles and of angles.

[1]) Dissertation offered to the Department of Pure Mathematics, University of Texas, U. S. A., in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June 1925. Presented before the American Mathematical Society at Ithaca, New York, September 10, 1925.

[2]) *Ueber die Grundlagen der Geometrie*, David Hilbert, Mathematische Annalen, Vol. 56 (1902), pp. 381—422. This paper will be referred to hereafter as „H. G.“

[3]) Cf. American Journal of Mathemetics, Vol. 41 (1919), pp. 299—319. We shall referr to this paper hereafter as „L. R. H. H.“

We encounter the problem noted by Hilbert[1]) at the end of his paper; that is, the problem of proving the congruence of the base angles of an isosceles triangle. Hilbert[2]) has solved this problem for the Euclidean case. The treatment for the Elliptic case seems even more difficult. In the solution of this problem we shall show that we can follow the treatment of Young[3]) in deriving the Non-Euclidean Trigonometry, and in particular the formulas for the solution of triangles; these formulas will enable us to prove the desired result.

It has been suggested to the author that there should be a simpler method for the solution of this problem of the angle-congruence in isosceles triangles; for instance, that perhaps it could be solved without the use of differential equations. The author would be interested in such a solution; and here takes the opportunity of raising the question of its possibility.

We have mentioned before that the problem we are solving seems more difficult for the geometry treated in this paper than for the Euclidean case. Hilbert[4]) in his treatment of the Euclidean case makes use of a theory of proportion, which depends upon the the parallel postulate. In his paper, H. G., he could have used this theory of proportion, could have determined the trigonometric functions of angles as we have done, and could have proceeded to the solution of triangles. In summary, we may say that trigonometry may be regarded as an analogy of the theory of proportion; this theory is more complicated for a Non-Euclidean Geometry than it is for the case of the Euclidean.

It will be assumed the reader is familiar with R. L. Moore's paper, „*On the Foundations of Plane Analysis Situs*“ [5]) and with the paper L. R. H. H. mentioned above.

[1]) See H. G.

[2]) *Ueber den Satz von der Gleichheit der Basis Winkel im Gleichschlenkligen Dreieck,* David Hilbert, Proceedings of the London Mathematical Society, Vol. 35, (1902—1903), pp. 50—68.

[3]) *On the Analytical Basis of Non-Euclidean Geometry,* W. H. Young, American Journal of Mathematics, Vol. 33, (1911), pp. 249—286. We shall refer to this paper hereafter as „Young“.

[4]) See article in *Proceedings of London Mathematical Society* referred to above.

[5]) *On the Foundations of Plane Analysis Situs,* R. L. Moore, Transactions of the American Mathematical Society, Vol. 17, No. 2, (April, 1916), pp. 131 to 164. This paper will be referred to hereafter as F. A.

In conclusion, I wish to thank Professor R. L. Moore for interesting me in the subject of the Foundations of Mathematics, and for arousing in me a desire to contribute to the extension of this field.

Axioms.

All the axioms except Axioms 5 and 6 of L. R. H. H. will be assumed; the reader is therefore referred to paragraphs 1 and 2 of this paper of R. L. Moore's for preliminary explanations and definitions. In general we shall not give definitions for terms that are in L. R. H. H., or for well known terms in Analysis Situs. In this paper we shall use „S“ in the same way that $\bar{S}$ is used in L. R. H. H., that is, to denote the set or class of all points. Also, we shall not regard S as a region.

Axiom 1. *There exists at least one region.*

Axiom 2. *If R and K are regions and R' is a subset of K' then R is a subset of K.*

Axiom 3. *If the region R_1 contains the point O in common with the region R_2, there exists a region R containing O such that R' is common to R_1 and R_2.*

Axiom 4. *If R_1 and R_2 are regions and R_2' is a subset of R_1 then $R_1 - R_2'$ is a non-vacuous, connected point-set.*

Axiom 5. *If R_1 and R_2 are regions such that $R_1' + R_2'$ does not contain all of S, then there exists a region R that contains both R_1' and R_2'.*

Axiom 6. *Every simple closed curve is the boundary of at least two regions.*

Axiom 7. *If O is a point and L and N are closed, bounded*[1]) *point-sets with no point in common, there exists a region K containing O such that if P is a point in K then every region that contains both a point of L and a point of N can be transformed, by a motion that carries some point of L into O, into a point set that contains both O and P.*

Axiom 8. *If R is a region and M is a motion then $M(R)$ is a region.*

Axiom 9. *If A, B, C, A', B', C', are points distinct or otherwise, such that every three regions that contain A, B, and C, respectively, can be transformed by some motion into regions containing A', B',*

[1]) As in L. R. H. H., page 300, line 24, we say that a point set is bounded provided that there is a region that contains it.

and C', respectively, there exists a motion that transforms A into A', B into B', and C into C'.

Axiom 10. *If M is a motion there exists a motion M^{-1} such that if $M(A) = B$ then $M^{-1}(B) = A$.*

Axiom 11. *If M and N are motions there exist a motion MN such that, for every point P, $M(N(P)) = MN(P)$.*

Axiom 12[1].) *If R_1 and R_2 are regions bounded respectively by the simple closed curves J_1 and J_2, R_1' and R_2' have no point in common, A_1, B_1, and C_1, are three distinct points on J_1, and A_2, B_2, and C_2, are three distinct points on J_2, and there exist three simple continuous arcs A_1XA_2, B_1YB_2, and C_1ZC_2 such that no two of these arcs have a point in common and no one of them has any point other than an end-point in common either with R_1' or with R_2' and M is a motion such that R_1' and $M(R_2')$ have no point in common and there exist three arcs $A_1\overline{X}M(A_2)$, $B_1\overline{Y}M(B_2)$, and $C_1ZM(C_2)$ from A_1 to $M(A_2)$, from B_1 to $M(B_2)$, and from C_1 to $M(C_2)$, respectively, then there exist three such arcs such that no two of them have a point in common and no one of them has any point other than end-point in common either with R_1' or with $M(R_2')$.*

We shall number the theorems of this paper in the same way that the corresponding theorems of L. R. H. H. are numbered. When we say that one of these theorems is true or false we shall mean that it is true or false (as the case may be), for the space we are considering in this paper. Theorems 1 to 21 and Theorem 23 are true.

We give now several theorems which differ in statement or in proof from the theorems of L. R. H. H.

Theorem 21. *If R is a region then $S-R'$ is a connected set of points*

Proof. Suppose that there are no points in $S-R'$. Then, by the convention we have made with respect to S[2]), R has at least one boundary point. By theorem 11 we are then led to a contradiction. The proof of the theorem may now be continued by methods analogous to those used in the proof of Theorem 21 of L. R. H. H.

Theorem 21 A. *Every simple closed curve, J, is the boundary of exactly two mutually exclusive regions, whose sum is $S-J$.*

[1]) Cf. J. R. Kline „*A Definition of Sense on Closed Curves in Non-Metrical Plane Analysis Situs*", Annals of Mathematics, Vol. XIX, (1918), pp. 185—200.

[2]) See paragraph preceding Axiom 1.

Proof. By Axiom 6 and Theorem 2, $S-J$ contains at least two regions R_1 and R_2 whose boundaries are J. Let D be the exterior of R_1. By Theorem 3, R_2 contains a point P in $S-J$; let H be that one of the two point sets R_1 and D which contains P. By theorems 10 and 21, H is connected. If one of the connected point sets R_2 and H is not a subset of the other, one must contain a boundary point of the other. But this is impossible, since neither contains a point of J. Hence $R_2 = H$. Further, by Axiom 6, $R_2 = D$.

Definition. The regions R_1 and R_2 mentioned in the proof of Theorem 21 A are the complementary domains of the curve J there mentioned and are called Jordan regions or Jordan domains.

Theorem 24. *If O is a point in a region R, there exists a simple closed curve in R one of whose complementary domains is a subset of R and contains O.*

By methods analogous to those used in the proof of Theorem 21, F. A., we can prove the following theorem.

Theorem 21 B. *If K and R are regions and the boundary of R is a subset of K' then either R or $S-R'$ is a subset of K.*

We give the following modification of the statement and of proof of Theorems 24 and 25 *of the paper F. A.*

Theorem 21 C. *If the points A and B separate the points C and D on the simple closed curve J, R is a complementary domain of J, AXB is an arc such that the segment AXB (written $\underset{\smile}{AXB}$) is a segment of R, and R_1 and R_2 are those complementary domains of the simple closed curves AXBCA and AXBDA respectively which are subsets of R then (1) ADB is entirely without R_1, (2) R_1 and R_2 have no points in common, (3) $R = \underset{\smile}{AXB} + R_1 + R_2$.*

Proof: Parts (1) and (2) offer no difficulty. In the proof of part (3) follow the proof of Theorem 25 F. A. Let $\overline{R}$, the interior of $\overline{J}$, be that complementary domain of $\overline{J}$ which is a subset of R.

Theorem 22 of L. R. H. H. is not true in S. Instead, we have the following theorem, which denies the theorem mentioned in the preceding sentence:

Theorem 22. *Every point set is compact.*

Proof. The theorem is true for a point set consisting of a finite number of points. Suppose that there exists a point set, K, which contains infinitely many points, but which has no limit point. Let P be a point of K. From the definition of a limit point and from

Theorem 24[1]) it follows that there exists a Jordan region R such that R' contains no point of $K-P$. By Theorem 21 A, D, the exterior of R, is a region. But $K-P$ is a subset of D, and hence by Theorem 14 has a limit point, which is then also a limit point of K.

Hence, K does not exist, and the truth of the theorem is established.

Definitions[2]): If J is a simple closed curve and K is a point set, the expressions „J encloses K“ and „K is within J“ mean that K is a subset of the interior of J. The term „interior of a simple closed curve“ has a definite meaning for the spaces considered in L. R. H. H. and H. G. but is ambigous in the sense that it may be either one of two complementary domains of the given simple closed curve in the case of the space we are considering in this paper. Theorems 25 and 26 remain true for our space no matter which complementary domain of J we take; in the case of Theorem 26 the hypothesis should include the condition that the interior of $\overline{J}$ contains P, and $\overline{J}$ contains points in both complementary domains of J, a condition that follows from the hypothesis of the theorem as stated in L. R. H. H. for the space there considered, but which need not follow for the space considered in this paper. In an analogous manner the term „interior of a simple closed curve J“ as used in the statement of theorems in L. R. H. H. may mean, when used in this paper, an arbitrary one or it may mean a definite one of the two complementary domains of J; in cases of the latter type we shall specify which is to be regarded as the interior only when the interpretation is not obvious.

Theorem 26. *If I and $\overline{I}$ are Jordan domains with boundaries J and $\overline{J}$ respectively, P is a point of J and of $\overline{I}$, and $\overline{J}$ has points in both complementary domains of J, then there exist two Jordan domains q and $\overline{q}$ with boundaries Q and $\overline{Q}$ respectively such that (1) every point of Q belongs to either J or to $\overline{J}$ and so does every point of $\overline{Q}$, (2) the curves Q and $\overline{Q}$ contain in common a segment of J that contains P, (3) q and $\overline{q}$ are both subsets of $\overline{I}$, (4) q is a subset of I and $\overline{q}$ is a subset of $S-I'$.*

Proof: Make the following modifications of the proof as given in L. R. H. H. Let H and K be defined as those complementary

[1]) The proof of Theorem 24 does not depend upon Theorem 22.

[2]) Cf. Theorems 21 A and 21 B.

domains of the closed curves $A\overline{E}BPA$ and $A\overline{F}BPA$ which are subsets of $\overline{I}$ (cf. Theorem 21 C). Let $\overline{H}$ and $\overline{K}$ be defined as those complementary domains of $\overline{h}$ and $\overline{k}$ which are subsets of the domains H and K respectively. Let R be that complementary domain of α which contains the segment APB of J. By the definition of $\overline{H}$ and $\overline{K}$ the segment APB is a subset of the boundary of $\overline{H}$ and $\overline{K}$. It follows easily by Theorem 21 C that $R=\overline{H}+\overline{K}+APB$.

Definitions: By T_A we mean a rotation about the point A; that is, a motion that transforms A into itself. By C_{AB} we mean the set of all points $[X]$ such that B can be transformed into X by a rotation about A. This point set is said to be a *circle* having a *center* at A. In the space S we are considering, as we shall prove later, every circle has exactly two centers. If C_{AB} is a simple closed curve, by I_{AB} we shall mean that complementary domain of C_{AB} which contains A, and by E_{AB} that complementary domain of C_{AB} which does not contain A. The domain I_{AB} is the interior with respect to A of C_{AB} and similarly E_{AB} is the exterior with respect to A of C_{AB}.

If C_{AB} is a Jordan curve, C_{AB} will be said to be a proper[1]) circle with respect to A provided that for every point X in I_{AB}, distinct from A, C_{AX} is a simple closed curve and I_{AX} is a subset of I_{AB}. The expression „a circle C_{AX} has a proper interior I_{AX}“ will mean that this circle is a proper circle with respect to A and that for every point Y within $I_{AX}-A$, C_{AY} in a proper circle with respect to A. A Jordan circle is a circle which is a simple closed curve.

Theorems 30, 31, and 32 are true if we make the following changes: The terms „proper circle“ and „proper interior“ as used in the statement of Theorems 31 and 32 are „proper circle“ and „proper interior“ with respect to the point O; further, add to the hypothesis of Theorem 31 the following statement, „There exists a point $\overline{P}$ such that $C_{O\overline{P}}$ is a subset of $S-\overline{R}'$“.

On pages 397 to 399 H. G. let the interiors of the Jordan curves there mentioned be those complementary domains of these curves which are subsets of the interior of the circle f; and let

[1]) At the present stage of the paper it is conceivable that C_{AB} has more than one center in I_{AB}. Cf. Theorem 62 A, part 2. A Jordan curve is a simple closed curve.

the interior of $\mathfrak{k}$ be that complementary domain of $\mathfrak{f}$ which contains the point M.

We shall now show that Theorem 33 holds for the space we are considering. (See proof given in L. R. H. H.). Let Q be a point of E_{OP}. But C_{OQ} is closed. There exists by Theorem 37 (a theorem the proof of which does not depend on this theorem) a Jordan region R containing I'_{OP}, but containing no point of C_{OQ}: Instead of the region R mentioned in the proof given in L. R. H. H. use the region R just defined. The statement made in lines 22 and 23, page 315, L. R. H. H., does not involve a contradiction in this space. Let X^* be a point of $\overline{k}$, distinct from P. Let K have the properties mentioned in R. L Moore's proof, and let it have the further property that no motion can transform it into a region that contains X^* and P. Now follow the argument of L. R. H. H.

Theorem 34 as stated for L. R. H. H. and Theorems 35 and 36 are not true for our space.

Theorem 34. *If $C_{O\overline{P}}$ is a Jordan circle then it is a proper circle having a proper interior with respect to O.*

Proof of Theorem 34: Let the point $\overline{P}$ mentioned in the proof of Theorem 34, L. R. H. H. be interpreted as the point $\overline{P}$ mentioned in this theorem; (suppose that the theorem does not hold true for $C_{O\overline{P}}$, and limit the discussion given in L. R. H. H., pages 315—316 to the set of points composed of the circle just mentioned and its interior with respect to O). The set of points which is composed of all the circles with centers at O which contain points on the interval OX and belong to S_1 is a set of points whose boundary contains X. We can easily prove that this pointset contains at least one other boundary point and then can prove by the methods used in the proof of Case 1 of Theorem 42 that C_{OX} is a Jordan circle. This part of Theorem 42 does not depend upon Theorem 34. Further, C_{OX} is on the exterior with respect to O of every circle C which intersects the segment OX. It follows by methods used in the proof of Theorem 31 that C_{OX} is a proper circle with a proper interior. We are thus led to a contradiction.

Theorem 37. *If M_1 is a maximal connected subset of a closed point set M, $S-M_1$ is connected, and K is a closed point set having no points in common with M, then there exists a Jordan domain containing M_1 but no points of K, whose boundary contains no point of $K+M$.*

Proof. See Theorem 29 and a theorem by R. L. Moore[1]).

Theorem 41. *Hypothesis: A connected domain D containing the point O has the following properties: (1) The boundary L of D is connected; (2) if X is a point of L, $C_{OX}=L$; (3) under a rotation about O, D goes into itself. Conclusion: Either L is a single point or L is a Jordan circle and $D=I_{OX}$ where X is a point of L.*

Proof. There exists for each point x on L an arc Ox such that $Ox-x$ is a subset of D. The truth of this statement may be estublished as follows: There exists by Theorem 15, F. A., an arc joining some point Q of L to O. In the order OQ let x be the first point of this arc. The proposition follows by (2) and (3) hypothesis. By Theorem 5 and Theorem 24 there exists a Jordan domain E^* containing O such that $E^{*\prime}$ cannot be thrown by a rotation about O into a point-set containing points of L. Let E be the complementary domain of $E^{*\prime}$; then E contains L. Let J^* be the boundary of E. Suppose that L contains at least two points and let x and y be any two points of L. From the property mentioned at the beginning of the proof we can show that there exists an arc xt such that t is a point of J^* but such that $xt-x-t$ is a subset of E and of D. By methods that will not be difficult for one familiar with Analysis Situs it follows that there exists an arc yu such that yu has no points in common with xt, u is a point of J^*, and the segment uy is a subset of E and of D. If z is a point common to E and to the pointset composed of D' minus the intervals yu and xt, we can prove in the same way the existence of an arc zv where v is a point of J^* and the segment zv is a subset of E and of D and contains no points of the intervals xt and yu.

Suppose that $L-x-y$ is connected. J^*-u-t consists of two mutually exclusive segments; call these M ad N. Let M^* consist of all those points z of $L-x-y$ such that a corresponding v is a point of M; similarly define N^* as those points z, of $L-x-y$, for which a corresponding v is a point of N. With the help of Theorems 32, 33, and 40 of F. A., there exists an arc pqr where p is a point of M, r is a point of N, and such that pqr is a subset of $E'-xt-yu$. It may easily be proved that pqr contains points of M^* and of N^*.

[1]) *Concerning the Separation of Point Sets by Curves*, R. L. Moore, Proceedings of the National Academy of Sciences, Vol. 11, No. 8, pp. 469—476, August, 1925.

Let z be a point of $L-x-y$, and let $z_1, z_2, z_3, \dots$ be a subset of N^* that has z as a sequential limit point. Let R be a region containing z such that R' does not contain any points of the arc xt, the arc yu or of $E^{*\prime}$. Let R^* be a region which contains z and such that $R^{*\prime}$ is a subset of R. Let R^{**} be a region containing z such that R^{**} cannot be thrown by a motion into a point-set which containts points of both the sets $R^{*\prime}$ and $S-R$ (Theorem 5); there exists a subinterval zw of zv which is also a subset of R^{**}.

Suppose that M^* and N^* have no points in common. There exists infinitely many points z_i, i a positive integer, of the set $z_1, z_2, \dots$ in R^{**}, for each z_i there exists a rotation T_i about O $T_{io}(z) = z_i$. Let w_i be the transform under T_i of w. The set of $w_i's$ will have a limit point W, which by (3), hypothesis, and Axiom 9 will be a point of D, and by definition of w_i and Axiom 9 a point of R'. Let C be a region containing W such that C is a subset of $S-E^{*\prime}-xt-yu-L$. By Axiom 9 there exists T'_o such that $T_o(zw) =$ an arc zW which has properties analogous to those of zw. C contains a point w_i; there exists in C an arc w_iW; from the definition of z_i, from the assumption at the beginning this paragraph, and from the statement at the end of the first paragraph, there exists an arc w_ip where p is a point of N, and w_ip is a subset of D, E', and $S-xt-yu$ It follows easily that z is a point of N^*. Thus M^* contains no limit point of N^*. Similarly N^* contains no limit point of M^*. Thus the supposition that $L-x-y$ is connected and that M^* and N^* have no points in common leads to a contradiction, since $M^* + N^* = L-x-y$.

It follows that if $L-x-y$ is connected there exists a point z and an arc vzw containing z and z only in common with L, such that the segment vzw is a subset of E and of $S-xt-yu$ and such that v is a point of M and w is a point of N. The Jordan curves $vzwtv$ and $vzwuv$ where wtv and wuv are sub-intervals of the boundary of E, are by Theorem 21 C the boundaries of mutually exclusive Jordan domains H and I, respectively, both subsets of E; where H contains x and I contains y, and no point of L besides z is a point of the boundaries of the domains H or I. It follows that $L-z$ is not connected. Hence by (2) hypothesis, if t is any point whatsoever of L then $L-t$ is not connected. It follows that $L-x-y$ is not connected. Hence L satisfies the definition of a Jordan curve. It follows easily that $D = I_{ox}$.

Theorem 42. *If p is a point distinct from O then there exists a sequence of points $w_1, w_2, w_3, \dots$ (1) where w_i lies on a proper Jordan circle, C_{w_i}, with center at O and a proper interior I_{ow_i}; (2) no two w's are on the same circle; (3) p is the sequential limit point of the sequence of w's, (4) C_{op} is either a proper Jordan circle with a proper interior I_{op} or a single point.*

Proof. The theorem is true by Theorem 34 in case p is a point of the interior with respect to O of some Jordan circle or is a point of some Jordan circle. Suppose hereafter, then in this proof that p is in the exterior with respect to O of every Jordan circle with center at O.

Case 1. Suppose that p satisfies conditions (1), (2), (3), of the conclusion of the theorem. Suppose that C_{op} contains a point q distinct from p. Let D be the set of all points x such that x is on the interior with respect to O or on the boundary of some circle, C_{w_i} mentioned in the hypothesis, and let L be the boundary of D. L contains both p ad q. Let t be any point of L. Then it is easily shown that t is the limit point of a sequence of points $t_1, t_2, t_3, \dots$ where the t's with distinct subscripts are distinct points, and each t is the transform under some rotation about O of some w. It follows by Axiom 9 that there exists a rotation about O that transforms p into t. It follows by a theorem of Janiszewski's that L is connected[1]). By Theorem 41. L is then a Jordan circle.

Case 2. Suppose that p does not satisfy (1), (2), (3), of the theorem. Let M be the set of all points x which do satisfy (1), (2), (3), and let N be $S-M$. Since S is connected, one of the sets N and M contains a limit point q of the other. It is easy to see that M is closed. Hence, q belongs to M. By Case 1, C_{oq} is either a single point or is a Jordan circle. The latter possibility leads to a contradiction with the help of Theorems 34 and 33. If C_{oq} is q there exists by Theorem 5 a region R containing q such that R' cannot be transformed by a rotation about O (which is therefore also a rotation abot q) into a point-set containing p. There exists a Jordan circle with center at O which lies entirely in R, whose exterior with respect to O is a subset of R, and hence the interior with

[1]) *Sur les continus irréductibles entre deux points*, S. Janiszewski, Journal de l'Ecole Polytechnique, 2-e Série, Seizième Cahier (1912), page 98.

respect to O of this circle contains p. By Theorem 34 this leads to a contradiction.

Theorem 43. *There exists not more than one point p distinct from O such that C_{Op} is p.*

Proof: Suppose there are two distinct points p and q besides O which remain invariant for every rotation about O. To obtain a contradiction we may proceed precisely as indicated in the last three sentences of the proof of Case 2 under Theorem 42.

Theorem 44. *There exists a point $\overline{O}$ distinct from O such that $C_{O\overline{O}}$ is $\overline{O}$; for every point x of $S-(O+\overline{O})$ C_{Ox} is a Jordan circle.*

Proof: Suppose that for each point x of $S-O$, C_{Ox} is a Jordan circle. Since S is separable there exists a countable set of points $X_1, X_2, X_3, \ldots$ such that if Y is a point of S then Y either belongs tho this set or is a limit point of it. By an argument similar to that used in the proof of Theorem 35 of L. R. H. H. it may be shown that there exists a sequence of circles $k_1, k_2, k_3, \ldots$ with center at O, such that every point is in the interior with respect to O of some k_n. Let M_n denote k_n plus its exterior with respect to O. By Theorem 34, and Theorem 14 F. A., there exists a point set G common to all the M's. Thus we are led to a contradiction Hence by Theorems 42 and 43 the theorem follows.

Definition. The point O defined in Theorem 44 is the *pole* or *opposite* of $\overline{O}$.

Theorem 45. *(1) If $\overline{O}$ is the pole of O then O is the pole of O. (2) If x is a point of $S-O-\overline{O}$ and M is a rotation about O that leaves x fixed, then M is the identity motion. (3) If x ond y are distinct points, but neither in the pole of the other, and M and N are motions such that $M(x)=N(x)$ and $M(y)=N(y)$, then for all points Z, $M(Z)=N(Z)$* [1].

Proof. Suppose that y is a point of $S-O-O$ that does not remain invariant under M. By Theorems 34 and 44 $S-O-O$ contains a point z such that I_{Oz} contains both x and y. By Theorem 32 we are led to a contradiction. Part (1) follows with the help of Theorems 43 and 44; part (3) is a consequence of part (2) and Axioms 10 and 11.

[1]) Compare this theorem with the first theorem in § 22, page 409, H. G. and with Theorem 32, part (2), L. R. H. H.

Theorem 46. *If O, P, Q, and X are points and M is a motion such that $M(O)=Q$ and $M(P)=X$, then $M(C_{OP})=C_{QX}=C_{M(O)M(P)}$.*

This theorem can easily be proved with the use, in particular, of Axioms 10 and 11.

Theorem 47. *If O and $\overline{O}$ are poles and M is a motion, then $M(O)$ and $M(\overline{O})$ are poles; in particular if $M(O)=\overline{O}$, then $M(\overline{O})=O$.*

Theorem 48. *If M is any motion whatever, there exists a point P such that $M(P)=P$.*

Proof. Let K be the surface of a sphere in a three dimensional Euclidean number space, let $\overline{J}$ be a great circle of K, and let $\overline{D}$ and $\overline{E}$ be the two Jordan domains into which $\overline{J}$ divides K. Let J be a Jordan curve in S with complementary domains D and E. With the help of Theorem 29, we can establish the existence of a correspondence that is continuous (in the sense there defined) between D and $\overline{D}$, between J and $\overline{J}$, between E and $\overline{E}$, and finally between K and S. It follows by a theorem of Brouwer's [1]) that there exists a point P which is left invariant by M.

Theorem 49. *If M is a motion such that $M(\overline{O})$ is P and $M(X)$ is Y, then for any motion N such that $N(\overline{O})$ is P, $N(C_{\overline{O}X})$ is C_{PY}.*

For proof see Axioms 10 and 11 and Theorem 46.

Theorem 50. *If O and $\overline{O}$ are poles there exists exactly one Jordan circle C with centers at O and $\overline{O}$ such that if M is any motion such that $M(O)=\overline{O}$, then (1) $M(C)=C$; (2) if x is a point such that $M(C_{Ox})$ contains a point of C_{Ox} then C_{Ox} is C, and if y is a point of I_{Ox} then $M(y)$ is a point of E_{Ox}.*

Proof: Suppose that there exists a motion M such that $M(O)=\overline{O}$ and such that if O is any circle with center at O, then $M(C)$ is not C. Then $S-O-\overline{O}$ consists of two mutually exclusive point-sets L and N which are defined as follows: L consists of those points x of S for which $M(C_{Ox})$ is a subset of I_{Ox}. N consists of those points X of S for which $M(C_{Ox})$ is a subset of E_{Ox}. (See Theorem 34, Axiom 8 and Theorems 47 and 46). Let C_1 be a Jordan circle with center at O. By Theorem 5 there exists a region R containing O such that R cannot be transformed by a motion into a region which contains points of more than one of the three following point-sets: C_1, O, and $\overline{O}$. If Y is a point of R, it is easily seen by Theorem 47 and 34 and Axiom 8, that Y is a point of N

[1]) Math. Ann., Vol. 71, pp. 114, 324; Amsterd. Ber. XVII, p. 750, XIX, p. 48.

and that $M(Y)$ is a point of L. Since both L and M exist, and since $S - O - \bar{O}$ is connected [1]) it follows that there exists a point Z which belongs to one of the sets L and N and is a limit point of the other. Suppose that Z is a point of L and hence that $W = M(Z)$ is a point of I_{OZ}. Then by Theorem 34 there exists a point y which belongs to both I_{OZ} and to E_{OW}. By Theorem 5 there exists a region H containing Z such that if $\bar{M}$ is any motion whatever then $\bar{M}(H)$ contains points of at most one of the point-sets C_{OW}, C_{OZ}, C_{Oy}. By definition of Z, H will contain points of N. Let Q be such a point. By Theorems 47 and 46 $M(C_{OQ}) = C_{\bar{O}q} = C_{Oq}$ where q is $M(Q)$. By definition of H, q is a point of I_{Oy}. By definition of y, Theorem 34 and definition of H, I_{OQ} contains I_{Oy}. This contradicts the definition of N, since q is a point of I_{Oy} and hence of I_{OQ}. Further, if we suppose that Z is a point of N, we get a contradiction by a similar argument. Hence, there must exist a circle C such that $M(C) = C$. Let X be a point of C. By Theorems 47 and 46 and Axiom 8 it follows that if $M(O) = \bar{O}$ then $M(I_{OX}) = E_{OX}$ and $M(E_{OX}) = I_{OX}$.

By Theorem 49, if M_1 is a motion distinct from M such that $M_1(O) = \bar{O}$ then $M_1(C) = C$. Hence Theorem 50 is true.

Definitions. The circle C uniquely determined by the points O and $\bar{O}$ according to Theorem 50 is said to be the great circle with centers at O and $\bar{O}$. The notations GC_O and $GC_{\bar{O}}$ are symbols that indicate this circle. The points O and $\bar{O}$ are poles of this circle. GI_O means I_{OX}, where X is a point of C; similarly, GE_O means E_{OX}. In general if P is a point, $\bar{P}$ will indicate the pole or opposite of P.

Theorem 51. *If C is GC_O and $M(O)$ is P, then $M(C)$ is GC_P.*

For proof see Theorems 46, 47, and 50, and Axioms 10 and 11.

Theorem 52. *If $C_{OP} = GC_O$ then (1) $C_{PO} = C_{P\bar{O}} = GC_P$; (2) $\bar{P}$, the pole of P, lies on C_{OP}.*

Proof: Let M and N be motions such that

$M(O) = \bar{O}$	$N(O) = O$	Then $NM(O) = \bar{O}$
$M(\bar{O}) = O$	$N(\bar{O}) = \bar{O}$	$NM(\bar{O}) = O$
$M(P) = Q$	$N(Q) = P$	$NM(P) = P$.

Hence $NM(\bar{P}) = \bar{P}$.

Therefore, by Theorem 50, $\bar{P}$ is a point of C_{OP}. Hence there exists a rotation about O which transforms P into $\bar{P}$. By Theorem 50

[1]) This statement can be proved by the help of Theorem 34, F. A.

this rotation carries GC_P into itself. Further, since O and $\bar{O}$ remain fixed under this rotation it is easily seen by Theorem 50 that O and $\bar{O}$ are points of GC_P.

Theorem 53. *If x and y are two distinct points there exists a point z such that GC_z contains both x and y.*

Proof: Let $C_{xP} = GC_x$. Then GC_P contains $\bar{x}$ and x (Theorem 52). If $y = \bar{x}$ the theorem is proved. If y is not the pole of x, by Theorem 44, x is a point of I_{xy} and $\bar{x}$ is a point of E_{xy}. Hence GC_P contains a point z of C_{xy}. There exists a rotation M about x that transforms z into y. By Theorem 51, $M(GC_P)$ is a great circle containing x and y.

Theorem 54. *If C_{Ox} is a Jordan circle there exists on this circle exactly one point $\bar{x}$ distinct from x such that if M is any rotation about O such that $M(x) = \bar{x}$ then $M(\bar{x}) = x$; if C_{Ox} is GC_O then x and $\bar{x}$ are poles.*

Proof: See H. G., page 405, and Theorems 47, 52.

Definition. The notation being that employed in the preceding theorem, the complementary intervals $xy\bar{x}$ and $xz\bar{x}$ of C_{Ox} are semi-circles of C_{Ox}. If C_{Ox} is a great circle these circles are great semi-circles.

Theorem 55. *If abx in an interval of a Jordan circle C_{Ox}, there exists exactly one point y on abx such that if M is any rotation about O that transforms a into y then $M(y) = x$ and if ay and yx are the intervals ay and yx of abx then $M(ay) = yx$.*

Proof: See H. G., page 405.

Definition. The notation being that employed in the preceding theorem, the point y is said to be the midpoint of the interval abx of C_{Ox}.

Theorem 56. *If $C_{OP} = GC_O$, $\bar{P}$ is the pole of P and Q is a point of C_{OP} distinct from P and from $\bar{P}$, and $\bar{Q}$ is the pole of Q, then the pair of points P and $\bar{P}$ separate the pair of points Q and $\bar{Q}$ on C_{OP}.*

Proof: See pages 403, 405, H. G. and Theorems 52 and 54.

Theorem 57. *If O is a point of $GC_P = C$ and $OY\bar{O}$ and $OZ\bar{O}$ are the two intervals of C with their end-points O and $\bar{O}$ and only these end-points in common, and x is any point of $S - O - \bar{O}$ and t is common to C_{Ox} and to $OY\bar{O}$, then C_{Ox} contains a point T of $OZ\bar{O}$ such that if M is a rotation about O such that one of the following statements is true, then the others are also true: $M(t) = T$, $M(T) = t$, $M(OY\bar{O}) = OZ\bar{O}$, $M(P) = \bar{P}$.*

Proof: The circle GC_o contains a point X of $OY\bar{O}$. By Theorems 52 and 56, $\bar{X}$ lies on $OZ\bar{O}$ and on GC_o. By Theorem 52, GC_o contains P and $\bar{P}$. There exists a rotation M obout O such that $M(X)=\bar{X}$. By page 405, H. G., $M(\bar{X})=X$, $M(P)=\bar{P}$, and $M(\bar{P})=P$. Hence $M(C)=C$ and $M(OY\bar{O})=OZ\bar{O}$.

If C_{Ox} is a Jordan circle, then C_{Ox} contairs a point t on $OY\bar{O}$. $M(t)=T$ is then a point of $OZ\bar{O}$. By Theorem 45 M^2 is the identity motion. Hence $M(T)=t$. It follows by Theorem 45 that if M_1 is a motion such that $M_1(O)=O$ and M_1 satisfies one of the four statements at the end of the theorem. then M_1 is identical with M.

Theorem 58. *If AB is a simple continuous arc and T is a motion that carries AB into a subset of itself, then at least one point of AB remains fixed under the transformation T.*

Proof. This theorem follows easily, with the help of Theorem 5, from the fact that AB is a simple continuous arc.

Theorem 59. *If ABC is an interval of a great circle $\bar{C}$ with a center at O and X is a point of ABC and there exists a motion T such that $T(O)=O$, $T(A)=X$, $T(X)=C$, $T(AX)=XC$, where AX and XC are the intervals AX and XC of the interval ABC, then (1) there exists a motion M which leaves X fixed and carries A into C; (2) if N is a motion such that $N(X)=X$ and either $N(A)=C$ or $N(C)=A$, then both $N(A)=C$ and $N(C)=A$; (3) if A and C are not poles and $\bar{M}$ is any motion such that $\bar{M}(A)=C$ and $\bar{M}(C)=A$, then $\bar{M}(X)=X$ and there exists exactly one point Y distinct from X such that $\bar{M}(Y)=Y$; furthermore, Y is the pole of X and is the midpoint of the interval ADC which is complementary to the interval ABC on $\bar{C}$.*

Proof. Let $\bar{X}$ be the pole of X and let $XE\bar{X}$ and $XF\bar{X}$ be complementary semicircles of $\bar{C}$. By Theorem 56, AX (see also hypothesis) is a subset of one of the two given semicircles, say of $XE\bar{X}$. Similarly XC must be a subset of one of the given semicircles; since XC and AX have only X in common, XC must be a subset of $XF\bar{X}$. By Theorem 57 there exists a rotation M about X such that $M(O)=\bar{O}$, and that $M(AX)=$ an interval HX of the semicircle $XF\bar{X}$. We wish to prove that $M(A)=H$ is the point C.

Case I. Suppose that HX is a proper subset of XC. Then $T^{-1}MT^{-1}M(O)=O$, and $T^{-1}MT^{-1}M(AX)$ is a proper subset of AX. By Theorem 58 this motion leaves fixed at least one point of AX. But it does not leave A for X fixed. Thus by Theorem 45 we are led to a contradiction.

Case II. If XC is a subset of HX, $M^{-1}(XC)$ is a subset of XA; this case may be handled like Case I. Thus we have $M(A)=C$; and $M(C)=A$, since M^2 ist the identity. (2), (3), (4) follow easily from Theorem 45 and the preceding argument.

Theorem 60. *Two distinct great circles have in common two and only two points and these points are poles; if O is a center and x is a point of a great circle C, and P is a point of I_{Ox} then $\bar{P}$, the pole of P, is a point of E_{Ox}.*

Proof: Suppose that C and K are distinct great circles. Let A and B be two distinct points that belong to both C and K. Suppose that A and B are not poles. By the preceding theorem the midpoints of each of the two complementary intervals AB of the circles C and K belong to both C and K. If we take the midpoints of the intervals on C and on K having as endpoints these midpoints and the midpoints of the intervals in C and K having as ends these midpoints and those defined before, and continue this process indefinitely we can prove by methods similar to those used in H. G., pages 415—417 that there exists a set of points V each point of which is the midpoint of an interval of K and an interval of C, the ends of these intervals being common to C and K; V is a common subset of C and K and is everywhere dense in the closed sets C and K. It follows that C is identical with K.

II. Let O be a center and let x be a point of C and suppose that P belongs to I_{Ox}. By Theorem 53 there exists a great circle H which contains O, P, and $\bar{O}$. By the preceding paragraph, since H is connected, H contains exactly two points Q and $\bar{Q}$, which are poles, in common with C. Hence the segment $Q\bar{O}\bar{Q}$ of H, which by Theorem 56 contains $\bar{P}$, is a subset of E_{Ox}. The theorem follows easily with the help of preceding theorems.

Theorem 61. *In the notation of Theorem 57, t and T are the only points common to GC_P and to C_{Ox}.*

Proof: If GC_P and C_{Ox} contain in common more than two points, then one of the complementary semicircles $Ot\bar{O}$ and $OT\bar{O}$, say $OT\bar{O}$, contains at least two points of C_{Ox}. In the order $OT\bar{O}$ let z be the first point common to the semicircle $OT\bar{O}$ and C_{Ox}, and let w be another point common to them. Let M be a rotation about O that transforms z into w. By Theorem 51 this rotation transforms $H = GC_P$ into a great circle H_1 which contains both O and w. By Theorem 60 H_1 is identical with H. The rotation M transforms the segment Oz of the semicircle $OT\bar{O}$ into a segment Ow of H. But the segment Ow of the semicircle $OT\bar{O}$ contains the point z. The interval on H which is complementary to the segment Ow just mentioned contains the point t of the semicircle $Ot\bar{O}$. Hence the segment Ow, which is the transform of the segment Oz of $OT\bar{O}$, contains at least one point of C_{Ox}. But this clearly involves a contradiction.

Theorem 62. *If $GC_P = C_{P\bar{O}} = C$, and if X and $\bar{X}$ are the two points common to C_{OP} and C (See Theorems 52 and 60) then (1) O is the midpoint of the great semicircle $XO\bar{X}$ of C; (2) GC_X contains O, $\bar{O}$, P, $\bar{P}$; (3) If R is a rotation about P that carries O into $\bar{O}$, then for any point Y of C, $R(Y) = \bar{Y}$, the pole of Y*

Proof: Part (2) follows from Theorem 52. (3) follows from Theorems 54 and 47, and H. G. page 405.

By Theorem 55 the semicircle $XO\bar{X}$ of C has a unique midpoint F. By Theorem 59 there exists a motion M such that $M(F) = F$ and that $M(X) = \bar{X}$. By Theorem 50 $GC_X = C_{XF}$. But by Theorem 52 GC_X contains both F and O. By Theorem 60 GC_X and $XO\bar{X}$ have in common only one point. Hence $F = O$. This proves part (1).

Theorem 62 A. *If C_{OX} contains two opposite points, then $C_{OX} = GC_O$. (2) If $C_{OX} = C_{PX}$, then P is either O or $\bar{O}$* [1]).

Proof: Suppose that $C_{OX} = C$ contains a pair of opposite points Y and $\bar{Y}$. Then there is a rotation M about O that transforms $\bar{Y}$ into Y and hence it transforms GC_Y into itself. By Theorem 50 O is then a point of GC_Y, and then by Theorem 52 Y is a point of GC_O. Since Y is common to C_{OX} and to GC_O, $C_{OX} = GC_O$.

(2) This is a result of Theorem 45 if X is O or $\bar{O}$. Hence if we suppose that the proposition is not true we may assume that C_{OX} is a Jordan circle, and that O and P are not poles. By Theorems 60 and 53 there exists exactly one great circle H which contains O and P. By Theorem 61 C contains exactly one point T of the semicircle $OT\bar{O}$ of H and exactly one point t of the semicircle $Ot\bar{O}$ of H, which is complementary to $OT\bar{O}$. There exists a rotation about O that carries t into T. By Theorems 61 and 57 this rotation

[1]) Part 2. of Theorem 62 A was first proved by Mr. C. M. Cleveland of the University of Texas, by methods differing from the ones used in this paper.

carries T into t. Since P also is a center of C, and since C contains only the points t and T in common with H, we can show in the same way that there exists a rotation about P that transforms T into t and transforms t into T. If T and t are not poles, it follows by Theorem 59 (Part 3) that P must be either O or the pole of O. If T and t are poles, it follows by Part (1) of the theorem that C is GC_O and also GC_P. It follows by Theorem 52 that O, P, $\bar{O}$, and $\bar{P}$ are points of GC_T. But these points are also points of H. By Theorem 60 we have then that GC_T is H and hence contains T. Thus we are led to a contradiction.

Definitions. If A and B are two distinct points which are not poles, it follows by Theorems 53 and 60 that there exists exactly one great circle K containing them. This great circle will be called *the line AB*. Of the two complementary intervals AB on K there is only one that is a subset of some great semicircle. (See Theorem 56) This interval will be called *the interval AB*. The segment AB of the interval AB will be called *the segment AB*. If M is a motion that leaves fixed a point X, and that transforms A into B and B into A, then M is called a *semi-rotation* about X. If there exists a motion that transforms A into a point E and transforms B into a point D then the interval AB is said to be congruent to the interval ED. It is easily shown by preceding theorems that such a motion transforms the interval AB into ED.

Theorem 63:

(1) If $AB = CD$ then $AB = DC$; in particular $AB = BA$.

(2) If $AB = CD$ and $CD = EF$ then $AB = EF$.

(3) If $M(x) = y$ then $M(C_{xy}) = C_{yx}$.

(4) If AB and CD are congruent intervals and M is any motion such that $M(AB)$ contains in common with CD two distinct points one of which is a common endpoint of $M(AB)$ and CD, then $M(AB)$ is CD or DC.

(5) No interval contains a pair of opposite points.

Proof: (1) is a consequence of the definition of congruence, the group-property of Motions and Theorems 55 and 59. (5) is a consequence of Theorem 56. (3) is a consequence of (1) and Theorem 49. (4) is the result of the definition of congruence, the group property of motions, part (1), and Theorems 60, 61, and 49;

(2) is a result of the group property of motions and the definition of congruence.

Definitions: If AB and CD are intervals and there exists a motion M such that $M(AB)$ is a proper subset of CD then AB is said to be less than CD and CD is said to be greater than AB. It may be shown by the preceding theorems that if AB is less than CD it is not congruent to CD or greater than CD. We may show as in H. G., pages 403 to 407, that by means of motions and midpoints we may set up a one to one reciprocal continuous correspondence between the points of a great circle C and the set of real numbres $[x]$, where $0 \leq x < 2\pi$, in such a way that if p, p_1, p', p'_1 are the coordinates of four points P, P_1, $M(P)$ and $M(P_1)$ respectively, where M is any rotation about the poles of C, then $p - p_1 = p' - p'_1$ (Mod 2π).

Of the rotations about the poles of AB there is one which transforms one of the two points A and B into the other, and which in the notation of H. G., page 405, has a parametric value w whose numerical value (mod 2π) is between zero and π. The numerical value of w is *the length* of the interval AB and will be denoted by the symbol $l(AB)$.

Theorem 64. *If X is the midpoint of AB and $M(AB) = CD$ then $M(X)$ is the midpoint of CD.*

Proof: Clearly $M(A)$ is one of the endpoints of CD and $M(B)$ is the other. The theorem follows easily with the use in particular of Axioms 10 and 11.

Theorem 65. *(1) AB and CD are intervals then $l(AB) \lesseqqgtr l(CD)$ according as $AB \lesseqqgtr CD$; (2) If C it an interior point of the interval AB then $l(AC) + l(CB) = l(AB)$.*

This theorem may be proved with the help in particular of Theorems 62, 64, and 47 and of the results of pages 403—405 of H. G.

Definitions. A point-set H which contains at least two points that are not poles is said to be collinear if it is a subset of a great circle C. The two complementary domains of C will be called the sides of H. If a set of points, X is a subset of one of the sides of H then that side will be called the X side of H. The poles of C will be called the poles of H If AB is an interval of C and $AB\bar{A}$ denotes the semicircle $AB\bar{A}$ of C then the point set $AB\bar{A} - \bar{A}$ will be called the ray AB, and A will be called the endpoint of this ray. Clearly the ray BA is not the same as the ray AB.

Definitions. If $C_{ox} = C$ is a Jordan circle then the intervals Ox and $\bar{O}x$ are *radii* of C. If C is not a great circle it is said to be a small circle By the *proper radii* or a small circle C_{ox} we shall mean those radii of C_{ox} which are *congruent* to the lesser of Ox and $\bar{O}x$. That center of a small circle which is an endpoint of *the proper radii* is said to be *the proper center* of C; that complementary domain of C which contains *the proper center* is said to be *the proper*[1]) *interior* of C, and the other one is *the proper exterior* of C. By the *radial length* of a circle we mean the length of one of its proper radii. Of two circles having unequal proper radii that one whose radial length is the greater is said to be the greater of the two circles. If two circles have congruent proper radii they are said to be congruent, and there exist motions that transform one into the other. An interval of length $\frac{1}{2}\pi$ is said to be a *quadrant*. A great circle has a radial length of $\frac{1}{2}\pi$ and is greater than any small circle. Two circles are said to be tangent if they have one and only one point in common. Two circles C and K are said to touch provided that they have points in common, but K does not contain points in common with both the interior and the exterior of C. If C and K are tangent small circles and their proper interiors have no point in common they are said to be tangent externally; otherwise they are tangent internally. Two small circles may be said to touch externally or to touch internally unter corresponding conditions

Theorem 66. *If $C_{ox} = C$ is a great circle and n is a positive number between O and $\frac{1}{2}\pi$ then there exists in I'_{ox} a circle K, and in E'_{ox} a circle k such that (1) K and k have radii of length n and are tangent externally and there exists a semi-rotation about t, the point of tangency, that transforms K into k, k into K, and the proper centers of k and K respectively into the proper centers of k and of K; (2) the proper interiors of K and of k contain no point of C, and are on opposite sides of C; (3) K and k touch C.*

Proof: Let H be a great circle passing through O and $\bar{O}$. For each point x of H let t_x be a point such that the length of xt_x is n.

[1]) Cf. the definitions following note on Theorem 29. Hereafter, when we use the word „proper" we shall use it n the sense just defined.

Let L_1 consist of all those points x of H such that I'_{xt_x} is a subset of I'_{OX}, and let L_2 consist of the remainder of H. Both classes exist. Let z be a point of division of the two classes L_1 and L_2. Suppose z belongs to L_2. Let zw be congruent to xt_x. Then I'_{zw} contains a point v of E_{OX}. Let $x_1, x_2, x_3, \ldots$ be a subset of L_1 which has z as a sequential limit point. Let P be a pole of H. Let $M_1, M_2, M_3, \ldots$ be a set of rotations about P such that for each integer m, $M_m(z) = x_m$. Then $M_m(v) = v_m$ is a point of I'_{OX} and so is V, a limit point of the set $v_1, v_2, v_3, \ldots$ If we apply the inverse motions of the M_i's and Axiom 9 we prove there exists a motion N such that $N(P) = P$; $N(z) = z$, and $N(V) = v$. But P and z are not poles and V and v are distinct points. Thus we have a contradiction by Theorem 45.

Hence z belongs to L_1. Let $K = C_{zw}$. By methods analagous to those used in the preceding paragraph we can prove that K touches C. Let y be a point GC_z. With the help of Theorem 60 we can prove that there exists a semicircle F of C which lies in E'_{zy}; and hence F contains none of the points that are common to C and K.

By methods similar to those used above we can prove that there exists a circe k' which is congruent to K, which plus its proper interior, is a subset of E'_{OX}, which touches C, and such that the set of points common to k' and C are a subset of F. We can establish the theorem by methods analogous to those of pages 411 to 414, H. G.

Let t be the point of tangency of K and k. In proving that K plus its interior is a subset of I'_{OX} we used merely the fact that the length of OX is greater than n. By Theorem 62 the length of OX is $\frac{1}{2}\pi$. Hence if m is a positive number less than n we can prove by a method similar to that used in the preceding theorem that there exist circles Z and z containing t such that Z plus its proper interior is a subset of K plus its proper interior, and z and k have the same relation. With the help of results given on page 413, H. G., we get the following theorem.

Theorem 67. *In the notation of Theorem 66 and the preceding paragraph we have; (1) Z and z are tangent to each other at t and to k, K and C and are (2) the only circles of radius m that are tangent to K, k, and C at t; (3) K and k are tangent to C at t and are the only circles of radius n that have this property; (3) K is the only circle of radius n that is tangent to k at t; (5) C is the only great circle tangent to K, k, Z and z at the point t.*

(1) is a consequence of § 28, H. G. We can in the same way prove that there exists a circle K' of radius n which is tangent to C at t. If K' is distinct from k and K we should have K' and one of the circles K and k, say K, tangent to the second, k; this is impossible by the theorem at the end of paragraph 27, H. G. (2) is a consequence of (3) and (4). (5) Follows easily.

Theorem 68. *If t is any point of S and C is any great circle containing t, then there exists a system W of circles such that (1) C is the only great circle of this system; (2) if n is any positive number less than $\frac{1}{2}\pi$ there exist two and only two circles K and k of W of radius n that are tangent to C at t; (3) if n, m, K, k, Z, z, t, and C be interpreted as in Theorems 66 and 67, the conclusions of these theorems hold. (4) Any two distinct circles of W are tangent at t, and any circle which touches or is tangent to a circle of W at t belongs to W.*

Also, if k is any small circle containing t, then there exists a great circle C which is tangent to k at t.

Proof. There exists a motion which transforms the point t of Theorems 66 and 67 into the t of this theorem and transforms the C of Theorem 66 and 67 into C of this theorem. The theorem follows, since, as may readily be seen from previous theorems, a motion transforms tangent circles into tangent circles.

Theorem 69. *(1) If two congruent circles are tangent they are tangent externally; (2) if two non-congruent small circles are tangent internally the smaller circle plus its proper interior is a subset of the larger circle plus its proper interior; (3) if t is a point of a small circle K and Q is any smaller circle, then there exist two circles Z and z congruent to Q such that z is tangent internally to K at t and Z is tangent externally to K at t*

Proof: (1) is a result of part (2) Theorem 66 and Theorem 68. (2) is a result of Theorems 68 and parts (1) and (2) of Theorem 67 (Note the paragraph preceding Theorem 67 in which the existence of Z and z is proved): (3) is a result of the definition of „tangent externally" and the properties of Jordan domains.

Definitions. If t is a point common to two circles C and K and every region R which contains t contains a point x of C which belongs to the proper interior of K and a point y of C which belongs to the proper exterior of K then C is said to intersect K, and t is a *point of intersection* of the circles C and K. (Note: If K is a great circle then „the proper interior of K" may be interpreted as one side of K and „the proper exterior of K" may be interpreted as the other side of K).

Theorem 70. *If x is the midpoint of the interval OP, then C_{xP} is tangent to C_{OP} at P: For proof* see H G., page 414.

Theorem 71. *If $C_{Px}=C$ and $C_{Qx}=K$ are not identical and have least two points in common then K intersects C at the point x.*

Proof. Suppose that K does not intersect C at x. There exists a region R containing X such that the set of points common to R and K is a subset either (a) of I'_{Px} or else of (b) of E'_{Px}. Consider case (b) and assume, as we can do without loss of generality that I_{Px} is the proper interior of C. There exists a circle $C_{xt}=k$ of radial length $2n$, where $2n$ is a positive number which is less than $\frac{1}{2}\pi$ and less than the length of Px such that k plus its proper interior is a subset of R. There exists by Theorem 69 a point p such that the length of px is n, that $C_{px}-x$ is a subset of the proper interior of C and by Theorem 70 is a subset of k plus its proper interior. It follows that C_{px} is tangent to K at x. But then by Theorems 68 and 69, K and C are tangent at x. Thus we are led to a contradiction. We can handle Case (a) in the same manner.

Theorem 72. *Two circles which are not identical have in common at most two points.*

Proof: Suppose that C and K are two distinct circles with a point x in common and having P and Q respectively as centers. The theorem has already been proved for the case where K and C are both great circles. (See Theorem 60). Hence we may assume that K is not a great circle or a single point. Then K does not contain both P and $\overline{P}$; we will assume that it does not contain $\overline{P}$ Let $I=I_{Px}$ and let $E=E_{Px}$. If we assume that K and C have at least three points in common it follows from Theorem 71 that there is a point t which is common to K and to E. There exists on the circle K an arc xty such that the points x and y are points of C but such that the segment xty of this arc is a subset of E.

Suppose that there is a point q distinct from x and from y which is common to C and K. The point q does not belong to the arc xty and hence there exists a region R containing q which contains no point of xty. By Theorem 71, R contains a point T which is common to K and E. The circle K contains an arc, XTY such that X and Y are points of C, and such that the segment XTY is a subset of E. Clearly the segments xty and XTY have no points in common.

Let L_1 consist of all those points w of E such that E'_{Pw} contains no point of the segment xty, and let $L_2=E-L_1$. It can easily be shown that both classes are non-vacuous. Since E is connected, there exists a point b which belongs to one of the sets L_1 or L_2 and is a limit point of the other Suppose that b is a point of L_1. Then by Theorem 5 there exists a region R containing b such that R cannot be thrown by a rotation about P into a region containing a point of the arc xty. But since b is a limit point of L_2, R contains a point a of L_2. By Theorem 34 and the definition of L_1, E'_{Pb} is a subset of E_{Pa}, and E'_{Pa} is a subset of E. Since a belongs to L_2, E'_{Pa} contains a point of the segment xty. But x is not a point of E'_{Pa}. Therefore C_{Pa} contains a point of the segment xty. But this contradicts the definition of R. Hence b must be a point of L_2 and it can be shown that C_{Pb} contains points of the arc xty Suppose that the circle C_{Pb} had at least two points in common with the circle K. Then by Theorem 71 there is a point f of xty which is a point of E_{Pb}. It is easily shown that every point of C_{Pb} is a limit point of L_1, an that there exists a point y of L_1. such that f is a point of E_{Py}. But this contradicts the definition of L_1. Hence C_{Pb} is tangent to K at some point of xty.

Hence the segment XTY is a subset of I_{Pb}. By an argument similar to the preceding we can show that there exists a point d such that C_{Pd} is tangent to K at a point of the segment XTY. By methods indicated earlier in this proof we can show that C is a subset of I_{Pd} while C_{Pb} is a subset of E_{Pd}. But the arc xty contains points in common with each of the circles C and C_{Pb}. But this in contrary to the supposition that C_{Pd} is tangent to K at a point of XTY[1]. Hence the supposition that the circles C and K have more than two points in common leads to a contradiction.

Definition. If C is a great circle with centers P and $\overline{P}$ and K is a great circle containing P and $\overline{P}$, then K is said to be perpendicular to C. Let O be a center of K. By Theorem 52, O is on C. Hence the following theorem

[1]) C_{Pd} must intersect the arc xty which contains no points of the segment XTY.

Theorem 73. *If C is perpendicular to K then K is perpendicular to C (C and K both great circles).*

Definition. If x is a point common to the perpendicular great circles C and K and H is a subset of C containing x and k is a subset of K such that k contains x, and each of the sets H and k contain at least two points which are not poles, then H and k are said to be perpendicular at x.

Theorem 73 A. *If Q is an interior point of a great semi-circle $PQ\bar{P}$ and x and y are the midpoints of the intervals PQ and $\bar{P}Q$ respectively, then C_{xP} and $C_{y\bar{P}}$ are tangent at Q.*

Proof. By Theorem 70, C_{xP} and $C_{y\bar{P}}$ are tangent to $C_{PQ} = C_{\bar{P}Q}$ at Q Hence by Theorem 68 the theorem follows.

Theorem 74. *If C is a great circle with center at O and $PO\bar{P}$ is a great semicircle whose endpoints P and $\bar{P}$, are points of C, and x is any point of this semicircle distinct from P, O, and $\bar{P}$, then C_{xP} is tangent to C at P. Conversely, if K is a small circle which is tangent to C at P, and the proper interior of K is on the O side of C, then the proper center of K is on the semi-circle $PO\bar{P}$.*

Proof. Set up a correspondence between the real numbers from O to 1 and the points of $PO\bar{P}$ as follows: To P assign the number O, to $\bar{P}$ the number 1, to the midpoint O of the semicircle assign the number $\frac{1}{2}$. Continue as in §§ 30 to 35, H. G. In the same way assign to the segment $\overline{POP}$ the numbers between O and -1. It follows from Theorems 70 and 68 that any circle with center at the point $(\frac{1}{2})^n$, (n a positive integer greater than 1), and through the point P is tangent to C at P. We shall now prove that *for all such values of n and for all positive integral values of k between O and 2^n (except the value 2^{n-1}) a circle with center at the point $k(\frac{1}{2})^n$ and passing through P is tangent to* C at P. For $n=1$ there are no corresponding values of k.

Let $K(u,v) = C_{uv}$. Suppose that k is greater than 2^{n-1}. Let $k(\frac{1}{2})^n - 1 = -h(\frac{1}{2})^n$. Then h is a positive integer between O and 2^{n-1} and $K(-h(\frac{1}{2})^n, P) = K(k(\frac{1}{2})^n, P)$. If $h = 2^{n-2}$, then by Theorems 70 and 68, $K(-h(\frac{1}{2})^n, P)$ is tangent to C at P. If h is not 2^{n-2} it is one of the values of k corresponding to $n-1$. By Theorem 70 $K(-h(\frac{1}{2})^{n-1}, P)$ is then tangent to $K(-h(\frac{1}{2})^n, P)$ at P. By the methods used at the end of § 27 H. G. we can show that $K(h(\frac{1}{2})^{n-1}, P)$ is tangent to $K(-h(\frac{1}{2})^{n-1}, P)$ at P and hence by Theorem 68, part 4, is tangent to $K(k(\frac{1}{2})^n, P$ at P.

If k is less than 2^{n-1}, it follows by Theorem 70 that $K(k(\frac{1}{2})^n, P)$ is tangent to $K(k(\frac{1}{2})^{n-1}, P)$ at P. In any case we have proved that either $K(k(\frac{1}{2})^n, P)$ is tangent to C at P or it is tangent to a circle $K(h(\frac{1}{2})^{n-1}, P)$ where h is between O and 2^{n-1} and is not 2^{n-2}. This last sentence is sufficient to verify the statement we made in the last sentences of the first paragraph, for $n=2$, and then to prove by mathematical induction that the statement holds for all larger integral values of n.

If x is any point whatever of $PO\bar{P}$, distinct from P, O, and $\bar{P}$, it can be proved by methods like those used in H. G., §§ 30 to 35 that x is a limit point of the set of points F, corresponding to the numbers of the type $k(\frac{1}{2})^n$. Let $a_1, a_2, a_3, \ldots$ be a subsequence of points of F having x as a sequential limit point. Let $\bar{n}$ be so large that twice the length of $a_n x$, for n greater than $\bar{n}$, is less than Px and $\bar{P}x$. It is easily shown that then x is a point of $I_{a_n P}$ and hence that any ray xQ having x as an endpoint will contain at least one point t_n of $C_{a_n P}$.

Let Q be a point of C_{xp} distinct from P. By Theorem 61 Q is the only point common to the ray xQ and to C_{xp}. Let $R_1, R_2, \ldots$ be a sequence of regions closing down on x, and for each value of n let b_n be a point of the sequence $a_n, a_{n+1}, a_{n+2}, \ldots$ such that b_n belongs to R_n. Let M_n be a rotation about b_n that transforms P into a point v_n of the ray xQ. Let $M_n(x) = X_n$. By Theorem 13 the set of points $X_1, X_2, X_3, \ldots$ has x for a limit point. The set of points $v_1, v_2, \ldots$ has a limit point V on the ray xQ. By Axiom 9 there exists a rotation about x that transforms P into V. Hence V is Q. But all of $C_{b_n P}$ except the point P is on the O side of C. (See definition of b_n). Hence Q is on the O side of C or is a point of C; this holds for every point Q of C_{xp}. By Theorem 71 C_{xp} and C must either be identical or must be tangent at P. By Theorem 62 A the two circles cannot be identical, since O is not x.

Let K be a small circle of radius n which is tangent to C at P and except for P is on the O side of C; there exists by the preceding argument on the same side of C, a point n belonging to $PO\bar{P}$ such that C_{nP} is tangent to C at P, and that the radius of the circle C_{nP} is n. By preceding theorems there exists exactly one such circle and it easily shown that this circle is K.

Definitions: If A, B, and C are distinct, non-collinear points, no two of which are poles, then the sum of the rays BA and BC

is called *the angle* ABC, and B is the *vertex* of this angle. If M is a motion and $M(B)=B'$, $M(Ray\,BA)=Ray\,B'A'$, and $M(Ray\,BC)=Ray\,B'C'$, then the angle ABC is said to be congruent to the angle $A'B'C'$. Let the circle W mentioned in § 41, H. G., be a great circle; we shall adopt for this paper the definitions and conventions given in this paragraph of H. G. If K is GC_B, and D and E are the points common to K and to the rays BA and BC respectively then the length of the interval DE is the numerical value of the parametric value of the angle ABC, as this parametric value has been defined in H. G., paragraph 41. This interval DE is called *the associated interval* and the great circle K is the *associated great circle* of the angle ABC. An angle of measure $\frac{1}{2}\pi$ is called a right angle. A point on the C side of the ray BA and on the A side of the ray BC, is said to be within the angle ABC and the set of all such points x is the *interior* of the angle ABC. It follows that the interior of the angle ABC is the same as the interior of either of the angles CBA and $A\bar{B}C$.

Theorems 75 and 76 follow without difficulty from preceding theorems and definitions.

Theorem 75. *(1) The interior I of the angle ABC is a Jordan domain whose boundary is the angle ABC plus $\bar{B}$, the pole of B; (2) if X is a point of I then the ray BX minus the point B is a subset of I; (3) the segment DE of the associated interval DE of the angle ABC is a subset of I but no point of the associated great circle minus this segment belongs to I; (4) if the ray BA is perpendicular to the ray BC at B then the angle ABC is a right angle, and conversely.*

Definitions. The interior of the angle ABC plus the boundary of this interior in called *the lune* ABC. Clearly the lune ABC is the same as the lune $A\bar{B}C$. $S-$(the lune ABC) is *the exterior of the angle* ABC, and any point of this set is said to be *without* the angle ABC.

If A, B, C, are distinct, non collinear points, no two of which are poles, then the intervals AB, BC, and CA are the *sides* of the triangle ABC, and the sum of these intervals is *the triangle* ABC. If x is a point *within* each of the angles ABC, BAC, BCA then x is *within* the triangle ABC and the set of such points x is the *interior* of the triangle ABC. Clearly x is within the triangle ABC if and only if it is on the A side of BC, on the B side of AC

and on the C side of AB. Also if it is within two of the angles of the triangle, it is within the third.

Theorem 76. *The interior of the triangle ABC is a Jordan domain whose boundary is the triangle ABC. (2) If x is a point within the angle ABC then the ray Bx contains exactly one point y in common with the segment AC of the side AC. (3) The segment By of the ray Bx is within the triangle ABC. (4) The segment AC of the side AC is within the angle ABC; if H is the line AC then no point of H minus the interval AC belongs to the lune ABC. (5) If the arc h of a great circle HF is a subset of the lune ABC, and not both B and $\bar{B}$ are endpoints of h, then h is an interval; that is, it is a subset of a great semicircle.*

Theorem 77. *If $C_{AD}=C$ and $C_{BD}=K$ have in common exactly two points D and d then D and d are not on the same side of AB.*

Proof. B and A are not poles, or the same point, since the circles C and K are not identical. Suppose that D and d are on the same side of AB. By Theorem 71 there exists a point y of K lying on the non-D side of AB. The interval yd of dyD of K contains a point h of the line AB while the interval yD of dyD contains a point g of AB. By Theorem 72 if Dxd is the complementary interval on K of Dyd, then Dxd contains no point of AB. By Theorems 71 and 72, if H is that complementary domain of C which contains the segment Dxd of K, H contains no point of the segment dyD of K. The segment Dxd contains neither A nor $\bar{A}$. By the methods used in the proof of Theorem 72, we can prove that there exists in H a circle k with centers at A and $\bar{A}$ such that k is tangent to K at a point z of the segment dxD. By Theorems 74 and 68 the intervals Az and Bz are both perpendicular to the common tangent of the circles K and k at z. Then A, z, and B are collinear. But there is only one great circle passing through A and B. Thus z is on AB and we are led to a contradiction.

Definitions. Two triangles ABD and $A_1B_1D_1$, are said to be congruent if there exists a motion M such that $M(A_1)=A$, $M(B_1)=B$, and $M(D_1)=D$. In the notation of Theorem 77 the triangles ABD and ABd are said to be symmetric triangles. If a triangle T_1 is congruent to a symmetric triangle of T_2 then T_1 and T_2 are said to be symmetrically congruent. It follows as in H. G. that T_1 is not congruent to T_2.

Theorem 78. *If the interval xy is less than a quadrant, and K is a great circle which is perpendicular to xy at x then (1) if z is a point such that yz is greater than yx and less than $y\bar{x}$, then C_{yz} contains two distinct points t and T in common with K such that the semi-circles $xt\bar{x}$ and $xT\bar{x}$ of K are complementary semicircles of K. (2) If W and Z are distinct points of $xt\bar{x}$ in the order $xWZ\bar{x}$ then yZ is greater than yW. (3) The converse of (2) is true.*

Proof. By Theorem 74 C_{yx} is tangent to K at x. It is easily shown that I'_{yx} does not contain z or a point of K distinct from x and that C_{yz} contains points of both the y side and the non-y side of K. Hence, by preceding theorems C_{yz} contains exactly two points of K. (1) follows from Theorem 77. In (2) if $W=x$, (2) follows from the second sentence in this proof. If W is distinct from x, then the interval yx, and the segment Wx of the semicircle $xW\bar{x}$ of K are subsets of I_{yW}; and the segment $W\bar{x}$, which contains Z, is a subset of E_{yW}. (2) and (3) follow easily.

Theorem 79. *If ABC is an acute angle, that is an angle whose measure is numerically less than $\pi/2$, and DE is its associated interval then (1) if a is a point distinct from A on the ray BA in the order $DAaB$ and C and c are points of the ray BC such that the angles ACB and acB are right angles, then in the right triangles ABC and aBc, CA is greater than ca and BC is greater than Bc; (2) if x is a point of the ray BA, there exists on the ray BC exactly one point y such that xy is perpendicular to BC, and further, $xy < \frac{1}{2}\pi$.*

Proof. The great circle K containing the associated interval DE contains O, the pole of BC which is on the A side of BC. Since angle ABC is less than a right angle, the interval DE is less than a quadrant and hence does not contain O. But all points of K which are within the angle ABC are on the interval DE. Hence O is without the lune ABC. There exists exactly one quadrant OAC. This quadrant is a subset of the great circle containing the interval AC. The interval AC is a proper subset of the quadrant OC. Similarly it can be proved that the interval ac is a proper subset of the quadrant Oc. From definition of the perpendicularity and Theorem 74, OD is perpendicular to BA at D. Since we have $DAaB$ it follows by Theorem 78, part (2) that OA is less than Oa. By Theorem 65 it follows that AC is greater than ac. Suppose that on the ray BC we have order BCc or that $C=c$. It is easily shown that in this case the intervals AC and ac have points in common. But then the quadrants OC and Oc would have at least two points in common and would be identical (Theorem 60). Then we would have $A=a$ which is contrary to the hypothesis. Hence Bc is less than BC. The proof of part (2) offers no difficulty.

Theorem 80. *In a right triangle a side opposite an acute angle is less than a quadrant.*

This follows from Theorem 79.

Theorem 81. *In the triangle ABC if angle C is a right angle and AC is less than a quadrant, the angle B is an acute angle* [1].

Theorem 82. *In the triangle ABC if C is a right angle and AB is less than a quadrant and either one of the angles B or A is acute, or one of the sides BC or AC is less than a quadrant, then both A and B are acute angles and both AC and BC are less than AB.*

Proof. From Theorem 81 and the hypothesis it follows that either A or B is less than a right angle. Suppose B is. From Theorem 79 it follows that both BC and AC are less than a quadrant. By Theorem 81 angles A and B are both acute. By Theorem 78, part (2), AC and BC are both less than AB.

Definition. The angle ABC is said to be symmetrically congruent to any angle which is congruent to the angle CBA.

It is our object in the remainder of this paper to prove that the base angles of an isoceles triangle are symmetrically congruent. We shall prove this proposition by showing that we can derive the formulas for the solution of triangles by methods analogous to those used by W. H. Young in his paper *On the Analytical Basis of Non-Euclidean Geometry* [2]. However, he assumes the symmetric congruence of the base angles of an isosceles triangle. This makes it necessary for us to modify his treatment considerably.

Henceforth when we use the term „the interval AB" we shall mean sometimes the set of points which constitutes the interval AB, and sometimes the number which is the length of this interval. Which interpretation is intendend will in general be evident from the context of the discussion.

[1]) For proof see M. Dehn, *„Ueber den Inhalt spärischer Dreiecke"*, page 169, *Mathematische Annalen,* Vol. 60, 1905.

[2]) American Journal of Mathematics, Vol. 33, 1911 (pp. 249—286). This paper will be referred to hereafter as „Young".

Theorem 83. *If A is an acute angle of the triangle ABC and x is a point of the segment AC, then the segment Bx is within the triangle and Bx is less than the greater of BA and BC.*

Proof. By Theorems 78, 72, 74, and 79 the segment AC and the interior of the triangle ABC are subsets of $I_{BA}+I_{BC}$.

Definitions: By *the sum* of a set of angles we mean the sum of the absolute values of the parametric values of these angles; in other words, the sum of the lengths of the associated intervals of these angles. The greater of two angles is the one having the greater measure. The *angle excess* or the *excess* of a triangle is the angle sum of this triangle diminished by π.

Theorem 84. *Every triangle has a positive angle excess.*

Proof. We can follow a proof given in a book by Carslaw[1]; consider the proof given on page 133. It can easily be shown, with the help of Theorem 79, from our definitions of the terms „mid-point of an interval" and „the congruence of triangles" that the triangle BDE is congruent to the triangle ADF.

Let AOB be a fixed angle. We shall suppose for convenience that OA and OB are quadrants; hence AB is the associated interval of the angle. On the great circle AB in the order $AB\bar{A}$ let A_1 be a point such that BA_1 is congruent to BA. BA_1 is the associated interval of the angle BOA_1; we shall call BOA_1 and any angle congruent to BOA_1 a symmetric angle of the angle BOA. If the variable angle XYZ is a function of a variable w, by the statement (1) $\lim\limits_{w\to a}$ (angle XYZ) = angle BOA, we mean that the limit of the absolute value of the measure of the angle XYZ as w approaches a is the absolute value of the measure of angle BOA. Let xOB be an angle which is congruent to the angle XYZ and let the point x be such that xB is the associated interval of the angle xOB. (Note that the angle xOB is *not* congruent to the angle BOx). From statement (1) it follows that at least one of the points A and A_1 and possibly both may be the limiting positions of the point x.

(2) By the statement $\lim\limits_{w\to a}$ (angle XYZ) = angle AOB *in the strict sense* we mean that the limit of the length of the interval Ax,

[1] H. S. Carslaw, *The Elements of Non-Euclidean Plane Geometry and Trigonometry*, Longmans, Green, and Co., (1916). Concerning the angle sum of polygons cf. M. Dehn, loc. cit.

as w approaches a, in zero. Definition (2) introduces a notion which evidently is a subcase of the case of definition (1).

Definition: If A, B, C, D are distinct points, no pair of which are poles, and no three of which are collinear, by the quadrilateral $ABCD$ we shall mean the sum of the intervals AB, BC, CD and DA. By a tri-rectangle we shall mean a quadrilateral, three of whose angles are right angles, and whose fourth vertex is not without any one of these angles.

Theorem 85. *If $ABNM$ is a tri-rectangle with right angles at N, B, and M, and C is a point on the ray MA such that MC is congruent to BN and BN and NM are both less than quadrants, then AM is less than BN and similarly BA is less than NM. The angles NBC, BCM, and BAM are each greater than a right angle.*

Proof. Let P be the pole of NM which is on the same side of NM as the points B and C. Let Q be the pole of BN which is on the M side of BN. It is easily seen that the quadrants BQ and PM contain in common the point A. Also, the quadrant PM contains the point C. Angle NPM is acute since NM is less than a quadrant. Hence by Theorem 82 angle PAB is acute and angle BAM is greater than a right angle. By Theorem 79 NM is greater than BA and similarly MA is less than BN. Since MC is congruent to BN the point C belongs to the segment PA. This segment, by a previous theorem, lies within the angle PBA, which is a right angle. Hence, angle PBC is an acute angle and angle CBN is greater than a right angle. Similarly, angle BCM is greater than a right angle[1].

Theorem 86. *In triangle ABC if $AB = BC$, then (1) angle CAB in greater than, equal to, or less than a right angle according to AB being greater than, equal to, or less than a quadrant. (2) In any case the limit of each of the angles A and C as angle B approaches zero and BA approaches a limit, is a right angle. (3) There exists an the segment AC, if AB is not a quadrant, exactly one point M such that the interval MB minus the point M is within the angle ABC, angle BMA is a right angle, and the lenght of AB is between*

[1] It is to be noted that since this is an elliptic geometry and since Young is developing a hyperbolic geometry, many inequalites occuring in Young's paper, as for instance, in the case of the theorem just proved, will be reversed in our treatment. In the future we shall mention this fact only where confusion is likely to occur.

that of MB and that of a quadrant. (4) The intervals MC and MA are less than a quadrant.

Proof. Suppose that BA is less than a quadrant. By methods used in the proofs of previous theorems, we can show that the segment AC is within I_{BA}, and that there exists on this segment a point M such that C_{BM} is tangent to AC at M. MB is perpendicular to AC at M, and since M is in I_{BA}, BM is less than AB. By Theorem 82 the conclusions for (1), (3), and (4) of this case hold. If now angle B approaches zero, (2) follows by Theorem 84. If BA is greater than a quadrant the triangle $A\bar{B}C$, satisfies the conditions of Case 1. Also angle ABC is congruent to angle CBA. The conclusion follows easily

For the case where AB is a quadrant there is no trouble.

Theorem 87. *In the triangle ABC if angle B is acute and angle C is greater than a right angle, then AC is less than AB.*

Proof. There exists by Theorem 79 on the ray BC a point N such that AN is perpendicular to BC at N and AN is less than a quadrant. Similarly, if E is a point of the ray BC in the order BCE, there exists on the ray CE a point N_1 such that AN_1 is less than a quadrant and AN_1 is perpendicular to BC at N_1. There is one and only one great circle through A which is perpendicular to CB, since A is not the pole of BC. Hence, if N_1 is distinct from N it must be $\bar{N}$, the pole of N. But $NA\bar{N}$ is a semicircle, and hence by Theorem 65 not both AN and AN_1 could be less than a quadrant. Hence N is N_1. Hence by Theorem 78, part (2) AC is less than AB. For, N is on the ray BC and is also on the ray CE. Evidently then we have the order BCN on the ray BC.

Theorem 87 A. *In a triangle ABC if angle B is acute, angle C in greater than a right angle, AB is not greater than a quadrant, and E is a point on the ray BC in the order BCE, then (1) angle ACE is greater than angle ABC; (2) angle BAC is acute.*

Proof. If the theorem is not true, there exists on the interval AC a point „a“ such that angle aBC has the same measure as angle ACE. Let O be that pole of BC which is on the A side of BC. By methods used in the proof of Theorem 87 we can prove the existence of a quadrant Oah such that h is a point of the ray BC in the order BCh. Let H be defined in the same way as the point N in the proof of Theorem 87. The ray AC falls within the angle BAH which by Theorem 82 is acute. Hence by Theorems 87

and 83 $BC < Ba \leq BA$. There exists a rotation M, about O that transforms C into B. By page 405 H. G., M transforms the ray CH into the ray BC. Hence it transforms the ray CA into the ray Ba. Let $M(a) = b$. By Theorem 87 aC is less than aB. Hence b is a point of the interval Ba and is distinct from a, since M is not the identity motion. It is easily shown by preceding theorems that the set of points common to the line Ba and to I_{Oa} is the segment ba, and that this segment is also a subset of the interval Ba. By methods used in the proof of Theorems 66 and 72, and by Theorem 67, there exists a point y on the segment ba such that C_{Oy} is tangent to Ba at y By Theorem 74 Oy is perpendicular to Ba. In the order Oyz on the line Oy there exists a point z such that Oyz is a quadrant which is perpendicular to BC at z, and that z is a point of the ray BC. Since y is a point on the interval Ba, By is less than a quadrant. By Theorem 82 angle Byz in then acute. Thus, if the theorem is false, angle Byz is both an acute angle and a right angle.

Theorem 88. *If DBC is an acute angle and H is a point on the ray BC and K is a great circle through H and perpendicular to BC then (1) K contains exactly one point A in common with the ray BD; (2) AH is less than a quadrant; (3) if HB is less than a quadrant, then so is AB, and if HB is greater than a quadrant, so is AB.*

Proof. It is evident by (3) Theorem 75, that Q, that pole of BC which is on the D side of BC, is not a point of the lune DBC and hence that Q is on the non-H side of DB. By definition of K, Q is a point of K. Hence it is easily shown that the quadrant HQ contains a point A of the ray BD. Let Z be GC_B. If one of the intervals AB and BH is greater than a quadrant and the other is less than a quadrant, then the endpoints A and H of the interval AH must be on opposite sides of Z, and Z and the interval AH contain in common a point X. The point X is evidently neither Q nor $\bar{Q}$: But by Theorem 52 Z contains Q and $\bar{Q}$. Hence K and Z contain in common three points and by Theorem 60 we are led to a contradiction. Hence we have proved (3).

Theorem 89. *In a triangle ABC the sum of the sides AB and AC is greater than the side BC.*

Proof. Suppose that the theorem is not true and that there exists a triangle ABC such that BC is greater than, or equal to the sum of AB and AC. Then there exist on the interval BC points X and x such that Cx is congruent to CA and BX is congruent to BA. If BC equals the sum of the other two sides then $X = x$. By Theorems 74 and 68 C_{Bx} and C_{Cx} are then tangent at x, and since these two circles have only one point in common it follows that $A = x$. Hence, B, C, and A are collinear, contrary to hypothesis

Hence, with the help of Theorem 65, we can prove that Cx and BX have no points in common. Then on BC we have the order $BXxC$. By the argument used above it follows that C_{CA} and C_{BA} are not tangent and since C and B are not poles the two circles are not identical (Theorem 62 A). Hence by Theorem 72, they have in common exactly two points. Let X_1 be the point distinct from X that is common to C_{BA} and the line BC. By Theorems 71 and 72 the segment XAX_1 of C_{BA} is a subset of the A side of BC.

Let x_1 be the point distinct from x common to the circle with center at C and containing A. The sum of the intervals Cx and XB is less than AB which is less than a semicircle. The intervals Cx_1 and BX_1 are subsets of that interval BC of the line BC which is greater than a great semicircle, and hence the intervals Cx_1 and BX_1 have no points in common. The points X and X_1 are thus both in E_{CA}. (By Theorems 71 and 72 the only point common to I_{CA} and the great circle passing through B and C are the points of the interval x_1Cx of this great circle). It follows by Theorem 71 that the segment XAX_1 of C_{BA} contains two points of C_{CA}. By Theorem 77 we are led to a contradiction.

Let OA[1]) be equal or less than a quadrant and let OB be less than OA. Then by preceding theorems AM, OM, BN, ON, OP, CP, are each not greater than a quadrant. With the help of Theorem 79 we can show that if H is the pole of OM on the A side of OM then there exists in the interval CH of the quadrant HCP a point E such that AE is perpendicular to HCP at E and EA is less than a quadrant. As in the proof of Theorem 84 we can show that by a semi rotation about C the triangle ECA goes into a triangle FCB, where F is a point of the ray CP The inequalities (reversed) of § 3 of Young follow without trouble. In § 4 Young applies the symmetry theorem and hence his argument is not valid for our purposes. We shall proceed as follows: In Figure 2 let B be the same as O. Then $OC = \frac{1}{2} OA$, and by inequality (1) § 3, Young $\frac{CP}{OC} > \frac{AM}{OA}$ Now we can proceed exactly as in paragraph 5, Young, noting, however, that the inequalities of Young must be reversed. We have by the above and previous theorems that CP is less than AM, which is less than EP, provided OC is less than OA, and OA is not greater than a quadrant. We have also by Theorem 82 that angle OCP and hence angle ECA is less than a right angle. Then in the right triangle EAC, since AC is less than a quadrant, it follows that EC is less than CA; hence (1) $\frac{CP}{OA} < \frac{AM}{OA} < \frac{EC+CP}{OA}$; (2) $OA = OC + CA$. Hence, if we let A approach C, we reach the conclusion at the end of § 7. In case C approaches A, multiply the inequality (1) above by $\frac{OA}{OC}$ and the same conclusion may be drawn.

We define and prove the existence of the sine of an angle AOM as in § 9[1]). We cannot prove at this stage that $\sin AOM = \sin MOA$. The same problem will arise in connection with the other trigonometric functions of an angle; it will be proved, after we have developed the addition formulas, that $F(\text{angle } AOM) = F(\text{angle } MOA)$ where F is any one of the six trigonometric functions.

Concerning § 10, Young. Let AOM be an acute angle and let OA be equal or less than a quadrant. Use the notation of § 10. Let F be that pole of OM which is on the A side of OM. By previous theorems the intervals BN, CP, AM are each less than a quadrant and hence are subsets of the quadrants FN, FP, and FM respectively. By Theorem 82 the angles OCP and OAM are acute. Hence by Theorem 79 there exists on the interval CP a point E such that the interval BE is less than OP and that BE and CP are perpendicular at E; similarly there exists on the interval AM a point H such that CH is perpendicular to AM at H, and CH, is less than a quadrant. Since P is the midpoint of NM there is a rotation about F that carries P into N and M into P. This rotation carries C into a point c of the quadrant FN and H into a point h of the quadrant FP. By Theorem 79, BN is less than CP which is congruent to Nc. Hence on the quadrant FN we have

[1]) See figure in § 3, Young.

[1]) The sine ratio is never greater than unity. See Theorem 82.

the order $NBcF$. Hence by Theorem 79 BE is greater than ch which is congruent to CH.

In triangle BCE angle E is a right angle. Hence angles EBC and BCE have a sum greater than a right angle. Angle PCH the fourth angle of the tri-rectangle $PCHM$, is greater than a right angle, and hence the sum of angles BCE and ACH is less than a right angle. It follows that the angle ACH is less than the angle EBC. Since E is a point of the ray CP, the interior of the triangle BEC is within the M side of OBA. Similarly the interior of the triangle CHA and all of this triangle except the side AC are on the M side of OBA. Let Q be a pole of OBA. There exists a rotation about Q that transforms C into B. Let the transforms under this rotation of A and H be A_1 and H_1 respectively. By the formulas on page 405, H. G., A_1 will be a point of the ray BC. Since the angle A_1BH_1 is congruent to the angle ACH which is less than the angle CBE, the ray BH_1-B falls within the angle CBE. Also, since BH_1 is congruent to CH which is less than BE, it follows by Theorem 78 (part 2) and Theorem 76 (part 3) that the interval BH_1-B is within the triangle BEC. Since the pole T of BE on the C side of BE is on the non-E side of BC (by an argument we have used several times) there is a point K common to the ray BC and the interval H_1T. The ray KH_1 will contain a point L of the ray BE. By Theorem 76, parts (4) and (5) the segment KL is a subset of the interior of the angle CBE and contains H_1. If K is a point of the ray CA, the interval KL must then contain a point of the side CE of the triangle CBE. But since the angles KLE and CEL are right angles and CE is less than a quadrant, such an intersection is evidently impossible. Also, evidently, K is distinct from B Hence K is a point of the segment BC. Since BK is less than BC which is less than a quadrant it follows that LK and hence LH_1 is less than a quadrant; also BH_1 is less than a quadrant. Hence from the right triangle BH_1L, the angle BH_1L is acute. Hence the angle BH_1K is greater than a right angle. Since the angles BH_1K and BH_1A_1 have a ray H_1B and a vertex H_1 in common either the interior of one is a subset of the interior of the other or their interiors have no points in common. By Theorem 76, part (4), the segment KB is within the angle BH_1K, and the segment BA_1 is within the angle BH_1A_1; the first of these angles is greater than a right angle, and the second is a right angle, and the segments BA_1 and BK are subsets of the ray BC. Hence K is without the angle BH_1A_1 and A_1 is a point of the segment BC and we have proved that BC is greater than BA_1 which is congruent to CA.

If we reverse inequalities, *§§ 7, 11, 12 and 14 follow with few or no modifications. § 13 will be considered later in connection with §§ 16–18.*

Concerning § 15 of Young. The inequalities for the sine, cosine, and tangent given in § 15 of Young hold good here. Young's proofs for the case of the cosine and tangent hold here. We shall now consider the case of the sine

Use the notation of Young in Figure 5. Assume that OA is less than a quadrant. By Theorem 78 (2) $OM < OA' < OA$. Angle AOA' is less than a right angle and hence by Theorem 79 and 82 there exists on the interval OA a point B such that $A'OB$ is a right triangle having a right angle at B and having the properties of the triangle mentioned in Theorem 82. We have then from the right triangle ABA' that AB is less than AA'. We also have $OB < OA' < OA$. Hence $OA - OA' = e\,AA'$ where e is a positive quantity less than unity. Also $A'M < OA'$. Hence.

$$\text{(1)} \qquad \frac{AM}{OA} = \frac{AA' + A'M}{OA' + e . AA'} > \frac{A'M}{OA'}.$$

From this inequality, if we let OA approach zero, it follows easily that sin (AOM) is not less than sin $(A'OM)$.

Paragraphs 16, 17 and 18. Since the sine function and the tangent function are monotone increasing functions of the decreasing OA (notation of § 10) it follows that the sines of acute angles are positive numbers, which, as pointed out by Young, are not greater than unity; and also that the tangents of acute angles are not zero. Refer now to Figure 7, §§ 17 and 18. Assuming that all the intervals mentioned are less than a quadrant, it follows that

$$\frac{OK}{OA} > \frac{AM}{OA} \quad \text{and} \quad \frac{KA}{OA} < \frac{OM}{OA}.$$

From the second of these inequalities it follows that if we let OA approach zero, cos (AOM) is greater than or equal to sin (AOK), which is a positive number. Since the cosine function is less than unity when OA is less than a quadrant and since it decreases monotonely with OA, it follows that the cosine of an acute angle is *between zero* and *one*. It follows that the six trigonometric functions of an acute angle exist, and are all positive numbers. In Figure 7, let OA have a definite value r less than a quadrant. Let OM have a length less than $e.r$, where e is a positive value which has been asigned in advance and which we will suppose is less than unity. Let Q be the pole of OM which is on the A side of OM. The point Q is without the circle C_{OA}. The circle C_{OA} has only the point A in common with the quadrant QM. Let N be the pole of OQ which is on the M side of OQ. Then by Theorem 52 and later theorems, angle QON is a right angle, and M is a point of the quadrant ON. The ray $OA-O$ is on the Q side of OM. Also A is on the M side of OQ. Hence the ray OA falls within the right angle $QOM = QON$ and contains a point B of the segment QN of the associated interval QN of that angle. By Theorem 76, part (2) the segments AQ and BQ each contain one point of any ray which falls within the angle QOB. Let y be a point belonging to the segment AQ and x a point common to the rays BQ and Oy. Qx is the difference between the measures of the angle xON and the right angle QON. (By Theorems 71 and 72, applied at the point x, the intervals Ox and xN have only x in common). Let Z be a point on the interval Oy such that OZ is equal to or less than OA. Let ZN' be the perpendicular to the ray OM at N'. By results we have established for the cosine ratio it follows that

(1) $$\frac{ON'}{OZ} < \frac{OM}{Oy} < \frac{OM}{OA} < e.$$

Those inequalities hold for all x's on the interval QB and all Z's on Oy such that OZ is less than $OA = r$. Hence in the notation of § 21 Young, it follows that $\lim\limits_{\substack{z=0\\ x=\frac{1}{2}\pi}} f(x,z) = 0$ in the strict sense where f stands for the cosine ratio. In the same construction let OM remain fixed, but let B approach Q. OBN and OxQ are right angles, since by preceding theorems O is a pole of the great circle QN. The limit of $\frac{BQ}{AQ}$ as AQ approaches zero is the cosine of angle MQN. But the cosine function for fixed angles is a monotone decreasing function of a decreasing argument and is never less than cosine of the angle, which is positive number; hence, as BQ approaches zero AQ approaches zero; and from the right triangle ABQ it follows that AB also approaches zero. Since OB and QM are quadrants $\frac{AM}{OA} = \frac{\frac{1}{2}\pi - AQ}{\frac{1}{2}\pi - AB}$. Whith the help of inequality (1) of the section concerning § 15 of Young it follows that if we define x, y, Z, N' as before we can find a position of B such that if e is a positive number

(2) $$1 > \frac{N'Z}{OZ} > \frac{yM}{Oy} > \frac{AM}{OA} > 1 - e.$$

In the notation of § 21 we have proved that $\lim\limits_{\substack{z=0\\ x=\frac{1}{2}\pi}} f(x,z) = 1$, where f stands for the sine ratio, and where the angle BON approaches the right angle QON in the strict sense. (See note following Theorem 84). In inequality (1) let OZ approach zero. Since the cosine function for a fixed angle decreases monotonely with the distance it follows that the cosine of angle xOM is less than e. Similarly in inequality (2), since the sine function increases with the decreasing distance it follows that sin xON is between 1 and $1-e$. Hence we can make the statement made at the foot of page 261, Young, provided we say that the cosine has the limit zero and the sine has the limit unity. Finally, it is not necessary that the angle AOM have the angle QOM as a limit in the strict sense. For, if the point B has both Q and $\overline{Q}$ as limiting positions it is evident that for a subsequence of the B's which have Q for a limiting position we can set up inequalities (1), and (2), while for that subsequence which has $\overline{Q}$ as limiting position, we get similar inequalites. Adopt again the notation of Figure 7, Young. Since $\frac{KO}{OA} > \frac{AM}{OA}$ it follows from what we have just proved that as angle AOK approaches a right angle and angle AOM approaches zero and the distance OA approaches zero, the sine function of angle AOM and the distance OM approaches zero. Also the cosine of angle AOK is not less than the sine of AOM. Hence, from what we have proved before about the cosine it follows that as angle AOM approaches zero its sine approaches zero.

From $\frac{AK}{OA} < \frac{OM}{OA}$ we can prove by methods analagous to those used above the statements in paragraphs 17 and 21 concerning the convergence of the cosine to the value unity and the uniform convergence of the cosine ratio to this same value as the angle AOM approaches zero and OA approaches zero. We can now affirm the statements of § 19 and accept the definitions there given.

We shall next establish some of the results given in §§ 20, 21, 23 and 24. See the construction in the section concerning § 15.

From $\frac{AM}{OA} > \frac{A'M}{OA'}$ it follows that

$$(1) \qquad 0 < \frac{AM}{OA} - \frac{A'M}{OA'} < \frac{AM - A'M}{OA'} = \frac{AA'}{OA'} = \frac{AA'}{A'B} \cdot \frac{A'B}{OA'}.$$

We wish to prove that if either the angle AOM or the angle $A'OM$ is held fixed and has a measure „a", as the angle AOA' approaches zero and the lengths OA and OA' approach zero, the right hand expression in the inequality above approaches zero. Since the angle AOM approaches a fixed value a, and the angle OMA is a right angle, it follows from the statement in § 1 on the lower half of page 251, Young, that the angle OAM approaches the value $\frac{1}{2}\pi - a$. By the same reasoning it follows in the right triangle ABA'[1]) that the angle $AA'B$ approaches the value a.

Consider now a fixed angle HIJ having a positive measure $\frac{1}{2}\pi - a - e$, where e is a positive number, and let its symmetric angle be called KIJ. (See note at the end of Theorem 84). Let the variable angle XIJ be congruent to the angle $A'AB$ and further assume that the triangle XIJ is congruent to the triangle $A'AB$ in such a way that for each value of the variable OA there is a motion that transforms A into I, A' into X, and B into J. Let OA be so small that angle $A'AB$ is greater than $\frac{1}{2}\pi - a - e$. (By hypothesis H and K are on opposite sides of IJ. The ray IJ is fixed and so is the point I but the point J is not fixed). Since XIJ is greater than either of the fixed angles HIJ and KIJ, if X is on the K side of IJ the ray IK must fall within the angle XIJ. If X falls upon the H side of IJ the ray IH will fall within the angle XIJ. We shall for the sake of simplicity consider the case where IK is within the angle XIJ. In this case by Theorem 76 the ray IK and the segment XJ contain a point L in common. Let r be a definite value of IL for which all the statements

[1]) See notes on paragraph 15 for notation.

above are true, and let k be the corresponding value of $\frac{JL}{IL}$. Let r' and k' be defined similarly in terms of IH and $\frac{JH}{IH}$. From the properties of the sine ratio it follows that when IL is less than r the ratio $\frac{JL}{IL}$ will be greater than k. But by inequality (1), in the section concerning § 15, $\frac{JX}{IX} > \frac{JL}{IL}$. We can make a similar statement for the case when X is on the H side of IJ. Hence, since as OA and OA' approach zero, AA' and IX approach zero, it follows that $\frac{IX}{JX} = \frac{(AA')}{(BA')}$ is always less than the smaller of the values $\frac{1}{k}$ and $\frac{1}{k'}$ and hence is bounded. $OM < OA' < OA$; therefore, as OA approaches O, so do OM and OA'. The variable angle AOM is less than some definite angle b which is less than a right angle. From what we proved in the section concerning §§ 16 and 17 concerning the cosine function it follows that $\frac{OM}{OA}$ is always greater than cos (b) which we have proved is a positive quantity. Hence, since OM approaches zero with OA', OA will approach zero with OA'. Hence, we have proved that if one of the quantities OA, OA' approaches zero the other does. But $\frac{A'B}{OA'}$ approaches zero with decreasing OA' and decreasing angle $A'OB$. Hence, if we keep either of the angles AOM and $A'OM$ fixed and let the other one vary, the sine function of the variable one and a decreasing modulus approaches the sine of the fixed one as limit as angle AOA' and the distance OA and OA' approach zero. Further, let OA be so small that the difference between sine ratio and the sine of the fixed angle is less than a positive number $\frac{1}{2}e$, where e has been assigned in advance, and let AOA' and OA be so small that for all smaller values of OA and AOA' the difference in inequality (1) will be less than $\frac{1}{2}e$. Now keep both angles fixed and let OA and OA' approach zero. It follows that the sines of the two angles differ by less than e. *Hence we have established the fact of the uniform convergence of the sine function and the continuity of the sine.* (Cf. § 21, Young).

Note. The assumption in this argument has been that the method of approach of these angles is in the strict sense

Concerning the uniform convergence of the cosine ratio and the continuity of the cosine: It has been shown that $OA - OA' = eAA'$ where $0 < e < 1$. We have

$$0 < \frac{OM}{OA'} - \frac{OM}{OA} = \frac{OM(OA - OA')}{OA' \,.\, OA} = \frac{OM}{OA} \cdot \frac{AA'}{OA'}\, e$$

is always less than unity. Hence from facts we have proved in the immediately preceding paragraphs and by methods analagous to those there used, we can prove the continuity of the cosine and the *uniform* convergence of the cosine ratio to the cosine. We have now proved properties that enable us to prove the results given in §§ 21, 20, 23, and 24. We are not yet ready for the developments in §§ 22, 25, and 26.

Concerning § 39 and following. Young's treatment of the existence of the length of a circle offers certain difficulties; his geometry is a hyperbolic geometry and so in § 27 he can use the fact that in that geometry the angle sum of a triangle is less than two right angles; in some of the following paragraphs he uses the symmetry relation. We can, of course, apply neither of these propositions. We shall change the order of the treatment he gives and prove first the addition formulas. Consider, then, (1) in Figure 12, paragraph 39, where A and N and the interiors of the angles AOB and BOM are on opposite sides of the ray OB and M belongs to the ray ON. Suppose AOM is acute. Its measure is the sum of the measures of AOB and BON. Let OA be less than a quadrant. Let Q be that pole of ON which is on the A side of ON. Then A is on the quadrant QM. (Let P be that pole of OQ which is on the A side of OQ and let T be that pole of QAM which is on the B side of QAM). By Theorem 52, T and P are points of GC_Q which by hypothesis contains O and M. In the triangle $\overline{Q}XB$ the side $\overline{Q}X$ is congruent to $\overline{Q}B$ and both are greater than a quadrant. By Theorem 86 the angles BXM and XBN are both greater than a right angle. Both of these angles have right angles as limits as the sides of the quadrilateral $XBNM$ approach zero.

We shall prove that if K is a point on the ray AM such that BKA is a right angle then the limit of $\frac{AK}{AB}$ as these intervals approach zero is the cosine of angle BOM. Let a and b be the measures of angles AOB and BOM respectively. For sufficiently small values of OA it follows by propositions proved by Young that the measure of angle OAM is as close as we please to $\frac{1}{2}\pi - (a+b)$ while that of angle OAB is as close as we please to $\frac{1}{2}\pi - a$. Hence for sufficiently small values of OA, the measure of angle BAK is as close as we please to b. The first of these angles is smaller than the second. It can be proved that M and B are on the same side of OA, the ray AM falls within the angle OAB and O and B are on opposite sides of AM. By Theorem 82 AB is less than OA. It follows easily that angle BAK is less than a right angle and hence from the triangle BAK that AK is less than a quadrant. Let R be a rotation about M that carries T into Q. Let A_1, B_1, K_1 be the transforms under R of A, B and K respectively. By Theorem 52, GC_M contains both T and Q. By Theorem 62, R carries Q into $\overline{T}$, the pole of T, and carries the quadrant MQ into the quadrant $M\overline{T}$ and carries the ray AM into the ray A_1M. Also, since on the ray OMN we have the order OMT, it follows that the ray MA_1 and the quadrant $M\overline{T}$ contain O. Since B and T are on the same side of AM it follows that B_1 and Q are on the same side of $O\overline{T}$, which is on GC_Q. Let S be a rotation about Q which carries A_1 into O. Let the transforms under S of the points B_1 and K_1 be Z and W respectively. It follows, (See H. G., pages 403 to 405), that S transforms the ray A_1M into the ray OM. Hence the motion SR transforms A into O, transforms B into a point Z which is on the Q side of OM and transforms K into a point W of the ray OM. Hence in the terms of the notation introduced at the end of Theorem 84, as OA approaches zero, the angle BAK approaches the angle BON in the strict sense, and, therefore, as we have proved in a preceding section, the limit of the ratio $\frac{AK}{AB}$ will be the cosine of the angle BOM. We can now prove with these additions to Young's arguments, the formula at the foot of page 272:

(1) $\sin AOM = \cos BON \sin AOB + \sin BON \cos AOB.$

Consider now the formula for the cosine of the sum of two angles. We have on the interval ON, the order OMN. From the tri-rectangle $BKMN$ we have BK is less than NM. Let Y be a point of the interval NM such that MY is congrunent to BK. Then $OM = ON - YN - MY$, from which

$$\frac{OM}{OA} = \frac{ON}{OB}\frac{OB}{OA} - \frac{KB}{AB}\frac{AB}{OA} - \frac{NY}{BN}\cdot\frac{BN}{OB}\cdot\frac{OB}{OA}.$$

With the help of results obtained above and methods used in § 39, Young, it follows

(2) $\cos AOM = \cos BON \cos AOB - \sin BON \sin AOB.$

The proof of these formulas for case (2), where AON is a right angle is along the same lines as the proof for the case where it is acute. If now angles BON and AOB have the same measure they are congruent. If we indicate these congruent angles by „x", the angle AOM will have a measure $2x$; if we keep in mind the construction we made in proving the formulas (1) and (2) we have

(3) $$\sin(2x) = 2\sin x \cos x$$

(4) $$\cos(2x) = \cos^2 x - \sin^2 x.$$

If we square and add we get

(5) $$\cos^2(2x) + \sin^2(2x) = (\cos^2 x + \sin^2 x)^2.$$

In the three formulas just proved substitute $\frac{1}{2}x$ in place of x. Wherever

(6) $\cos^2 x + \sin^2 x = 1$ we can prove from formulas (5) and (4) that

(7) $$\cos(\tfrac{1}{2}x) = \sqrt{\frac{1+\cos x}{2}}$$
$$\sin(\tfrac{1}{2}x) = \sqrt{\frac{1-\cos x}{2}}$$

where the radicals stand for the positive square root.

We shall now prove the following theorem:

Theorem 90. *If AOB is an angle whose measure is x, then (1) $\cos x = \mathrm{COS}\, x$ and $\sin x = \mathrm{SIN}\, x$, where $\mathrm{COS}\, w$ and $\mathrm{SIN}\, w$ are the cosine and sine of Euclidean geometry, and where the argument w is expressed in radian measure (Euclidean); (2) $f(AOB) = f(BOA)$, where AOB is an acute angle, and „f" is any one of the six trigonometric functions.*

Proof. When angle AON is a right angle, the ratio $\frac{AM}{OA}$ is constant and equals unity, and the ratio $\frac{OM}{OA}$ is constant and equals zero This agrees with our previous definitions of the sine and cosine of a right angle. If we let $x = \frac{1}{2}\pi$ it follows from formulas (6) and (7) that part (1) of Theorem 90 is true for x equal to $\frac{1}{4}\pi$ or $\frac{1}{2}\pi$. But the formula (6) is true for x equal to $\frac{1}{4}\pi$ and hence by (7) the theorem is true for $x = (\frac{1}{2})^3\pi$. By mathematical induction, with the help of formulas (6) and (7), it follows that when $x = (\frac{1}{2})^n\pi$, (n a positive integer), part (1) of the theorem is true. Finally we can prove by mathematical induction, with the help of formulas (1) and (2), that if k is any positive integer not greater than 2^n, then for $x = k(\frac{1}{2})^n\pi$, the theorem is true. But the numbers $x = k(\frac{1}{2})^n\pi$ are everywhere[1]) dense in the number interval from 0 to $\frac{1}{2}\pi$. Hence, from the continuity of the sine and cosine both in this (elliptic) space, and in the Euclidean space it follows that part (1) of the theorem holds for all values between zero and $\frac{1}{2}\pi$.

We have been assuming so far that we had the construction of Figure 12. However, the angle AOB has the same measure as the angle BOA. From what we have just proved, it follows that the trigonometric functions of these two angles are the same. We have thus proved the theorem. *The theorem we have just proved allows us to accept without question for this geometry the results of §§ 22, 25, 26, 39, to 45, Young. Also since the sine and cosine are the same. functions respectively as the sine and cosine of angles of the same measure in Euclidean geometry the formulas at the top of page 275 follow without further argument.*

Theorem 91. *See notation at end of Theorem 84. If XYZ is a variable triangle such that the angle Z approaches a right angle, and such that the measure of angle Y approaches the measure of the acute angle BOA, as g, the greatest side of the triangle XYZ, approaches zero, then $\frac{XZ}{XY}$ approaches the sine of BOA which is the same as the sine of AOB, and the ratio $\frac{YZ}{YX}$ approaches the cosine of BOA which is the same as the cosine of angle AOB. The*

[1]) See note at bottom of page 254 of Young.

approach need not be strict in the sense of the note following Theorem 84.

Proof. We shall assume that g is sufficiently small so that angle Y is acute. We shall assume first that Z is constant and is a right angle. In the notation of the note at the end of Theorem 84, for a certain subsequence of values of g the ray Ox is on the A side of OB and the angle xOB approaches the angle AOB in the strict sense; we have proved before that the sine ratios and the cosine ratios approach the sine and cosine of AOB (over this sequence). For a sequence where x is on the A_1 side of OB we get in the same way the sine of A_1OB and the cosine of A_1OB. But AOB and A_1OB have the same measure. Hence by Theorem 90 the first case follows.

In case angle Z is not a right angle, let W be a point on the ray YZ such that ZWX is a right angle. We may assume that YX is less than a quadrant, that angle X is acute, and that g, the greatest side of either of the triangles XZY and YWX is less than a quadrant. As g approaches zero, XZY and XZW approach right angles and hence, since angle YWX is a right angle, it follows from the triangle ZXW that angle ZXW approaches zero $\frac{XZ}{XY}=\frac{XZ}{XW}\cdot\frac{XW}{XY}$. By the first case it follows that the first expression on the right of the equality has unity as limit, while the second approaches sin AOB. Since Z and W are both points of the ray YZ it follows that either Z is a point of the interval YW or that W is a point of the interval YZ. Hence ZY is either the sum or the difference of YW and ZW. We shall consider the second case. $\frac{ZY}{XY}=\frac{WY}{XY}-\frac{WZ}{XW}\cdot\frac{XW}{XY}$. It can easily be proved by case one and previous theorems that this ratio has a limit cos AOB.

Theorem 92. *If ABC is an acute angle then there exists on the ray BC a point D such that the measures of the angles DAB and ADB have the same numerical value.*

We leave the proof of this theorem to the reader.

We are now ready to treat the questions of the length of circles and related topics. Let AOA' be an acute angle, and let OA and OA' be congruent and less than quadrant. By Theorems 88 and 76 there exists on the ray OA a point B and on the ray OA' a point B' such that $BA'O$ and $B'AO$ are right angles, such that all the sides

of the triangles $BA'O$ and $B'AO$ are less than a quadrant, and that the segments BA' and AB' are within the angle AOA'. By Theorem 82 both OB and OB' are greater than OA. It follows easily that the segments AB' and $A'B$ have in common exactly one point D. Let E be any point of C_{OA} which is within the angle AOA'. By Theorem 76 the ray OE contains exactly one point F on the segment AA', one point G on the segment AB', and one point J on $A'B$. By Theorems 78 and 83, OE is greater than OF but is less than the smaller of OG and OJ. Let K be a great circle which is perpendicular to OE at E. By the method we used in proving that $A'B$ and AB' intersect in a point D we can prove, (since BOE and EOA' are acute angles), that K contains a point I of the segment AG and a point L of the segment JA' of the interval JA'. We wish to prove next that L is a point of the segment $A'D$. Suppose that it is not. The point L cannot be A', since K is tangent to C_{OA} at E. Hence, since the intervals JA' and DA' are both subsets of the interval $A'B$, L is either D or a point of $A'J$ in the order $A'DL$. By Theorem 78 OJ is less than OB which is less than a quadrant. Since JOA' is acute, it follows by Theorem 82 that JA' and hence JD is less than a quadrant and that OJA' is acute. By the same theorem it follows that EL is less than a quadrant. Since E and L are within the angle AOA', the interval EL contains no point of OA or for OA'. Since OF is less than OE which is less than OG it follows that E is a point of the segment FG. By Theorem 76 E is then within the angle GAF which is the same as the angle DAA'. In the same way we can prove that E is within the angle $DA'A$ But from this it follows that E is on the A side of $A'D$ and on the A' side of AD, and hence is within the angle ADA' and hence within the triangle DAA'. If L is a point distinct from D in the order $A'DL$ it follows that L is on the opposite side of AD from E Hence the interval EL contains a point Z of the great circle passing through A and D. Since EL is entirely within the angle AOA', and since by Theorem 76, the interval AB' is the only part of the great circle passing through A and D which is a subset of the lune AOA', Z is a point of the interval AB'. Hence whether D is L of whether D is a point in the order $A'DL$ the interval EL contains a point Z of the interval AB' and also contains the point L of the interval $A'B$. Let $\bar{L}$ and $\bar{E}$ be the poles of L and E respectively. The point $\bar{E}$ is without the triangle ADA'. (Theorem 60) Hence the semicircle $E\bar{L}\bar{E}$ contains a point V of the triangle. But the point F of the segment AA' is within C_{OA}, (OF is less than OA by Theorem 83); by Theorem 78, since OE is perpendicular to K at E, if Q is any point of K distinct from E, OQ is greater than OE. Hence the side AA' contains no point in common with K. Hence V must be a point of one of the sides AD and $A'D$. But then one of the intervals AB' and $A'B$ must contain two points of K. This leads to a contradiction. Hence L is a point of the segment $A'D$ and similarly I is a point of the segment AD.

By Theorem 86 angle $AA'O$ is acute. Since angle AOA' is also acute it is easily proved that there exists on the rays OA' and $A'O$ (and hence on the interval OA') a point W such that AWO is a right angle and that AW is less than a quadrant and is also less than AA'. Also, in the right triangle DAB, AB and BD are both less than a quadrant, and hence by Theorem 82 AD is less than DB. With the help of Theorem 89 we can now prove the following inequality

(1) $AW < AA' < AE + EA' < AI + IEL + LA' < AD + DA' < A'B.$

From inequality (1) we get in the notation of Young, Figure 10, (assuming AK and $A'K$ to be tangent to C_{OA})

(2) $$A'A < AB + A'B < AK + A'K < AT < \frac{r}{n}.$$

Hence we can draw the conclusion of § 30, since by the inequality (1) above it follows that the inscribed polygons have an upper bound, and that the circumscribed polygons have a lower bound which is not less than the upper bound of the inscribed polygons. Also if we add new vertices to any inscribed polygon we increase the sum, while if we add vertices to a circumscribed polygon we decrease the sum.

From inequality (1) above it follows that

$$1 > \frac{n\,(A'A)}{n\,(A'D + AD)} > \frac{(AA')}{(A'B)}.$$

Angles $AA'O$ and $A'AO$ are acute, Hence, from the triangle AOA' it follows that when angle AOA' approaches zero the angles $AA'O$, $AA'B'$, $A'AO$, $A'AB$, must approach right angles.

Also, the associated interval of angle AOA' approaches zero with this angle; by Theorem 79 it follows that BA' is less than this associated interval Also angle $BA'B'$ is a right angle. Hence angle $AA'B$ approaches zero and we have by Theorem 91 that $\frac{AA'}{A'B}$ has cosine zero, that is unity, for a limit as angle AOA' approaches zero. *Now we can follow §§ 33, 34, 35 and the definition of length of circles in § 36.*

From inequality (1) we can prove that

(3) $$AW < \text{arc } AA' < A'B.$$

But when OA' approaches zero OB approaches zero, given that angle AOA' is fixed. Hence if we divide the above inequality (3) by OA, and denote by L the limit $\frac{\text{arc } AA'}{OA}$ as OA approaches zero it follows that if this limit exists and we let x be the measure of angle AOA', then

(4) $$\sin x \leq L \leq \tan x.$$

Even if this limit does not exists, if L is any number between the upper and lower limits of $\frac{\text{arc } AA'}{OA}$ as OA approaches zero, inequality (4) will still be true. Let C and C' be points on the rays OA and OA' respectively such that OC and OC' are quadrants. Then the interval CC' is the associated interval of the angle AOA'. Let C'' be the midpoint of this interval. Then by previous theorems C'' and the ray OC'' will be within angle AOA' and this ray contains exactly one point A'' of C_{OA}. There is a rotation about O that transforms C into C'' and transforms C'' into C'. Hence angle AOA'' is congruent to $A''OA'$. It is easily shown that each of these angles has a measure $\frac{1}{2}x$. Also (see paragraph 37, Young), the lengths of the arcs AA'' and $A''A'$ of C_{OA} are the same and are each equal to a half of the length of the arc AA'[1]) of this circle. Hence by an argument like that used in the proof of inequality (4) we can prove that $2 \sin \frac{1}{2}x \leq L \leq 2 \tan \frac{1}{2}x$. By a repetition of this argument it follows that $4 \sin \frac{1}{4}x \leq L \leq 4 \tan \frac{1}{4}x$. With the use of mathematical induction we can prove that

[1]) We are referring to those arcs which are within the angle AOA'.

$2^n \sin (\frac{1}{2})^n x \leq L \leq 2_n \tan (\frac{1}{2})_n x$ holds for all integral values of n. The left and right members of this inequality (see Theorem 90) approach the value x as n is increased indefinitely. *Hence*

$\lim\limits_{OA=0} \frac{(\text{arc } AA')}{(OA)}$ *exists and equals x, where x is the measure of angle AOA'. We have justified the definition of radian measure given in § 37, Young, and have identified this with the measure we have used previously for angles.*

§§ 48—51. Young here uses the symmetry theorem and we shall have to modify his methods.

Consider an acute angle BOB' whose measure is π/n, n a positive integer. Let OB and OB' be congruent. Let Y be a point on the interval BB' such that angle BOY is congruent to angle YOB'. (It follows from one of the immediately preceding arguments that there is such a point Y, that the ray OY is within the angle BOB', and that the measure of the angles BOY and YOB' is one-half that of BOB'). Assume that OB is less than a quadrant and let K and H be points on OY such that $B'KO$ and BHO are right angles. As we have proved in previous arguments, the triangles BYH, $B'YK$, BOH, and $B'OK$ satisfy the hypothesis of Theorem 82. Let $S(B,n) = n.(B'K + BH)$, where $B'K$ and BH are sides of the triangles $B'KY$ and BHY. Then we have from the foregoing and Theorem 89

(1) $$S(B,n) \leq n(BB') \leq S(B,n) + n.KY + n.YH.$$

But

(2) $$n.KY = \frac{KY}{YB'}(n.YB') < \frac{KY}{YB'}.(n.BB').$$

Now let angle BOB' approach zero. Then by Theorem 86 the angles OBB' and $OB'B$ approach right angles. It follows then from the triangle OYB that OYB approaches a right angle and that the angles $KB'Y$ and YBH approach zero. As angle BOB' approaches zero, so does its associated interval. Hence, by Theorems 86 and 79 as angle BOB' approaches zero BB' approaches zero, and hence $\frac{KY}{YB'}$ has the same limit. But $n.(BB')$ is the perimeter of an inscribed polygon, and hence

$$\lim_{\text{angle } BOB'=0} (n.BB') = L_{B'}, \text{ the length of } C_{OB}.$$

But it follows from what we have proved, including inequality (2), that $n.KY$ approaches zero and similary that $n.YH$ has the same limit. It follows from inequality (1) that

(3) $$\lim_{n=\infty} S(B,n) = L_B\,[1].$$

With the help of this result and from the continuity of the sine ratio of an angle, we can prove, by arguments largely similar to those used by Young in § 48, the continuity of $f(r)$. Using the notation of § 49, we can show by the argument he there uses, the following inequality:

(4) $$S(C,n) - S(B,n) > S(A,n) - S(C,n).$$

If we now proceed to the limit, we can make the statements which are given on page 276, Young, provided we reverse the inequality signs throughout. From inequality (1) in our treatment of the lengths of circles it follows that the length of that part of C_{ON} which is within the angle BON is less than the interval BN (assuming that OB is less than a quadrant and that angle BON is acute, see Figure 13). From this it is easily proved

$$\lim_{r=0} f(r) = O. \quad \text{Let } f(O) = O.$$

From this it follows by § 37 and by definition of $f(r)$, that

$$f'(O) = \lim_{r=0} \frac{f(r) - f(O)}{r - O} = 1.$$

But the argument we have given in the preceding sections to establish inequality (4) is valid if B is the point O. Since in this treatment the inequalities are reversed it follows from the argument given in Young that the incremental ratio of $f(r)$ is never greater than unity or less than zero for r less than or equal to $\frac{1}{2}\pi$. It follows from his argument that *the derivative of $f(r)$ exists for all values of r and is never greater than unity or less than zero, for r from O to $\frac{1}{2}\pi$ inclusive.*

§§ 52—55. In the figure on page 277, Young let ABC be a right triangle that satisfies the hypothesis of Theorem 82. Let P

[1]) We have given the proof for the case where OB is less than quadrant Obviously (3) holds where OB is a quadrant, and the argument following is valid for $r = \frac{1}{2}\pi$.

be that pole of BC which is on the A side of BC. Then by arguments we have used before it follows that all the triangle ABC plus its interior with the exception of the side BC is on the A side and P side of BC, and every quadrant from P to a point of BC contains a point of the side BA. By hypothesis, the point C_1 is the midpoint of the side BC. Hence there is a rotation about P that transforms B into C_1 and transforms C_1 into C, and that transforms the interval BC_1 into the interval C_1C. By Theorem 64 this rotation transforms C_2, the midpoint of the interval BC_1 into D_2, the midpoint of the interval C_1C. Hence it transforms the quadrant PA_2C_2 into the quadrant PQ_2D_2, the point A_2 into the point Q_2; hence the triangles A_2C_2B and $Q_2D_2C_1$ are congruent triangles and have the same angle sum. Since D_2 is the midpoint of C_1D_2C we can prove in the same way that there is a rotation about P which transforms C_1 into D_2, which transforms D_2 into C, and which transforms Q_2 into a point E_2 on the quadrant PAC. Let d_2 be the angle excess of each of the three congruent triangles A_2BC_2, $Q_2C_1D_2$, and E_2D_2C. Then we can write inequalities (1) and (2), page 277[1]), Young, as Young does, provided we replace „tan P_2BC" by $\frac{P_2C}{BC}$. By the method we have just used it is easily proved that we can construct 2^{n-1} triangles between C_1 and C which are congruent to the triangle A_nBC_n. Hence we can write as Young does

$$2^{n-1} . d_n < d'_n$$

$$\frac{P_nC}{BC} < \frac{A_nC_n}{(2^{n-1}+1)\,BC_n}.$$

From the second of these inequalities it follows from the argument given by Young that $\frac{P_nC}{BC}$ approaches zero. Let H_nK_n be the associated interval of the angle P_nBC. From the monotone character of the tangent ratio it follows that $\frac{P_nC}{BC}$ is greater than H_nK_n divided by $\frac{1}{2}\pi$. Hence the interval H_nK_n, and the angle of which it is the measure approaches zero as n approaches infinity.

1) The expression on the right in (2) Young should be „=", not „<".

We may now follow Young in §§ 52, 53, 54, 55. In connection with § 55, See Dehn, loc cit, page 169, Theorem 2.

In § 57, Figure 17 let M be a point such that OM is perpendicular to the interval AB at M. (It follows by Theorem 86, part (3) that if OA is less than a quadrant there exists exactly one such a point and that OM is less than AM). Let $C = C_{OA}$ and let $K = C_{OM}$. Let E be the point in which the ray OM intersects C. Let N and P be the points where the rays OA and OB respectively intersect K. In the immediately following, the term arc will be restricted to those arcs of C and K which are in the lune ABC. It follows from inequality (1) of our discussion of the length of a circle, (following the proof of Theorem 91), that arc NM is less than interval AM which is less than arc AE and arc MP is less than interval MB which is less than arc EB. Adding these inequalities we get the inequalily in § 57; however, we will replace the TM in Young by EM. To follow his argument we must then prove that EM is of a higher order than the arc AB. But the arc AB is greater than the chord AB which is greater than the interval AM. As AM and MB approach zero each of the angles AOM and MOB, and hence the angle AOB approaches zero. It follows by Theorem 86 that each of the angles OAM and OAE has a right angle as a limit and hence the limit of the angle EAM is zero. It follows that $\frac{EM}{AM}$ has a limit $\tan(O) = 0$. The argument in § 58 offers no difficulty. The main part of the remainder of this paper will consist in the proof of the differential equation at the foot of page 283, Young.

Concerning the triangle with equal base angles.

Let AA' be an interval less than a quadrant and let M and N be points on the same side of AA' such that angle MAA' has the same measure as $NA'A$ and let both angles be acute. Let Q be that pole of AA' which is on the same side of AA' as M and N. With the help of Theorem 76, parts (3) and (4) it follows that the rays AM and $A'N$ intersect in a point P such that all of the triangle PAA' except the side AA' is in the interior of the birectangular triangle QAA'. Let $PA = x$, $PA' = y$, and $AA' = a$. Let the measure of the angles PAA' and $PA'A$ be X, and let the measure of APA' be Y. The measure of angle AQA' is a, since AA' is the associated interval of this angle. The angle sum of triangle AQA'

is greater than that of APA' since the second triangle is contained within the first. Hence

(1) $$\pi + a > Y + 2X > \pi$$
$$\pi + a - 2X > Y > \pi - 2X.$$

Let X_1 denote a number such that

(2) $$0 < X_1 < X.$$

There exists a point P_1 such that each of the angles P_1AA' and $P_1A'A$ is of measure X_1 and the triangle P_1AA', except for the interval AA', is within the triangle PAA'. From inequality (1) we have

(3) $$\pi + a - 2X_1 > \pi + a - 2X > Y.$$

Theorem 93. *Consider a variable w and for each value $\overline{w}$ of w a pair of fixed points A and A', and a set $[APA']_{\overline{w}}$ of triangles such that (1) the interiors of all the triangles of the set $[APA']_{\overline{w}}$ are on the same side of AA'; (2) for each triangle APA' of the set $[APA']_{\overline{w}}$ the angles PAA' and $PA'A$ have the same measure X; (3) AA' is less than a quadrant; (4) the upper bound, for the set of triangles $[APA']_{\overline{w}}$, of X is $\frac{1}{2}\pi$ and the lower bound is a number $X_{\overline{w}}$; (5) $\lim_{w\to k} X_w = \frac{1}{2}\pi$ and $\lim_{w\to k} AA' = 0$. Then, if ε is a positive number, there exists a positive number δ_ε such that for any triangle APA' in the set $[APA']_w$, $\left|\frac{PA - PA'}{AA'}\right| < \varepsilon$, for all w's such that $|w - k| < \delta_\varepsilon$.*

Proof: The variable triangle APA' is a function of the variable X; the *range* of X is determined by the variable w; X_w and AA' are single-valued functions of w. Consider, during that part of this argument which precedes inequality (5), a fixed value of w for which $\frac{1}{2}\pi > \pi + a - 2X_w$, and the corresponding set $[APA']_w$ of triangles. Let the measure of angle APA' be Y, and let the lengths of the intervals AP, $A'P$, and AA' respectively be x, y, and a. Let Q be that pole of AA' which is on the P_w side of AA', where P_w is the vertex P of that triangle APA' of the set $[APA']_w$ whose base angles PAA' and $PA'A$ have the measure X_w. For each triangle APA' let H denote a point on the ray PA' and K a point on the ray PA such that PAH and $PA'K$ be right angles. It follows from the statement of the theorem and the paragraph preceding the statement of the theorem that when X is greater than X_w the ray $A'P$ falls without the angle $P_wA'A$ and hence the ray $A'H$ falls within the angle $AA'H_w$. Also, the ray AP falls without the angle P_wAA' and within the angle QAA'. Since the angles PAH and P_wAH_w are right angles, it follows that the ray AH falls within the angle $A'AH_w$. It can easily be shown that the point H is within the triangle $A'AH_w$ and hence that we have on the ray $A'H$ and on the segment AH_w a point Q_w such that the interval $A'H$ is less than the interval $A'Q_w$. From the triangle $AA'Q_w$, with angle $AA'Q_w$ greater than a right angle, it can be shown by Theorem 87 A that angle $A'Q_wA$ is an acute angle and angle $A'Q_wH_w$ is greater than a right angle. By inequality (3) preceding the statement of the theorem and the choice of w we have made, Y_w is less than a right angle. By Theorem 83 AP and $A'P$ are each less than a quadrant. By Theorem 88 triangle P_wAH_w satisfies the hypothesis of Theorem 82. By Theorem 87 $A'Q_w < A'H_w$ and hence

(4) $$A'H < A'H_w, \quad \text{and similarly} \quad AK < AK_w.$$

It is easily seen from the proceding inequalities and the argument that follows that the case where $A'P$ is less than AP can be handled in the same way as the case where AP is less than $A'P$; so we shall consider only the first of these two cases. We have then

(5) $$0 \leqslant x - y = AP - A'P < PH - A'P = A'H \leqslant A'H_w.$$

Divide every element in (5) by AA' and let w approach k. Since X_w approaches $\frac{1}{2}\pi$, angle $A'AH_w$ approaches zero, and angle $AA'H_w$ approaches a right angle. Since AA' also approaches zero, we can show by methods we have used before, (involving in particular Theorem 91), that $A'H_w$ and AH_w approach zero. Then by Theorem 91 $\frac{A'H_w}{AA'}$ approaches the tangent of zero, that is zero. We have thus established the conclusion of the theorem.

We shall now derive a differential equation involving the sides and angles of the triangles discussed above when the interval AA' is kept fixed.

For a fixed value $\overline{w}$ of the variable w discussed in Theorem 93 the triangles APA' of the set $[APA']_{\overline{w}}$ are functions of X alone; where X, as indicated before, is a variable whose range $[X]_{\overline{w}}$ is determined by $\bar{w}$. Consider then, during the argument preceding Theorem 94, the range $[X]_{\bar{w}}$ and the set $[APA']_{\bar{w}}$ for the fixed $\bar{w}$ just mentioned. Let APA' and AP_1A' be two triangles of the set $[APA']_{\bar{w}}$ such that angle PAA' is greater than P_1AA'; that is, X is greater than X_1; also let $\pi + a - 2X_1 < \frac{1}{2}\pi$. On the ray AP let E be a point such that $AE = AP_1 = x_1$ and on the ray $A'P$ let $A'F = A'P_1 = y_1$. In the order $A'P_1T$ let T be a point common to the ray $A'P_1$ and the interval AP. By Theorem 87 A ATA' is an acute angle and $Y_1 >$ angle $ATA' > Y$. By Theorem 87 $AP_1 < AT < AP$. Similarly $A'P_1 < A'P$.

On the ray FP_1 there exists in the order FP_1G a point G that belongs also to the ray PA. It follows by the statement made at the foot of page 251, Young, above the footnote, that as $X - X_1$ approaches zero the angle $Y - Y_1$ approaches zero. By Theorem 86 the following angles will under the same circumstances approach right angles: AEP_1, P_1EP, EP_1A, $A'P_1F$, $A'FP_1$, P_1FP. From the triangle PFG it follows that the angle PGF approaches the value $\frac{1}{2}\pi - Y_1$ and from the triangle GP_1E it follows that the angle GP_1E approaches the value Y_1.

By § 57. Young, we have thas except for infinitesimals of order higher than the second, $P_1F = (X - X_1) . f(y_1)$ and $P_1E = (X - X_1) . f(x_1)$.

Hence

$$(1) \qquad \lim_{x - x_1 = 0} . \frac{P_1F}{P_1E} = \frac{f(y_1)}{f(x_1)}.$$

Also as X approaches X_1, EP_1 and FP_1 approach zero. With the help of Theorem 91 we can show from the triangle EGP_1 that GP_1 and EG also approach zero. It follows that $GF = GP_1 + P_1F$ approaches zero. By an argument like that just given, and involving the triangle GFP we can show that the intervals GP and PF also approach zero. It follows by Theorem 91 that $\frac{GF}{FP}$ approaches $\tan (Y_1)$, and $\frac{GP_1}{EP_1}$ approaches $\sec (Y_1)$.

But

$$\frac{GF}{FP} = \frac{GP_1 + P_1F}{FP} = \frac{P_1F}{FP}\left[1 + \frac{GP_1}{EP_1} \cdot \frac{EP_1}{P_1F}\right].$$

From this we get that except for infinitesimals of order higher than the first

$$\frac{P_1F}{FP} = \frac{X - X_1}{y - y_1} . f(y_1) = \frac{\frac{GF}{FP}}{1 + \frac{GP_1}{EP_1} \cdot \frac{EP_1}{P_1F}}$$

Now let one of the angles X and X_1 remain fixed, and let $X - X_1$ approach zero. Dropping subscripts, we get

$$(2) \qquad \frac{dy}{dX} = \frac{f(y) + \sec Y . f(x)}{\tan Y}$$

Similarly

$$\frac{dx}{dX} = \frac{f(x) + \sec Y . f(y)}{\tan Y}$$

Subtracting the foregoing equations and simplifying, we get

$$(3) \qquad \frac{d(y - x)}{dX} = [f(x) - f(y)] . \tan \tfrac{1}{2} Y.$$

Theorem 94. *Hypothesis of Theorem 93. If* Y_w *and* AA' *are of the same order, the* $\lim_{w=k} \left[\frac{AP_w - A'P_w}{(AA')^2}\right] = 0.$

Proof. Differential equation (2) shows that, for w fixed, x and y are continuous functions of X for $X_w \leqslant X \leqslant \frac{1}{2}\pi$, whenever $\pi + a - 2X_w < \frac{1}{2}\pi$. We have then by differential equation (3), that if we keep w fixed and allow X to vary

$$y_w - x_w = \int_{x_w - y_w}^{0} d(x - y) = \int_{X_w}^{\pi/2} [f(y) - f(x)] . \tan \tfrac{1}{2} Y . dX.$$

For values of X such that $x = y$, the integrand is zero. For values of X such that $x \neq y$, we can write the integrand in the preceding equation as follows:

$$\left|\frac{x_w - y_w}{a^2}\right| = \left|\int_{X_w}^{\pi/2} \left(\frac{f(y) - f(x)}{y - x}\right) . \left(\frac{y - x}{a}\right) . \left(\frac{\tan \frac{1}{2} Y}{a}\right) . dX\right|.$$

But the incremental radio of $f(x)$ is never greater than unity. If ε is a positive number, there exists by Theorem 93 a positive

δ_ε such that whenever $|w-k|<\delta_\varepsilon$, then for any triangle APA' in the set $[APA']_w$, $\left|\frac{AP-A'P}{AA'}\right|=\left|\frac{x-y}{a}\right|<\varepsilon$. It follows from the proof proceding this theorem that $Y \leqslant Y_w$. Since Y_w and $AA'=a$ are of the same order, it follows that $\frac{\tan\frac{1}{2}Y}{a}$ has an upper bound M. Hence we have the following:

$$\left|\frac{x_w-y_w}{a^2}\right|<\varepsilon . M . (\tfrac{1}{2}\pi-X_w).$$

But M is fixed, and ε and $\frac{1}{2}\pi-X_w$ may be made as small as we please. Hence the conclusion of the theorem follows.

§ 59. Young. Make the following modifications in Young's treatment. Let Q' be a point on the ray OP' such that angle $OP'Q'$ and angle $OQ'P$ have the same measure χ and let Q be a point on the ray OP such that angles $OP'Q$ and OQP' have the same measure. Let OAP be a triangle satisfying the hypothesis of Theorem 82.

Let K be a point on the ray OQ' such that OK is congruent to OP, and let L be a point on the ray OP such that OP' is congruent to OL. By Theorems 86 and 84 the angles $OQ'P$, OPQ', OKP, and OPK approach a right angle as PP' and angle $P'OP$ approach zero. Then angle $Q'PK$ approaches zero, and we have by Theorem 91 that $\frac{Q'P}{KP}$ approaches unity. It follows by Young's argument that KP is of the same order as $\Delta\Phi$. Hence we have by Theorem 94, $Q'K$ is of order higher than the second with respect to $Q'P$. But $\Delta y = KP'$ which differs from $Q'P'$ by $Q'K$ Similarly, Δy differs from PQ by infinitesimals of order higher than the second.

Hence if we replace Δy by $Q'P'$ or by PQ, the result, in the limit, in the difference equation set up by Young, will be the same. Also we may replace $Q'P$ by $\Delta\Phi . f(y)$, and $P'Q$ by $\Delta\Phi . f(y+\Delta y)$. With these modifications we can proceed as at the foot of page 283. We can proceed as Young does in solving Killing's equation on pages 284 and 285. His argument at the top of page 286 shows that $f(x)=k . \sin\frac{x}{k}$. These formulas are sufficient to establish the categoricity of the space S.

Independence examples.

We give here a set of independence examples similar to those given in L. R. H. H., and shall make the following changes[1]) in the examples there given. Let S_2 and S_3 be the surface of a sphere of radius 10. Let S_4 be the set of rational points belonging to a circle and the regions for this space be segments of this set. All the spaces S_7 to S_{12} are the same as S_2. In the case of S_{10} let P and $\overline{P}$ be a pair of opposite points and let C be the great circle having these centers. Let the symbol y indicate the ordinate of the point Y of S_{10}; that is, $y=0$ if Y is a point of C; if Y is not on C, y is numerically equal to the length of the interval YH, not greater than a quadrant, of a great circle containing Y and perpendicular to C at H; further, y is positive or negative according as Y is on the P or the $\overline{P}$ side of C. Let $\overline{M}$ be a continuous, single-valued transformation of S_{10} into itself such that $y'=\frac{(y)^3}{25\pi^2}$, where y' is the ordinate of $\overline{M}(Y)$ and such that $\overline{M}(H)=H$. In E_{12} let a motion be a one to one continuous transformation of S_{12} into itself that transforms great circles into great circles an preserves distances.

[1]) Cf. L. R. H. H., pp. 318—319.

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