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Introduction.

In his paper *Ueber die Grundlagen der Geometrie*, Hilbert 1) formulates a set of axioms concerning a group of motions which is sufficient to necessitate that this group should be simply isomorphic with either the Euclidean or the Bolyai-Lobatschewskian group of rigid motions in a plane. He assumes, however, that the set of points which undergoes the transformation is a number manifold. In his paper *On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry*, R. L. Moore 2) gives a treatment in which this assumption is not made in advance, but in which there is a simultaneous analysis of the group of transformations and of the space which undergoes this transformation. In this paper we shall give a similar analysis for the Double-Elliptic case. After a group of preliminary theorems we shall prove that every motion distinct from the identity leaves fixed exactly two points, which we shall call poles. We shall then introduce the notions of great circles, intervals, congruence of intervals, of triangles and of angles.

2) *Ueber die Grundlagen der Geometrie*, David Hilbert, Mathematische Annalen, Vol. 56 (1902), pp. 381—429. This paper will be referred to hereafter as "H. G."
3) Cf. American Journal of Mathematics, Vol. 41 (1919), pp. 299—319. We shall refer to this paper hereafter as "L. R. H. H."
We encounter the problem noted by Hilbert⁴ at the end of his paper; that is, the problem of proving the congruence of the base angles of an isosceles triangle. Hilbert⁴ has solved this problem for the Euclidean case. The treatment for the Elliptic case seems even more difficult. In the solution of this problem we shall show that we can follow the treatment of Young⁵ in deriving the Non-Euclidean Trigonometry, and in particular the formulas for the solution of triangles; these formulas will enable us to prove the desired result.

It has been suggested to the author that there should be a simpler method for the solution of this problem of the angle-congruence in isosceles triangles; for instance, that perhaps it could be solved without the use of differential equations. The author would be interested in such a solution; and here takes the opportunity of raising the question of its possibility.

We have mentioned before that the problem we are solving seems more difficult for the geometry treated in this paper than for the Euclidean case. Hilbert⁴ in his treatment of the Euclidean case makes use of a theory of proportion, which depends upon the the parallel postulate. In his paper, H. G., he could have used this theory of proportion, could have determined the trigonometric functions of angles as we have done, and could have proceeded to the solution of triangles. In summary, we may say that trigonometry may be regarded as an analogy of the theory of proportion; this theory is more complicated for a Non-Euclidean Geometry than it is for the case of the Euclidean.

It will be assumed the reader is familiar with R. L. Moore's paper "On the Foundations of Plane Analysis Situs"⁶ and with the paper L. R. H. H. mentioned above.

⁴ See H. G.
⁶ On the Analytical Basis of Non-Euclidean Geometry, W. H. Young, American Journal of Mathematica, Vol. 33, (1911), pp. 249—260. We shall refer to this paper hereafter as "Young".
⁷ See articles in Proceedings of London Mathematical Society referred to above.
⁸ On the Foundations of Plane Analysis Situs, R. L. Moore, Transactions of the American Mathematical Society, Vol. 17, No. 2, (April, 1916), pp. 111 to 164. This paper will be referred to hereafter as F. A.

In conclusion, I wish to thank Professor R. L. Moore for interesting me in the subject of the Foundations of Mathematics, and for arousing in me a desire to contribute to the extension of this field.

**Axioms.**

All the axioms except Axioms 5 and 6 of L. R. H. H. will be assumed; the reader is therefore referred to paragraphs 1 and 2 of this paper of R. L. Moore's for preliminary explanations and definitions. In general we shall not give definitions for terms that are in L. R. H. H., or for well known terms in Analysis Situs.

In this paper we shall use "S" in the same way that S is used in L. R. H. H., that is, to denote the set or class of all points. Also, we shall not regard S as a region.

**Axiom 1.** There exists at least one region.

**Axiom 2.** If R and K are regions and R' is a subset of K' then R is a subset of K.

**Axiom 3.** If the region R₁ contains the point O in common with the region R₂, there exists a region R containing O such that R' is common to R₁ and R₂.

**Axiom 4.** If R₁ and R₂ are regions and R₃ is a subset of R₁ then R₃ — R₃ is a non-vacuous, connected point-set.

**Axiom 5.** If R₁ and R₂ are regions such that R₂ — R₁ does not contain all of S, then there exists a region R that contains both R₁ and R₂.

**Axiom 6.** Every simple curve is the boundary of at least two regions.

**Axiom 7.** If O is a point and L and N are closed, bounded⁴ point-sets with no point in common, there exists a region K containing O such that if P is a point in K then every region that contains both a point of L and a point of N can be transformed, by a motion that carries some point of L into O, into a point set that contains both O and P.

**Axiom 8.** If R is a region and M is a motion then M(R) is a region.

**Axiom 9.** If A, B, C, A', B', C', are points distinct or otherwise, such that every three regions that contain A, B, and C, respectively, can be transformed by some motion into regions containing A', B', C'.
and $C'$, respectively, there exists a motion that transforms $A$ into $A'$, $B$ into $B'$, and $C$ into $C'$.

**Axiom 10.** If $M$ is a motion there exists a motion $M^{-1}$ such that if $M(A) = B$ then $M^{-1}(B) = A$.

**Axiom 11.** If $M$ and $N$ are motions there exists a motion $MN$ such that, for every point $P$, $M(N(P)) = MN(P)$.

**Axiom 12.** If $R_1$ and $R_2$ are regions bounded respectively by the simple closed curves $J_1$ and $J_2$, $R_1$ and $R_2$ have no point in common, $A_1$, $B_1$, and $C_1$ are three distinct points on $J_1$, and $A_2$, $B_2$, and $C_2$ are three distinct points on $J_2$, and there exist three simple continuous arcs $A_1X_1A_2$, $B_1Y_1B_2$, and $C_1Z_1C_2$ such that no two of these arcs have a point in common and no one of them has any point other than an end-point in common either with $R_1$ or with $R_2$ and $M$ is a motion such that $R_1$ and $M(R_2)$ have no point in common and there exist three arcs $A_1\widebar{X}M(A_2), B_1\widebar{Y}M(B_2),$ and $C_1\widebar{Z}M(C_2)$ from $A_1$ to $M(A_2)$, from $B_1$ to $M(B_2)$, and from $C_1$ to $M(C_2)$, respectively, then there exist three arcs such that no two of them have a point in common and no one of them has any point other than end-point in common either with $M(R_1)$ or with $M(R_2)$.

We shall number the theorems of this paper in the same way that the corresponding theorems of L. R. H. H. are numbered. When we say that one of these theorems is true or false we shall mean that it is true or false (as the case may be), for the space we are considering in this paper. Theorems 1 to 21 and Theorem 23 are true.

We give now several theorems which differ in statement or in proof from the theorems of L. R. H. H.

**Theorem 21.** If $R$ is a region then $S\neq R'$ is a connected set of points.

**Proof.** Suppose that there are no points in $S\neq R'$. Then, by the convention we have made with respect to $S^*$, $R$ has at least one boundary point. By theorem 11 we are then led to a contradiction. The proof of the theorem may now be continued by methods analogous to those used in the proof of Theorem 21 of L. R. H. H.

**Theorem 21A.** Every simple closed curve, $J$, is the boundary of exactly two mutually exclusive regions, whose sum is $S-J$.


9. See paragraph preceding Axiom 1.
Theorem 24: it follows that there exists a Jordan region $R$ such that $R'$ contains no point of $K-P$. By Theorem 21 A, $D$, the exterior of $R'$, is a region. But $K-P$ is a subset of $D$, and hence by Theorem 14 has a limit point, which is then also a limit point of $K$.

Hence, $K$ does not exist, and the truth of the theorem is established.

Definitions: If $J$ is a simple closed curve and $K$ is a point set, the expressions $J$ encloses $K$ and $K$ is within $J$ mean that $K$ is a subset of the interior of $J$. The term "interior of a simple closed curve" has a definite meaning for the spaces considered in L. R. H. H. and H. G. but is ambiguous in the sense that it may be either one of two complementary domains of the given simple closed curve in the case of the space we are considering in this paper. Theorems 26 and 26 remain true for our space no matter which complementary domain of $J$ we take; in the case of Theorem 26 the hypothesis should include the condition that the interior of $J$ contains $P_1$ and $J$ contains points in both complementary domains of $J$, a condition that follows from the hypothesis of the theorem as stated in L. R. H. H. for the space there considered, but which need not follow for the space considered in this paper. In an analogous manner the term "interior of a simple closed curve $J$" as used in the statement of theorems in L. R. H. H. may mean, when used in this paper, an arbitrary one or it may mean a definite one of the two complementary domains of $J$; in cases of the latter type we shall specify which is to be regarded as the interior only when the interpretation is not obvious.

Theorem 26. If $I$ and $I$ are Jordan domains with boundaries $J$ and $J$ respectively, $P$ is a point of $J$ and of $J$, and $J$ has points in both complementary domains of $J$, then there exist two Jordan domains $q$ and $q$ with boundaries $Q$ and $Q$ respectively such that (1) every point of $Q$ belongs to either $J$ or $J$ and so does every point of $Q$, (2) the curves $Q$ and $Q$ contain in common a segment of $J$ that contains $P_1$ (3) $q$ and $q$ are both subsets of $I$, (4) $q$ is a subset of $I$ and $q$ is a subset of $S-P'$.

Proof: Make the following modifications of the proof as given in L. R. H. H. Let $H$ and $K$ be defined as those complementary domains of the closed curves $AEBPA$ and $AFBPA$ which are subsets of $I$ (cf. Theorem 21 C). Let $H$ and $K$ be defined as those complementary domains of $h$ and $k$ which are subsets of the domains $H$ and $K$ respectively. Let $R$ be that complementary domain of $a$ which contains the segment $ABP$ of $J$. By the definition of $H$ and $K$ the segment $APB$ is a subset of the boundary of $H$ and $K$. It follows easily by Theorem 21 C that $R = H + K + APB$.

Definitions: By $T_A$ we mean a rotation about the point $A$; that is, a motion that transforms $A$ into itself. By $C_A$ we mean the set of all points $[X]$ such that $B$ can be transformed into $X$ by a rotation about $A$. This point set is said to be a circle having a center at $A$. In the space $S$ are we considering, as we shall prove later, every circle has exactly two centers. If $C_{AB}$ is a simple closed curve, by $I_A$ we shall mean that complementary domain of $C$, which contains $A$, and by $E_A$ that complementary domain of $C_A$ which does not contain $A$. The domain $I_A$ is the interior with respect to $A$ of $O_A$ and similarly $E_A$ is the exterior with respect to $A$ of $O_A$.

If $C_{AB}$ is a Jordan curve, $C_{AB}$ will be said to be a proper circle with respect to $A$ provided that for every point $X$ in $I_A$, distinct from $A$, $C_{AB}$ is a simple closed curve and $L_A$ is a subset of $I_A$. The expression "a circle $C_{AB}$ has a proper interior $I_{AB}$" will mean that this circle is a proper circle with respect to $A$ and that for every point $Y$ within $I_A-A$, $C_{AB}$ in a proper circle with respect to $A$. A Jordan circle is a circle which is a simple closed curve.

Theorems 30, 31, and 32 are true if we make the following changes: The terms "proper curve" and "proper interior" as used in the statement of Theorems 31 and 32 are "proper curve" and "proper interior" with respect to the point $O$; further, add to the hypothesis of Theorem 31 the following statement, "There exists a point $P$ such that $C_{O}P$ is a subset of $S-R^2$.

On pages 397 to 399 H. G. let the interiors of the Jordan curves there mentioned be those complementary domains of these curves which are subsets of the interior of the circle $t$; and let

1) The proof of Theorem 24 does not depend upon Theorem 22.

2) Cf. Theorems 21 A and 21 B.
the interior of I be that complementary domain of f which contains the
point M.

We shall now show that Theorem 35 holds for the space we are
considering. (See proof given in L. R. H. B.) Let Q be a point of E_0^p.
But C_0^Q is closed. There exists by Theorem 37 a (theorem the proof
of which does not depend on this theorem) a Jordan region R con-
taining J_0^P; but containing no point of C_0^Q. Instead of the region R
mentioned in the proof given in L. R. H. H. use the region R just
defined. The statement made in lines 22 and 23, page 315, L. R.
H. H., does not involve a contradiction in this space. Let X be a
point of k, distinct from P. Let K have the properties mentioned
in R. L. Moore's proof, and let it have the further property that
no motion can transform it into a region that contains X and P.
Now follow the argument of L. R. H. H.

Theorem 34 as stated for L. R. H. H. and Theorems 35 and 36
are not true for our space.

Theorem 34. If C_0^P is a Jordan circle then it is a proper
circle having a proper interior with respect to O.

Proof of Theorem 34: Let the point P mentioned in the
proof of Theorem 34, L. R. H. H. be interpreted as the point P
mentioned in this theorem; (suppose that the theorem does not hold
ture for C_0^P, and limit the discussion given in L. R. H. H., pages
315—316 to the set of points composed of the circle just mentioned
and its interior with respect to O). The set of points which is com-
posed of all the circles with centers at O which contain points
on the interval Ox and belong to S, is a set of points whose
boundary contains X. We can easily prove that this pointset con-
tains at least one other boundary point and then can prove by the
methods used in the proof of Case 1 of Theorem 42 that C_0^P is
a Jordan circle. This part of Theorem 42 does not depend upon
Theorem 34. Further, C_0^P is on the exterior with respect to O
every circle O which intersects the segment Ox. It follows by
methods used in the proof of Theorem 31 that C_0^P is a proper
circle with a proper interior. We are thus led to a contradiction.

Theorem 37. If M is a maximal connected subset of a closed
point set M, S — M, is connected, and K is a closed point set having
no points in common with M, then there exists a Jordan domain
containing M, but no points of K, whose boundary contains no point
of K + M.

Proof. See Theorem 29 and a theorem by R. L. Moore.

Theorem 41. Hypothesis: A connected domain D containing the
point O has the following properties: (1) The boundary L of D is
connected; (2) if X is a point of L, C_0^X = L; (3) under a rotation
about O, D goes into itself. Conclusion: Either L is a single point
or L is a Jordan circle and D = I_0^X where X is a point of L.

Proof. There exists for each point x on L an arc Ox such
that Ox—x is a subset of D. The truth of this statement may be
established as follows: There exists by Theorem 15, F. A., an
arc joining some point Q of L to O. In the order OQ let x be the
first point of this arc. The proposition follows by (2) and (3)
hypothesis. By Theorem 5 and Theorem 24 there exists a Jordan do-
main E containing O such that E cannot be thrown by a rotation
about O into a point-set containing points of L. Let E be the
complementary domain of E; then E contains L. Let J be the
boundary of E. Suppose that L contains at least two points
and let x and y be any two points of L. From the property mentioned
at the beginning of the proof we can show that there exists an
arc at such that t is a point of J but such that at—x is a subset
of E and of D. By methods that will not be difficult for one
familiar with Analysis Situs it follows that there exists an arc
yu such that yu has no points in common with xt, u is a point of J,
and the segment uy is a subset of E and of D. If x is a point
common to E and to the pointset composed of D minus the inter-
vals yu and xt, we can prove in the same way the existence of
an arc zv where v is a point of J and the segment zv is a subset
of E and of D and contains no points of the intervals xt and yu.

Suppose that L = x — y is connected. J = u — v consists of two mutually
exclusive segments; call these M ad N. Let M consist of all those
points z of L such that a corresponding v is a point of M; simi-
larly define N as those points z of L such that a corresponding
v is a point of N. With the help of Theorems 32, 33, and 40 of F. A.,
there exists an arc pqr where p is a point of M, r
is a point of N, and such that pqr is a subset of E = xt — yu. It
may easily be proved that pqr contains points of M and of N.

1) Concerning the Separation of Point Sets by Curves, R. L. Moore,
Proceedings of the National Academy of Sciences, Vol. 11, No. 9, pp. 469—476,
August, 1925.
Let \( z \) be a point of \( L - x - y \), and let \( z_1, z_2, z_3, \ldots \) be a subset of \( N^* \) that has \( z \) as a sequential limit point. Let \( R \) be a region containing \( z \) such that \( R \) does not contain any points of the arc \( zt \), the arc \( yu \) or of \( E^w \). Let \( R^* \) be a region which contains \( z \) and such that \( R^{**} \) is a subset of \( R \). Let \( R^{**} \) be a region containing \( z \) such that \( R^{**} \) cannot be thrown by a motion into a point-set which contains points of both the sets \( R^* \) and \( S - R \) (Theorem 5); there exists a subinterval \( xu \) of \( x \) which is also a subset of \( R^{**} \).

Suppose that \( M^* \) and \( N^* \) have no points in common. There exists infinitely many points \( z_i \), \( i \) a positive integer, of the set \( z_1, z_2, \ldots \) in \( R^{**} \), for each \( z_i \), there exists a rotation \( T_i \) about \( O \) \( T_i(z) = z_i \). Let \( w_i \) be the transform under \( T_i \) of \( w \). The set of \( w_i \)'s will have a limit point \( W \), which by (3), hypothesis, and Axiom 9 will be a point of \( D \), and by definition of the \( w_i \)'s and Axiom 9 a point of \( R^* \). Let \( C \) be a region containing \( W \) such that \( C \) is a subset of \( S - E^w - xt - yu - L \). By Axiom 9 there exists \( T_a \) such that \( T_a(xw) = \) an arc \( xW \) which has properties analogous to those of \( xw \). \( C \) contains a point \( w_i \); there exists in \( C \) an arc \( w_iW \), from the definition of \( z_i \), from the assumption at the beginning this paragraph, and from the statement at the end of the first paragraph, there exists an arc \( w,p \) where \( p \) is a point of \( N \), and \( w,p \) is a subset of \( D, E^t \), and \( S - xt - yu \). It follows easily that \( z \) is a point of \( N^* \). Thus \( M^* \) contains no limit point of \( N^* \). Similarly \( N^* \) contains no limit point of \( M^* \). Thus the supposition that \( L - x - y \) is connected and that \( M^* \) and \( N^* \) have no points in common leads to a contradiction, since \( M^* + N^* = L - x - y \).

It follows that if \( L - x - y \) is connected there exists a point \( z \) and an arc \( uv \) containing \( z \) and \( x \) only in common with \( L \) such that the segment \( zuv \) is a subset of \( E \) and of \( S - xt - yu \) and such that \( v \) is a point of \( M \) and \( u \) is a point of \( N \). The Jordan curves \( uv \) and \( uvw \) are sub-intervals of the boundary of \( E \), are by Theorem 21C the boundaries of mutually exclusive Jordan domains \( H \) and \( I \), respectively, both subsets of \( E \), where \( H \) contains \( x \) and \( I \) contains \( y \), and no point of \( L \) besides \( z \) is a point of the boundaries of the domains \( H \) or \( I \). It follows that \( L - x \) is not connected. Hence by (2) hypothesis, if \( x \) is any point whatsoever of \( L \) then \( L - t \) is not connected. It follows that \( L - x - y \) is not connected. Hence \( L \) satisfies the definition of a Jordan curve.

It follows easily that \( D = I_{ax} \).

**Theorem 42.** If \( p \) is a point distinct from \( O \) then there exists a sequence of points \( w_1, w_2, w_3, \ldots \) (1) where \( w_i \) lies on a proper Jordan circle, \( C_{O} \), with center at \( O \) and a proper interior \( I_{O} \); (2) no two \( w_i \)'s are on the same circle; (3) \( p \) is the sequential limit point of the sequence of \( w_i \); (4) \( C_{O} \) is either a proper Jordan circle with a proper interior \( I_{O} \) or a single point.

**Proof.** The theorem is true by Theorem 34 in case \( p \) is a point of the interior with respect to \( O \) of some Jordan circle or is a point of some Jordan circle. Suppose hereafter, then in this proof that \( p \) is in the exterior with respect to \( O \) of every Jordan circle with center at \( O \).

Case 1. Suppose that \( p \) satisfies conditions (1), (2), (3), of the conclusion of the theorem. Suppose that \( C_{O} \) contains a point \( q \) distinct from \( p \). Let \( D \) be the set of all points \( z \) such that \( z \) is on the interior with respect to \( O \) or on the boundary of some circle, \( C_{O} \), mentioned in the hypothesis, and let \( L \) be the boundary of \( D \). \( L \) contains both \( p \) and \( q \). Let \( t \) be any point of \( L \). Then it is easily shown that \( t \) is the limit point of a sequence of points \( t_1, t_2, t_3, \ldots \) where the \( t_i \)'s with distinct subscripts are distinct points, and each \( t_i \) is the transform under some rotation about \( O \) of some \( w \). It follows by Axiom 9 that there exists a rotation about \( O \) that transforms \( p \) into \( t \). It follows by a theorem of Janiszewski that \( L \) is connected 1). By Theorem 41. \( L \) is then a Jordan circle.

Case 2. Suppose that \( p \) does not satisfy (1), (2), (3), of the theorem. Let \( M \) be the set of all points \( z \) which do satisfy (1), (2), (3), and let \( N = S - M \). Since \( S \) is connected, one of the sets \( N \) and \( M \) contains a limit point \( q \) of the other. It is easy to see that if \( M \) is closed. Hence, \( q \) belongs to \( M \). By Case 1, \( C_{O} \) is either a single point or is a Jordan circle. The latter possibility leads to a contradiction with the help of Theorems 34 and 35. If \( C_{O} \) is \( q \) there exists by Theorem 5 a region \( R \) containing \( q \) such that \( R \) cannot be transformed by a rotation about \( O \) (which is therefore also a rotation about \( q \)) into a point-set containing \( p \). There exists a Jordan circle with center at \( O \) which lies entirely in \( R \), whose exterior with respect to \( O \) is a subset of \( R \), and hence the interior with

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Theorem 43. There exists not more than one point p distinct from O such that C_{O} is p.

Proof: Suppose there are two distinct points p and q besides O which remain invariant for every rotation about O. To obtain a contradiction we may proceed precisely as indicated in the last three sentences of the proof of Case 2 under Theorem 42.

Theorem 44. There exists a point O distinct from O such that C_{O} is O; for every point x of S — (O + O) C_{O} is a Jordan circle.

Proof: Suppose that for each point x of S — O, C_{Ox} is a Jordan circle. Since S is separable there exists a countable set of points x_1, x_2, x_3,... such that if y is a point of S then y either belongs to this set or is a limit point of it. By an argument similar to that used in the proof of Theorem 36 of L. R. H. H. it may be shown that there exists a sequence of circles k_1, k_2, k_3,..., with center at O, such that every point is in the interior with respect to O of some k_i. Let M_i denote k_i plus its exterior with respect to O. By Theorem 34, and Theorem 14 F. A., there exists a point set G common to all the M_i's. Thus we are led to a contradiction. Hence by Theorems 42 and 43 the theorem follows.

Definition. The point O defined in Theorem 44 is the pole or opposite of O.

Theorem 45. (1) If O is the pole of O then O is the pole of O. (2) If x is a point of S — (O + O) and M is an identity motion, then M is the identity motion. (3) If x and y are distinct points, but neither in the pole of the other, and M and N are motions such that M(x) = N(x) and M(y) = N(y), then for all points Z, M(Z) = N(Z).

Proof: Suppose that y is a point of S — O — O that does not remain invariant under M. By Theorems 34 and 44 S — O — O contains a point x such that I_{Ox} contains both x and y. By Theorem 32 we are led to a contradiction. Part (1) follows with the help of Theorems 43 and 44; part (3) is a consequence of part (2) and Axioms 10 and 11.

Theorem 46. If O, P, Q, and X are points and M is a motion such that M(O) = Q and M(P) = X, then M(C_{OQ}) = C_{PX}.

This theorem can easily be proved with the use, in particular, of Axioms 10 and 11.

Theorem 47. If O and O are poles and M is a motion, then M(O) and M(O) are poles; in particular if M(O) = O, then M(O) = O.

Theorem 48. If M is any motion whatever, there exists a point P such that M(P) = P.

Proof. Let K be the surface of a sphere in a three dimensional Euclidean number space, let J be a great circle of K, and let D and E be the two Jordan domains into which J divides K. Let J be a Jordan curve in S with complementary domains D and E. With the help of Theorem 29, we can establish the existence of a correspondence that is continuous (in the sense there defined) between D and D, between J and J, between E and E, and finally between K and S. It follows by a theorem of Brouwer's that there exists a point P which is left invariant by M.

Theorem 49. If M is a motion such that M(O) is P and M(X) is Y, then for any motion N such that N(O) is P, N(C_{O}) is C_{Y}.

For proof see Axioms 10 and 11 and Theorem 46.

Theorem 50. If O and O are poles there exists exactly one Jordan circle C with centers a O and O such that if M is any motion such that M(O) = O, then (1) M(O) = O; (2) if x is a point such that M(C_{O}) contains a point of C_{O}, then C_{O} is C, and if y is a point of I_{Ox} then M(y) is a point of E_{Ox}.

Proof: Suppose that there exists a motion M such that M(O) = O and such that if O is any circle with center at O, then M(O) is not C. Then S — O — O consists of two mutually exclusive point sets L and N which are defined as follows: L consists of those points x of S for which M(C_{O}) is a subset of I_{Ox}. N consists of those points X of S for which M(C_{O}) is a subset of E_{Ox}. (See Theorem 34, Axiom 8 and Theorems 47 and 48). Let C be a Jordan circle with center at O. By Theorem 5 there exists a region R containing O such that R cannot be transformed by a motion into a region which contains points of more than one of the following point sets: C, O, and O. If Y is a point of R, it is easily seen by Theorem 47 and 34 and Axiom 8, that Y is a point of N

1) Compare this theorem with the first theorem in § 22, page 409, H. G. and with Theorem 32, part (3), L. R. H. H.

and that \( M(Y) \) is a point of \( L \). Since both \( L \) and \( M \) exist, and since \( S - O - \overline{O} \) is connected 1) it follows that there exists a point \( Z \) which belongs to one of the sets \( L \) and \( N \) and is a limit point of the other. Suppose that \( Z \) is a point of \( L \) and hence that \( W = M(Z) \) is a point of \( I_{o_o} \). Then by Theorem 34 there exists a point \( y \) which belongs to both \( I_{o_o} \) and to \( E_{o_o} \). By Theorem 5 there exists a region \( H \) containing \( Z \) such that if \( M \) is any motion whatever then \( M(H) \) contains points of at most one of the point-sets \( C_{o_o}, C_{o_o}, C_{o_o}, C_{o_o} \). By definition of \( Z \), \( H \) will contain points of \( N \). Let \( Q \) be such a point.

By Theorems 47 and 48 \( M(C_{o_o}) = C_{o_o} = C_{o_o} \) where \( q \) is \( M(Q) \). By definition of \( H \), \( q \) is a point of \( I_{o_o} \). By definition of \( y \), Theorem 54 and definition of \( H \), \( I_{o_o} \) contains \( I_{o_o} \). This contradicts the definition of \( N \), since \( q \) is a point of \( I_{o_o} \) and hence of \( I_{o_o} \). Further, if we suppose that \( Z \) is a point of \( N \), we get a contradiction by a similar argument. Hence, there must exist a circle \( C \) such that \( M(C) = C \).

Definition. The circle \( C \) uniquely determined by the points \( O \) and \( \overline{O} \) according to Theorem 50 is said to be the great circle with centers at \( O \) and \( \overline{O} \). The notations \( G_{C_o} \) and \( G_{C_{o_o}} \) are symbols that indicate this circle. The points \( O \) and \( \overline{O} \) are poles of this circle. \( G_{I_{o_o}} \) means \( I_{o_o} \), where \( X \) is a point of \( C \); similarly, \( G_{E_{o_o}} \) means \( E_{o_o} \).

In general if \( P \) is a point, \( P \) will indicate the pole or opposite of \( P \).

Theorem 51. If \( C \) is \( G_{C_o} \) and \( M(O) \) is \( P \) then \( M(C) \) is \( G_{C_{o_o}} \).

For proof see Theorems 46, 47, and 50, and Axioms 10 and 11.

Theorem 52. \( G_{o_o} = G_{C_o} \) then (1) \( C_o_o = C_o_o \); (2) \( \overline{P} \), the pole of \( P \), lies on \( C_o_o \).

Proof: Let \( M \) and \( N \) be motions such that

\[
\begin{align*}
M(O) &= \overline{O} & N(O) &= 0 & \text{Then } NM(O) &= \overline{O} \\
M(O) &= 0 & N(O) &= \overline{O} & NM(O) &= 0 \\
M(P) &= Q & N(Q) &= P & NM(P) &= P.
\end{align*}
\]

Hence \( NM(P) = \overline{P} \).

Therefore, by Theorem 50, \( \overline{P} \) is a point of \( C_o_o \). Hence there exists a rotation about \( O \) which transforms \( P \) into \( \overline{P} \). By Theorem 50

1) This statement can be proved by the help of Theorem 34, F. A.
Proof: The circle $GC_0$ contains a point $X$ of $OY\bar{O}$. By Theorems 52 and 56, $X$ lies on $OZ\bar{O}$ and on $GC_0$. By Theorem 52, $GC_0$ contains $P$ and $\bar{P}$. There exists a rotation $M$ about $O$ such that $M(X) = \bar{X}$. By page 405, H. G., $M(\bar{X}) = X$, $M(P) = \bar{P}$, and $M(\bar{P}) = P$. Hence $M(C) = C$ and $M(OY\bar{O}) = OZ\bar{O}$.

If $C_0$ is a Jordan circle, then $C_0$ contains a point $t$ on $OY\bar{O}$. $M(t) = T$ is then a point of $OZ\bar{O}$. By Theorem 45 $M^3$ is the identity motion. Hence $M(T) = t$. It follows by Theorem 45 that if $M$ is a motion such that $M_0(O) = O$ and $M$ satisfies one of the four statements at the end of the theorem, then $M_0$ is identical with $M$.

Theorem 58. If $AB$ is a simple continuous arc and $T$ is a motion that carries $AB$ into a subset of itself, then at least one point of $AB$ remains fixed under the transformation $T$.

Proof. This theorem follows easily, with the help of Theorem 5, from the fact that $AB$ is a simple continuous arc.

Theorem 59. If $ABC$ is an interval of a great circle $\bar{C}$ with a center at $O$ and $X$ is a point of $ABC$ and there exists a motion $T$ such that $T(O) = O$, $T(A) = X$, $T(X) = C$, $T(AX) = XC$, where $AX$ and $XC$ are the intervals $AX$ and $XC$ of the interval $ABC$, then (1) there exists a motion $M$ which leaves $X$ fixed and carries $A$ into $C$; (2) if $N$ is a motion such that $N(X) = X$ and either $N(A) = C$ or $N(C) = A$, then both $N(A) = C$ and $N(C) = A$; (3) if $A$ and $C$ are not poles and $M$ is any motion such that $M(A) = A$ and $M(C) = C$, then $M(X) = X$ and there exists exactly one point $Y$ distinct from $X$ such that $M(Y) = Y$; furthermore, $Y$ is the pole of $X$ and is the midpoint of the interval $ADC$ which is complementary to the interval $ABC$ on $\bar{C}$.

Proof. Let $\bar{X}$ be the pole of $X$ and let $XE\bar{X}$ and $XF\bar{X}$ be complementary semicircles of $\bar{C}$. By Theorem 56, $AX$ (see also hypothesis) is a subset of one of the two given semicircles, say $XE\bar{X}$. Similarly $XC$ must be a subset of one of the given semicircles; since $XC$ and $AX$ have only $X$ in common, $XC$ must be a subset of $XF\bar{X}$. By Theorem 57 there exists a rotation $M$ about $X$ such that $M(O) = O$, and that $M(AB) = X$ of the semicircle $XF\bar{X}$. We wish to prove that $M(A) = H$ is the point $C$.

Case I. Suppose that $HX$ is a proper subset of $XC$. Then $T^{-1}MT^{-1}M(O) = O$, and $T^{-1}MT^{-1}M(AX)$ is a proper subset of $AX$. By Theorem 58 this motion leaves fixed at least one point of $AX$. But it does not leave $A$ for $X$ fixed. Thus by Theorem 45 we are led to a contradiction.

Case II. If $XC$ is a subset of $HX$, $M^{-1}(XC)$ is a subset of $XC$; this case may be handled like Case I. Thus we have $M(A) = C$; and $M(C) = A$, since $M^3$ is the identity. (2), (3), (4) follow easily from Theorem 45 and the preceding argument.

Theorem 60. Two distinct great circles have in common two and only two points and these points are poles; if $O$ is a center and $x$ is a point of a great circle $C$, and $P$ is a point of $I_\infty$, then $\bar{P}$, the pole of $P$, is a point of $E_\infty$.

Proof: Suppose that $C$ and $K$ are distinct great circles. Let $A$ and $B$ be two distinct points that belong to both $C$ and $K$. Suppose that $A$ and $B$ are not poles. By the preceding theorem the midpoints of each of the two complementary intervals $AB$ of the circles $C$ and $K$ belong to both $C$ and $K$. If we take the midpoints of the intervals on $C$ and on $K$ having as endpoints these midpoints and the midpoints of the intervals in $C$ and $K$ having as ends these midpoints and those defined before, and continue this process indefinitely we can prove by methods similar to those used in H. G., pages 415–417 that there exists a set of points $V$ each point of which is the midpoint of an interval of $K$ and an interval of $C$, the ends of these intervals being common to $C$ and $K$; $V$ is a common subset of $C$ and $K$ and is everywhere dense in the closed sets $C$ and $K$. It follows that $C$ is identical with $K$.

II. Let $O$ be a center and let $x$ be a point of $C$ and suppose that $P$ belongs to $I_\infty$. By Theorem 53 there exists a great circle $H$ which contains $O$, $P$, and $\bar{O}$. By the preceding paragraph, since $H$ is connected, $H$ contains exactly two points $Q$ and $\bar{Q}$, which are poles, in common with $C$. Hence the segment $QO\bar{Q}$ of $H$, which by Theorem 56 contains $P$, is a subset of $E_\infty$. The theorem follows easily with the help of preceding theorems.

Theorem 61. In the notation of Theorem 57, $T$ and $T$ are the only points common to $GC_0$ and to $C_0$.

Proof: If $GC_0$ and $C_0$ contain in common more than two points, then one of the complementary semicircles $OT\bar{O}$ and $OT\bar{O}$, say $OT\bar{O}$, contains at least two points of $C_0$. In the order $OT\bar{O}$ let $z$ be the first point common to the semicircle $OT\bar{O}$ and $C_0$, and let $w$ be another point common to them. Let $M$ be a rotation about $O$ that transforms $z$ into $w$. By Theorem 51 this rotation
transforms $H = GC_r$ into a great circle $H_r$ which contains both $O$ and $T$. By Theorem 60 $H_r$ is identical with $H$. The rotation $M$ transforms the segment $Ox$ of the semicircle $OT\overline{O}$ into a segment $Ow$ of $H$. But the segment $Ow$ of the semicircle $OT\overline{O}$ contains the point $x$. The interval on $H$ which is complementary to the segment $Ox$ just mentioned contains the point $t$ of the semicircle $OT\overline{O}$. Hence the segment $Ow$, which is the transform of the segment $Ox$ of $OT\overline{O}$, contains at least one point of $C_m$. But this clearly involves a contradiction.

Theorem 62. If $GC_r = C_m = C$, and if $X$ and $\overline{X}$ are the two points common to $C_m$ and $C$ (See Theorems 52 and 60) then (1) $O$ is the midpoint of the great semicircle $XO\overline{X}$ of $C$; (2) $GC_r$ contains $O, \overline{O}, P, \overline{P}$; (3) If $B$ is a rotation about $P$ that carries $O$ into $\overline{O}$, then for any point $Y$ of $C, R(Y) = \overline{Y}$, the pole of $X$.

Proof: Part (2) follows from Theorem 52. (3) follows from Theorems 54 and 47, and H. G. p. 405.

By Theorem 55 the semicircle $XO\overline{X}$ of $C$ has a unique midpoint $F$. By Theorem 59 there exists a motion $M$ such that $M(F) = F$ and that $M(X) = \overline{X}$. By Theorem 50 $GC_r = C_m$. But by Theorem 52 $GC_r$ contains both $F$ and $O$. By Theorem 60 $GC_r$ and $XO\overline{X}$ have in common only one point. Hence $F = O$. This proves part (1).

Theorem 62A. If $C_m$ contains two opposite points, then $C_m = GC_r$.

Proof: Suppose that $C_m = C$ contains a pair of opposite points $Y$ and $\overline{Y}$. Then there is a rotation $M$ about $O$ that transforms $\overline{Y}$ into $Y$ and hence it transforms $GC_r$ into itself. By Theorem 50 $O$ is then a point of $GC_r$, and then by Theorem 52 $Y$ is a point of $GC_r$. Since $Y$ is common to $C_m$ and to $GC_r$, $C_m = GC_r$.

(2) This is a result of Theorem 45 if $X$ is $O$ or $\overline{O}$. Hence if we suppose that the proposition is not true we may assume that $C_m$ is a Jordan circle, and that $O$ and $P$ are not poles. By Theorem 60 and 53 there exists exactly one great circle $H$ which contains $O$ and $P$. By Theorem 61 $C$ contains exactly one point $t$ of the semicircle $OT\overline{O}$ of $H$ and exactly one point $t$ of the semicircle $OT\overline{O}$ of $H_r$, which is complementary to $OT\overline{O}$. There exists a rotation about $O$ that carries $t$ into $T$. By Theorems 61 and 57 this rotation carries $T$ into $t$. Since $P$ also is a center of $C$, and since $C$ contains only the points $t$ and $T$ in common with $H$, we can show in the same way that there exists a rotation about $P$ that transforms $T$ into $t$ and transforms $t$ into $T$. If $T$ and $t$ are not poles, it follows by Theorem 59 (Part 3) that $P$ must be either $O$ or the pole of $O$. If $T$ and $t$ are poles, it follows by Part (1) of the theorem that $C$ is $GC_r$ and also $GC_r$. It follows by Theorem 52 that $O, P, \overline{O}, \overline{P}$ are points of $GC_r$. But these points are also points of $H$. By Theorem 60 we have then that $GC_r$ is $H$ and hence contains $T$. Thus we are led to a contradiction.

Definitions. If $A$ and $B$ are two distinct points which are not poles, it follows by Theorems 55 and 60 that there exists exactly one great circle $K$ containing them. This great circle will be called the line $AB$. Of the two complementary intervals $AB$ on $K$ there is only one that is a subset of some great semicircle. (See Theorem 56) This interval will be called the interval $AB$. The segment $AB$ of the interval $AB$ will be called the segment $AB$. If $M$ is a motion that leaves a point $X$, and that transforms $A$ into $B$ and $B$ into $A$, then $M$ is called a semi-rotation about $X$. If there exists a motion that transforms $A$ into a point $E$ and transforms $B$ into a point $D$ then the interval $AB$ is said to be congruent to the interval $ED$. It is easily shown by preceding theorems that such a motion transforms the interval $AB$ into $ED$.

Theorem 63:

(1) If $AB = CD$ then $AB = DC$; in particular $AB = BA$.

(2) If $AB = CD$ and $CD = EF$ then $AB = EF$.

(3) If $M(x) = y$ then $M(C_m) = C_m$.

(4) If $AB$ and $CD$ are congruent intervals and $M$ is any motion such that $M(AB)$ contains in common with $CD$ two distinct points one of which is a common endpoint of $M(AB)$ and $CD$, then $M(AB)$ is $CD$ or $DC$.

(5) No interval contains a pair of opposite points.

Proof: (1) is a consequence of the definition of congruence, the group-property of Motions and Theorems 55 and 59. (2) is a consequence of Theorem 56. (3) is a consequence of (1) and Theorem 49. (4) is the result of the definition of congruence, the group property of motions, part (1), and Theorems 60, 61, and 49;
(2) is a result of the group property of motions and the definition of congruence.

**Definitions:** If $AB$ and $CD$ are intervals and there exists a motion $M$ such that $M(AB)$ is a proper subset of $CD$ then $AB$ is said to be less than $CD$ and $CD$ is said to be greater than $AB$. It may be shown by the preceding theorems that if $AB$ is less than $CD$ it is not congruent to $CD$ or greater than $CD$. We may show as in H. G., pages 403 to 407, that by means of motions and midpoints we may set up a one to one reciprocal continuous correspondence between the points of a great circle $C$ and the set of real numbers $[x]$, where $0 \leq x < 2\pi$, in such a way that if $p, p_1, p', p'_1$ are the coordinates of four points $P, P_1, M(P)$ and $M(P_1)$ respectively, where $M$ is any rotation about the pole of $C$, then $p - p_1 = p' - p'_1$ (mod $2\pi$).

Of the rotations about the poles of $AB$ there is one which transforms one of the two points $A$ and $B$ into the other, and which in the notation of H. G., page 405, has a parametric value $w$ whose numerical value (mod $2\pi$) is between zero and $\pi$. The numerical value of $w$ is the length of the interval $AB$ and will be denoted by the symbol $l(AB)$.

**Theorem 64.** If $X$ is the midpoint of $AB$ and $M(AB) = CD$ then $M(X)$ is the midpoint of $CD$.

**Proof:** Clearly $M(A)$ is one of the endpoints of $CD$ and $M(B)$ is the other. The theorem follows easily with the use in particular of Axioms 10 and 11.

**Theorem 65.** (1) $AB$ and $CD$ are intervals then $l(AB) \leq l(AB)$ according as $AB \leq CD$; (2) If $C$ is an interior point of the interval $AB$ then $l(AC) + l(CB) = l(AB)$.

This theorem may be proved with the help in particular of Theorems 62, 64, and 47 and of the results of pages 403 – 405 of H. G.

**Definitions.** A point-set $H$ which contains at least two points that are not poles is said to be collinear if it is a subset of a great circle $C$. The two complementary domains of $C$ will be called the sides of $H$. If a set of points, $X$ is a subset of one of the sides of $H$ then that side will be called the $X$ side of $H$. The poles of $C$ will be called the poles of $H$. If $AB$ is an interval of $C$ and $AB\bar{A}$ denotes the semicircle $ABA$ of $C$ then the point set $AB\bar{A} = \bar{A}$ will be called the ray $AB$, and $A$ will be called the endpoint of this ray. Clearly the ray $BA$ is not the same as the ray $AB$.

**Definitions.** If $C_{\infty} = C$ is a Jordan circle then the intervals $Ox$ and $O\bar{x}$ are radii of $C$. If $C$ is not a circle it is said to be a small circle. By the proper radius or a small circle $C_{\infty}$ we shall mean those radii of $C_{\infty}$ which are congruent to the lesser of $Ox$ and $O\bar{x}$.

That center of a small circle which is an endpoint of the proper radii is said to be the proper center of $C$; that complementary domain of $C$ which contains the proper center is said to be the proper exterior of $C$, and the other one is the proper interior of $C$. By the radial length of a circle we mean the length of one of its proper radii. Of two circles having unequal proper radii that one whose radial length is the greater is said to be the greater of the two circles. If two circles have congruent proper radii they are said to be congruent, and there exist motions that transform one into the other. An interval of length $\frac{\pi}{4}$ is said to be a quadrant. A great circle has a radial length of $\frac{\pi}{4}$ and is greater than any small circle. Two circles are said to be tangent if they have one and only one point in common. Two circles $C$ and $K$ are said to touch provided that they have points in common, but $K$ does not contain points in common with both the interior and the exterior of $C$. If $C$ and $K$ are tangent small circles and their proper interiors have no point in common they are said to be tangent externally; otherwise they are tangent internally. Two small circles may be said to touch externally or to touch internally under corresponding conditions.

**Theorem 66.** If $C_{\infty} = C$ is a great circle and $n$ is a positive number between 0 and $\frac{\pi}{4}$ then there exists in $T_{\pi, e}$ a circle $K$, and in $E_{2\pi}$ a circle $k$ such that (1) $K$ and $k$ have radii of length $n$ and are tangent externally and there exists a semi-rotation about $t$, the point of tangency, that transforms $K$ into $k$, $k$ into $K$, and the proper centers of $k$ and $K$ respectively into the proper centers of $k$ and of $K$; (2) the proper interiors of $K$ and of $k$ contain no point of $C$, and are on opposite sides of $C$; (3) $K$ and $k$ touch $C$.

**Proof:** Let $H$ be a great circle passing through $O$ and $C$. For each point $x$ of $H$ let $t_x$ be a point such that the length of $x_t$ is $n$. 1) Of the definitions following note on Theorem 29. Hereafter, when we use the word "proper" we shall use it in the sense just defined.
Let $L_1$ consist of all those points $x$ of $H$ such that $I'_x$ is a subset of $I'_{ab}$, and let $L_2$ consist of the remainder of $H$. Both classes exist. Let $z$ be a point of division of the two classes $L_1$ and $L_2$. Suppose $z$ belongs to $L_1$. Let $zw$ be congruent to $zt$. Then $I'_w$ contains a point $v$ of $E_{ab}$. Let $x_1, x_2, x_3, \ldots$ be a subset of $L_1$ which has $z$ as a sequential limit point. Let $P$ be a pole of $H$. Let $M_1, M_2, M_3, \ldots$ be a set of rotations about $P$ such that for each integer $n$, $M_n(z) = x_n$. Then $M_n(v) = v_n$ is a point of $I'_{ab}$ and so is $V$, a limit point of the set $v_1, v_2, v_3, \ldots$. If we apply the inverse motions of the $M_n$'s and Axiom 9 we prove there exists a motion $N$ such that $N(P) = P; N(x) = x_n$ and $N(V) = v$. But $P$ and $z$ are not poles and $V$ and $v$ are distinct points. Thus we have a contradiction by Theorem 45.

Hence $z$ belongs to $L_1$. Let $K = C_m$. By methods analogous to those used in the preceding paragraph we can prove that $K$ touches $C$. Let $y$ be a point $G_y$. With the help of Theorem 60 we can prove that there exists a semicircle $F$ of $C$ which lies in $E_{ab}$ and hence $F$ contains none of the points that are common to $C$ and $K$.

By methods similar to those used above we can prove that there exists a circle $k'$ which is congruent to $K$, which plus its proper interior, is a subset of $E_{ab}$, which touches $C$, and such that the set of points common to $k'$ and $C$ are a subset of $F$. We can establish the theorem by methods analogous to those of pages 411 to 414, H. G.

Let $t$ be the point of tangency of $k$ and $k$. In proving that $K$ plus its interior is a subset of $I'_{ab}$ we used merely the fact that the length of $OX$ is greater than $n$. By Theorem 62 the length of $OX$ is $\frac{1}{2} n$. Hence if $m$ is a positive number less than $n$ we can prove by a method similar to that used in the preceding theorem that there exist circles $Z$ and $S$ containing $t$ such that $Z$ plus its proper interior is a subset of $K$ plus its proper interior, and $Z$ and $k$ have the same relation. With the help of results given on page 413, H. G., we get the following theorem.

**Theorem 67.** In the notation of Theorem 66 and the preceding paragraph we have: (1) $Z$ and $S$ are tangent to each other at $t$ and to $k$, $K$ and $C$ and are (2) the only circles of radius $m$ that are tangent to $K$, $k$, and $C$ at $t$; (3) $K$ and $k$ are tangent to $C$ at $t$ and are the only circles of radius $n$ that have this property; (4) $K$ is the only circle of radius $n$ that is tangent to $k$ at $t$; (5) $C$ is the only great circle tangent to $K$, $k$, $Z$ and $S$ at the point $t$.

(1) is a consequence of § 28, H. G. We can in the same way prove that there exists a circle $K'$ of radius $n$ which is tangent to $C$ at $t$. If $K'$ is distinct from $k$ and $K$ we should have $K'$ and one of the circles $K$ and $k$, say $K$, tangent to the second, $k'$; this is impossible by the theorem at the end of paragraph 27, H. G. (2) is a consequence of (3) and (4). It follows easily.

**Theorem 68.** If $t$ is any point of $S$ and $C$ is any great circle containing $t$, then there exists a system $W$ of circles such that (1) $C$ is the only great circle of this system; (2) if $n$ is any positive number less than $\frac{1}{2} n$ there exist two and only two circles $K$ and $k$ of $W$ of radius $n$ that are tangent to $C$ at $t$; (3) if $n, m, K, k, Z, S, t, C$ are interpreted as in Theorems 60 and 67, the conclusions of these theorems hold. (4) Any two distinct circles of $W$ are tangent at $t$, and any circle which touches or is tangent to a circle of $W$ at $t$ belongs to $W$.

Also, if $k$ is any small circle containing $t$, there exists a great circle $C$ which is tangent to $k$ at $t$.

**Proof.** There exists a motion which transforms the point $t$ of Theorems 66 and 67 into the $t$ of this theorem and transforms the $C$ of Theorem 66 and 67 into $C$ of this theorem. The theorem follows, since, as may readily be seen from previous theorems, a motion transforms tangent circles into tangent circles.

**Theorem 69.** (1) If two congruent circles are tangent they are tangent externally; (2) if two non-congruent small circles are tangent internally the smaller circle plus its proper interior is a subset of the larger circle plus its proper interior; (3) if $t$ is a point of a small circle $K$ and $Q$ is any smaller circle, then there exist two circles $Z$ and $S$ congruent to $Q$ such that $Z$ is tangent internally to $K$ at $t$ and $S$ is tangent externally to $K$ at $t$.

**Proof:** (1) is a result of part (2) Theorem 66 and Theorem 68. (2) is a result of Theorems 68 and parts (1) and (2) of Theorem 67 (Note the paragraph preceding Theorem 61 in which the existence of $Z$ and $S$ is proved): (3) is a result of the definition of tangent externally and the properties of Jordan domains.

**Definitions.** If $t$ is a point common to two circles $C$ and $K$ and every region $R$ which contains $t$ contains a point $x$ of $C$ which belongs to the proper interior of $K$ and a point $y$ of $C$ which
belongs to the proper exterior of $K$, then $C$ is said to intersect $K$, and $i$ is a point of intersection of the circles $C$ and $K$. (Note: If $K$ is a great circle then "the proper interior of $K$ may be interpreted as one side of $K$ and "the proper exterior of $K$" may be interpreted as the other side of $K").

**Theorem 70.** If $x$ is the midpoint of the interval $OP$, then $C_{op}$ is tangent to $C_{op}$ at $P$. For proof see H. G. p. 414.

**Theorem 71.** If $C_{in}=C$ and $C_{in}=K$ are not identical and have two points in common then $K$ intersects $C$ at the point $x$.

*Proof:* Suppose that $K$ does not intersect $C$ at $x$. There exists a region $R$ containing $X$ such that the set of points common to $R$ and $K$ is a subset either (a) of $I_{in}$ or else of (b) of $E_{in}$. Consider case (b) and assume, as we can do without loss of generality that $I_{in}$ is the proper interior of $C$. There exists a circle $C_{in}=k$ of radial length $2n$, where $2n$ is a positive number which is less than $\frac{1}{2}p$ and less than the length of $Px$ such that $k$ plus its proper interior is a subset of $R$. There exists by Theorem 69 a point $p$ such that the length of $px$ is $n$, that $C_{in}=x$ is a subset of the proper interior of $C$ and by Theorem 70 is a subset of $k$ plus its proper interior. It follows that $C_{in}$ is tangent to $K$ at $x$. Then by Theorems 68 and 69, $K$ and $C$ are tangent at $x$. Thus we are led to a contradiction. We can handle Case (a) in the same manner.

**Theorem 72.** Two circles which are not identical have in common at most two points.

*Proof:* Suppose that $C$ and $K$ are two distinct circles with a point $x$ in common and having $P$ and $Q$ respectively as centers. The theorem has already been proved for the case where $K$ and $C$ are both great circles. (See Theorem 60). Hence we may assume that $K$ is not a great circle or a single point. Then $K$ does not contain both $P$ and $Q$. We will assume that it does not contain $P$. Let $I=I_{in}$ and let $E=E_{in}$. If we assume that $K$ and $C$ have at least three points in common it follows from Theorem 71 that there is a point $t$ which is common to $K$ and to $E$. There exists on the circle $K$ an arc $xty$ such that the points $x$ and $y$ are points of $C$ but such that the segment $xty$ of this arc is a subset of $E$.

Suppose that there is a point $q$ distinct from $x$ and from $y$ which is common to $C$ and $K$. The point $q$ does not belong to the arc $xty$ and hence there exists a region $R$ containing $q$ which contains no point of $xty$. By Theorem 71, $R$ contains a point $T$ which is common to $K$ and $E$. The circle $K$ contains an arc $XTY$ such that $X$ and $Y$ are points of $C$, and such that the segment $XTY$ is a subset of $E$. Clearly the segments $xty$ and $XTY$ have no points in common.

Let $L_{a}$ consist of all those points $t$ of $E$ such that $E_{in}$ contains no point of the segment $xty$, and let $L_{a} = E - L_{a}$. It can easily be shown that both classes are non-vacuous. Since $E$ is connected, there exists a point $b$ which belongs to one of the sets $L_{a}$ or $L_{b}$ and is a limit point of the other. Suppose that $b$ is a point of $L_{a}$. Then by Theorem 5 there exists a region $R$ containing $b$ such that $R$ cannot be thrown by a rotation about $P$ into a region containing a point of the arc $xty$. But since $b$ is a limit point of $L_{a}$, $R$ contains a point $a$ of $L_{a}$. By Theorem 34 and the definition of $L_{a}$, $E_{in}$ is a subset of $E_{in}$, and $E_{in}$ is a subset of $E$. Since $a$ belongs to $L_{a}$, $E_{in}$ contains a point of the segment $xty$. But $x$ is not a point of $E_{in}$. Therefore $C_{in}$ contains a point of the segment $xty$. But this contradicts the definition of $R$. Hence $b$ must be a point of $L_{a}$, and it can be shown that $C_{in}$ contains points of the arc $xty$. Suppose that the circle $C_{in}$ had at least two points in common with the circle $K$. Then by Theorem 71 there is a point $f$ of $xty$ which is a point of $E_{in}$. It is easily shown that every point of $C_{in}$ is a limit point of $L_{a}$, and that there exists a point $q$ of $L_{a}$, such that $f$ is a point of $E_{in}$. But this contradicts the definition of $L_{a}$. Hence $C_{in}$ is tangent to $K$ at some point of $xty$.

Hence the segment $XTY$ is a subset of $L_{a}$. By an argument similar to the preceding we can show that there exists a point $d$ such that $C_{in}$ is tangent to $K$ at a point of the segment $XTY$. By methods indicated earlier in this proof we can show that $C$ is a subset of $L_{a}$ while $C_{in}$ is a subset of $E_{in}$. But the arc $xty$ contains points in common with each of the circles $C$ and $C_{in}$. But this in contrary to the supposition that $C_{in}$ is tangent to $K$ at a point of the segment $XTY$.

**Definition.** If $C$ is a great circle with centers $P$ and $Q$ intersecting, then $K$ is said to be perpendicular to $C$. Let $O$ be a center of $K$. By Theorem 52, $O$ is on $C$. Hence the following theorem.

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1) $C_{in}$ must intersect the arc $xty$ which contains no points of the segment $XTY$. 

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If \( k \) is less than \( 2^{n-1} \), it follows by Theorem 70 that \( K(k(\frac{1}{2^n}, P) \) is tangent to \( K(k(\frac{1}{2^n-1}, P) \) at \( P \). In any case we have proved that either \( K(k(\frac{1}{2^n}, P) \) is tangent to \( C \) at \( P \) or it is tangent to a circle \( K(k(\frac{1}{2^n-1}, P) \) where \( h \) is between \( O \) and \( 2^{n-1} \) and is not \( 2^{n-1} \). This last sentence is sufficient to verify the statement we made in the first sentences of the first paragraph, for \( n = 2 \), and then to prove by mathematical induction that the statement holds for all larger integral values of \( n \).

If \( x \) is any point whatever of \( P \) distinct from \( P, O \), and \( P \), it can be proved by methods like those used in H. G., §§ 30 to 35 that \( x \) is a limit point of the set of points \( F \), corresponding to the numbers of the type \( k(\frac{1}{2^n}) \). Let \( a_1, a_2, a_3, \ldots \) be a subsequence of points of \( F \) having \( x \) as a sequential limit point. Let \( n \) be so large that twice the length of \( a_n x \), for \( n \) greater than \( n \), is less than \( Px \) and \( Px \). It is easily shown that then \( x \) is a point of \( I_n \), and hence that any ray \( xQ \) having \( x \) as an endpoint will contain at least one point \( z \) of \( C_n \).

Let \( Q \) be a point of \( C_n \) distinct from \( P \). By Theorem 61 \( Q \) is the only point common to the ray \( xQ \) and to \( C_n \). Let \( B_1, B_2, \ldots \) be a sequence of regions closing down on \( x \), and for each value of \( n \) let \( b_n \) be a point of the sequence \( a_n, a_{n+1}, a_{n+2}, \ldots \) such that \( b_n \) belongs to \( B_n \). Let \( M_n \) be a rotation about \( b_n \) that transforms \( P \) into a point \( v_n \) of the ray \( xQ \). Let \( M_n(x) = X_n \). By Theorem 13 the set of points \( X_1, X_2, X_3, \ldots \) has \( x \) as a limit point. The set of points \( v_1, v_2, \ldots \) has a limit point \( V \) on the ray \( xQ \). By Axiom 9 there exists a rotation about \( x \) that transforms \( P \) into \( V \). Hence \( V \) is \( Q \). But all of \( C_n \), except the point \( P \) is on the \( O \) side of \( C \). (See definition of \( b_n \).) Hence \( Q \) is on the \( O \) side of \( C \) or is a point of \( C \); this holds for every point \( Q \) of \( C_n \). By Theorem 71 \( C_n \) and \( C \) must either be identical or must be tangent at \( P \). By Theorem 62 \( C_n \) and \( C \) are identical, since \( O \) is not \( x \).

Let \( K \) be a small circle of radius \( n \) which is tangent to \( C \) at \( P \) and except for \( P \) is on the \( O \) side of \( C \); there exists by the preceding argument on the same side of \( C \), a point \( n \) belonging to \( PO \) such that \( C_n \) is tangent to \( C \) at \( P \), and that the radius of the circle \( C_n \) is \( n \). By preceding theorems there exists exactly one such circle and it easily shown that this circle is \( K \).

Definitions: If \( A, B, \) and \( C \) are distinct, non-collinear points, no two of which are poles, then the sum of the rays \( BA \) and \( BC \).
is called the angle $ABC$, and $B$ is the vertex of this angle. If $M$ is a motion and $M(B) = B'$, $M(Ray\ BA) = Ray\ B'A'$, and $M(Ray\ BC) = Ray\ B'C'$, then the angle $ABC$ is said to be congruent to the angle $A'B'C'$. Let the circle $W$ mentioned in § 41, H. G., be a great circle; we shall adopt for this paper the definitions and conventions given in this paragraph of H. G. If $K$ is $GC_a$, and $D$ and $E$ are the points common to $K$ and to the rays $BA$ and $BC$ respectively then the length of the interval $DE$ is the numerical value of the parametric value of the angle $ABC$, as this parametric value has been defined in H. G., paragraph 41. This interval $DE$ is called the associated interval and the great circle $K$ is the associated great circle of the angle $ABC$. An angle of measure $\frac{1}{4}\pi$ is called a right angle. A point on the $C$ side of the ray $BA$ and on the $A$ side of the ray $BC$, is said to be within the angle $ABC$ and the set of all such points $x$ is the interior of the angle $ABC$. It follows that the interior of the angle $ABC$ is the same as the interior of either of the angles $CBA$ and $ABC$.

Theorems 75 and 76 follow without difficulty from preceding theorems and definitions.

Theorem 75. (1) The interior $I$ of the angle $ABC$ is a Jordan domain whose boundary is the angle $ABC$ plus $\overline{B}$, the pole of $B$; (2) if $X$ is a point of $I$ then the ray $BX$ minus the point $B$ is a subset of $I$; (3) the segment $DE$ of the associated interval $DE$ of the angle $ABC$ is a subset of $I$ but no point of the associated great circle minus this segment belongs to $I$; (4) if the ray $BA$ is perpendicular to the ray $BC$ at $B$ then the angle $ABC$ is a right angle, and conversely.

Definitions. The interior of the angle $ABC$ plus the boundary of this interior in called the lune $ABC$. Clearly the lune $ABC$ is the same as the lune $ABC$, $S$ — (the lune $ABC$) is the exterior of the angle $ABC$, and any point of this set is said to be without the angle $ABC$.

If $A$, $B$, $C$, are distinct, non-collinear points, no two of which are poles, then the intervals $AB$, $BC$, and $CA$ are the sides of the triangle $ABC$, and the sum of these intervals is the triangle $ABC$. If $x$ is a point within each of the angles $ABC$, $BAC$, $BCA$ then $x$ is within the triangle $ABC$ and the set of such points $x$ is the interior of the triangle $ABC$. Clearly $x$ is within the triangle $ABC$ if and only if it is on the $A$ side of $BC$, on the $B$ side of $AC$ and on the $C$ side of $AB$. Also if it is within two of the angles of the triangle, it is within the third.

Theorem 76. The interior of the triangle $ABC$ is a Jordan domain whose boundary is the triangle $ABC$. (2) If $x$ is a point within the angle $ABC$ then the ray $Bx$ contains exactly one point $y$ in common with the segment $AC$ of the side $AC$. (3) The segment $By$ of the ray $Bx$ is within the triangle $ABC$. (4) The segment $AC$ of the side $AC$ is within the angle $ABC$; if $H$ is the line $AC$ then no point of $H$ minus the interval $AC$ belongs to the lune $ABC$. (5) If the arc $h$ of a great circle $HF$ is a subset of the lune $ABC$, and not both $B$ and $B$ are endpoints of $h$, then $h$ is an interval; that is, it is a subset of a great semicircle.

Theorem 77. If $C_a = C$ and $C_{a_0} = K$ have in common exactly two points $D$ and $d$ then $D$ and $d$ are not on the same side of $AB$.

Proof. $B$ and $A$ are not poles, or the same point, since the circles $C$ and $K$ are not identical. Suppose that $D$ and $d$ are on the same side of $AB$. By Theorem 71 there exists a point $y$ of $K$ lying on the non-$D$ side of $AB$. The interval $yd$ of $dyD$ of $K$ contains a point $h$ of the line $AB$ while the interval $yD$ of $dyD$ contains a point $q$ of $AB$. By Theorem 72 if $Dxd$ is the complementary interval on $K$ of $DyD$, then $Dxd$ contains no point of $AB$. By Theorems 71 and 72, if $H$ is that complementary domain of $C$ which contains the segment $Dxd$ of $K$, $H$ contains no point of the segment $dyD$ of $K$. The segment $Dxd$ contains neither $A$ nor $A$. By the methods used in the proof of Theorem 72, we can prove that there exists in $H$ a circle $k$ with centers at $A$ and $A$ such that $k$ is tangent to $K$ at a point $z$ of the segment $DxD$. By Theorems 74 and 68 the intervals $Ax$ and $Az$ are both perpendicular to the common tangent of the circles $K$ and $k$ at $z$. Then $A$, $z$, and $B$ are collinear. But there is only one great circle passing through $A$ and $B$. Thus $z$ is on $AB$ and we are led to a contradiction.

Definitions. Two triangles $ABD$ and $A_{1}B_{1}D_{1}$, are said to be congruent if there exists a motion $M$ such that $M(A) = A$, $M(B) = B$, and $M(D) = D$. In the notation of Theorem 77 the triangles $ABD$ and $ABD_{1}$ are said to be symmetric triangles. If a triangle $T_{1}$ is congruent to a symmetric triangle of $T_{2}$ then $T_{1}$ and $T_{2}$ are said to be symmetrically congruent. It follows as in H. G. that $T_{1}$ is not congruent to $T_{2}$.
Theorem 78. If the interval \( xy \) is less than a quadrant, and \( K \) is a great circle which is perpendicular to \( xy \) at \( x \) then (1) if \( z \) is a point such that \( yz \) is greater than \( yx \) and less than \( y\bar{x} \), then \( C_{yz} \) contains two distinct points \( i \) and \( T \) in common with \( K \) such that the semi-circles \( xiz \) and \( xzi \) of \( K \) are complementary semi-circles of \( K \). (2) If \( W \) and \( Z \) are distinct points of \( xiz \) in the order \( xWZx \) then \( yZ \) is greater than \( yW \). (3) The converse of (2) is true.

Proof. By Theorem 74 \( C_{wy} \) is tangent to \( K \) at \( x \). It is easily shown that \( I_{wy} \) does not contain \( z \) or a point of \( K \) distinct from \( x \) and that \( C_{wy} \) contains points of both the \( y \) side and the non-\( y \) side of \( K \). Hence, by preceding theorems \( C_{wy} \) contains exactly two points of \( K \). (1) follows from Theorem 77. In (2) if \( W = x \), (2) follows from the second sentence in this proof. If \( W \) is distinct from \( x \) then the interval \( yz \), and the segment \( Wz \) of the semi-circle \( xWz \) of \( K \) are subsets of \( I_{wy} \); and the segment \( Wz \), which contains \( z \), is a subset of \( E_{wy} \). (2) and (3) follow easily.

Theorem 79. If \( ABC \) is an acute angle, that is an angle whose measure is numerically less than \( \pi/2 \), and \( DE \) is its associated interval then (1) if \( a \) is a point distinct from \( A \) on the ray \( BA \) in the order \( DAA \) and \( C \) and \( c \) are points of the ray \( BC \) such that the angles \( ADB \) and \( aCB \) are right angles, then in the right triangles \( ABC \) and \( aBC \) \( CA \) is greater than \( ca \) and \( BC \) is greater than \( cB \); (2) if \( x \) is a point of the ray \( BA \), there exists on the ray \( BA \) exactly one point \( y \) such that \( xy \) is perpendicular to \( BC \), and further, \( xy < \frac{1}{2} \pi \).

Proof. The great circle \( K \) containing the associated interval \( DE \) contains \( O \), the pole of \( BC \) which is on the \( A \) side of \( BC \). Since angle \( ABC \) is less than a right angle, the interval \( DE \) is less than a quadrant and hence does not contain \( O \). But all points of \( K \) which are within the angle \( ABC \) are on the interval \( DE \). Hence \( O \) is without the lune \( ABC \). There exists exactly one quadrant \( OAC \). This quadrant is a subset of the great circle containing the interval \( AC \). The interval \( AC \) is a proper subset of the quadrant \( OAC \). From definition of the perpendicularity and Theorem 74, \( OD \) is perpendicular to \( BA \) at \( D \). Since we have \( DAA \) it follows by Theorem 78, part (2) that \( OA \) is less than \( Oa \). By Theorem 65 it follows that \( AC \) is greater than \( ac \). Suppose that on the ray \( BC \) we have order \( BCo \) or that \( C = c \). It is easily shown that in this case the intervals \( AC \) and \( ac \) have points in common. But then the quadrants \( OC \) and \( Oe \) would have at least two points in common and would be identical (Theorem 60). Then we would have \( A = a \) which is contrary to the hypothesis. Hence \( Bc \) is less than \( BC \). The proof of part (2) offers no difficulty.

Theorem 80. In a right triangle a side opposite an acute angle is less than a quadrant.

This follows from Theorem 79.

Theorem 81. In the triangle \( ABC \) if angle \( C \) is a right angle and \( AC \) is less than a quadrant, the angle \( B \) is an acute angle 1).

Theorem 82. In the triangle \( ABC \) if \( C \) is a right angle and \( AB \) is less than a quadrant and either one of the angles \( B \) or \( A \) is acute, or one of the sides \( BC \) or \( AC \) is less than a quadrant, then both \( A \) and \( B \) are acute angles and both \( AC \) and \( BC \) are less than \( AB \).

Proof. From Theorem 81 and the hypothesis it follows that either \( A \) or \( B \) is less than a right angle. Suppose \( B \) is. From Theorem 79 it follows that both \( BC \) and \( AC \) are less than a quadrant. By Theorem 81 angles \( A \) and \( B \) are both acute. By Theorem 78, part (2), \( AC \) and \( BC \) are both less than \( AB \).

Definition. The angle \( ABC \) is said to be symmetrically congruent to any angle which is congruent to the angle \( CBA \).

It is our object in the remainder of this paper to prove that the base angles of an isosceles triangle are symmetrically congruent. We shall prove this proposition by showing that we can derive the formulas for the solution of triangles by methods analogous to those used by W. H. Young in his paper "On the Analytical Basis of Non-Euclidean Geometry" 2). However, he assumes the symmetric congruence of the base angles of an isosceles triangle. This makes it necessary for us to modify his treatment considerably.

Henceforth when we use the term "the interval \( AB \)" we shall mean sometimes the set of points which constitutes the interval \( AB \) and sometimes the number which is the length of this interval. Which interpretation is intendend will in general be evident from the context of the discussion.


2) American Journal of Mathematics, Vol. 33, 1911 (pp. 249—266). This paper will be referred to hereafter as "Young".

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Theorem 83. If \( A \) is an acute angle of the triangle \( ABC \) and \( x \) is a point of the segment \( AC \), then the segment \( Bx \) is within the triangle and \( Bx \) is less than the greater of \( BA \) and \( BC \).

Proof. By Theorems 78, 72, 74, and 79 the segment \( AC \) and the interior of the triangle \( ABC \) are subsets of \( I_A + I_{BC} \).

Definitions: By the sum of a set of angles we mean the sum of the absolute values of the parametric values of these angles; in other words, the sum of the lengths of the associated intervals of these angles. The greater of two angles is the one having the greater measure. The angle excess or the excess of a triangle is the angle sum of this triangle diminished by \( \pi \).

Theorem 84. Every triangle has a positive angle excess.

Proof. We can follow a proof given in a book by Carlaw; consider the proof given on page 133. It can easily be shown, with the help of Theorem 79, from our definitions of the terms "midpoint of an interval" and "the congruence of triangles" that if the triangle \( BDE \) is congruent to the triangle \( ADF \).

Let \( AO \) be a fixed angle. We shall suppose for convenience that \( OA \) and \( OB \) are quadrants; hence \( AB \) is the associated interval of the angle. On the great circle \( AB \) in the order \( ABA \) let \( A_1 \) be a point such that \( BA_1 \) is congruent to \( BA \). \( BA_1 \) is the associated interval of the angle \( BOA_1 \); we shall call \( BOA_1 \) and any angle congruent to \( BOA_1 \), a symmetric angle of the angle \( BOA \). If the variable angle \( XYZ \) is a function of a variable \( x \), by the statement (1) limit (angle \( XYZ \) = angle \( BOA \), we mean that the limit of the absolute value of the measure of the angle \( XYZ \) as \( x \) approaches \( a \) is the absolute value of the measure of angle \( BOA \). Let \( xOB \) be an angle which is congruent to the angle \( XYZ \) and let the point \( x \) be such that \( xB \) is the associated interval of the angle \( xOB \). (Note that the angle \( xOB \) is not congruent to the angle \( BOx \).) From statement (1) it follows that at least one of the points \( A \) and \( A_1 \) and possibly both may be the limiting positions of the point \( x \).

(2) By the statement limit (angle \( XYZ \) = angle \( AOB \) in the strict sense we mean that the limit of the length of the interval \( Ax \), as \( x \) approaches \( a \), in zero. Definition (2) introduces a notion which evidently is a subcase of the case of definition (1).

Definition: If \( A, B, C, D \) are distinct points, no pair of which are poles, and no three of which are collinear, by the quadrilateral \( ABCD \) we shall mean the sum of the intervals \( AB, BC, CD \) and \( DA \). By a tri-rectangle we shall mean a quadrilateral, three of whose angles are right angles, and whose fourth vertex is not without any one of these angles.

Theorem 85. If \( ABNM \) is a tri-rectangle with right angles at \( N, B, \) and \( M \), and \( C \) is a point on the ray \( MA \) such that \( MC \) is congruent to \( BN \) and \( MN \) and \( NM \) are both less than quadrants, then \( AM \) is less than \( BN \) and similarly \( BA \) is less than \( NM \). The angles \( NBC, BCM, \) and \( BAM \) are each greater than a right angle.

Proof. Let \( P \) be the pole of \( NM \) which is on the same side of \( NM \) as the points \( B \) and \( C \). Let \( Q \) be the pole of \( BN \) which is on the \( M \) side of \( BN \). It is easily seen that the quadrants \( BQ \) and \( PM \) contain in common the point \( A \). Also, the quadrant \( PM \) contains the point \( C \). Angle \( NPM \) is acute since \( NM \) is less than a quadrant. Hence by Theorem 82 angle \( PAB \) is acute and angle \( BAM \) is greater than a right angle. By Theorem 79 \( NM \) is greater than \( BA \) and similarly \( MA \) is less than \( BN \). Since \( MC \) is congruent to \( BN \), the point \( C \) belongs to the segment \( PA \). This segment, by a previous theorem, lies within the angle \( PBA \), which is a right angle. Hence, angle \( PBC \) is an acute angle and angle \( CAB \) is greater than a right angle. Similarly, angle \( BCM \) is greater than a right angle.

Theorem 86. In triangle \( ABC \) if \( AB = BC \), then (1) angle \( CAB \), in greater than, equal to, or less than a right angle according to \( AB \) being greater than, equal to, or less than a quadrant. (2) In any case the limit of each of the angles \( A \) and \( C \) as angle \( B \) approaches zero and \( BA \) approaches a limit, is a right angle. (3) There exists an angle \( AC \), if \( AB \) is not a quadrant, exactly one point \( M \) such that the interval \( MB \) minus the point \( M \) is within the angle \( ABC \), angle \( BMA \) is a right angle, and the length of \( AB \) is between

that of $MB$ and that of a quadrant. (4) The intervals $MC$ and $MA$ are less than a quadrant.

Proof. Suppose that $BA$ is less than a quadrant. By methods used in the proofs of previous theorems, we can show that the segment $AC$ is within $I_{N_1}$, and that there exists on this segment a point $M$ such that $MC$ is tangent to $AC$ at $M$. $MB$ is perpendicular to $AC$ at $M$, and since $M$ is in $I_{N_1}$, $BM$ is less than $AB$.

By Theorem 82 the conclusions for (1), (3), and (4) of this case hold. If now angle $B$ approaches zero, (2) follows by Theorem 84. If $BA$ is greater than a quadrant the triangle $ABC$ satisfies the conditions of Case 1. Also angle $ABC$ is congruent to angle $CBA$. The conclusion follows easily.

For the case where $AB$ is a quadrant there is no trouble.

Theorem 87. In the triangle $ABC$ if angle $B$ is acute and angle $C$ is greater than a right angle, then $AC$ is less than $AB$.

Proof. There exists by Theorem 79 on the ray $BC$ a point $N$ such that $AN$ is perpendicular to $BC$ at $N$ and $AN$ is less than a quadrant. Similarly, if $E$ is a point of the ray $BC$ in the order $BCE$, there exists on the ray $CE$ a point $N_1$ such that $AN_1$ is less than a quadrant and $AN_1$ is perpendicular to $BC$ at $N_1$. There is one and only one great circle through $A$ which is perpendicular to $CE$, since $A$ is not the pole of $BC$. Hence, if $N_1$ is distinct from $N$ it must be $N$, the pole of $N$. But $AN_1$ is a semi-circle, and hence by Theorem 65 not both $AN$ and $AN_1$ could be less than a quadrant. Hence $N$ is $N_1$. Hence by Theorem 78 part (2) $AC$ is less than $AB$. For, $N$ is on the ray $BC$ and is also on the ray $CE$. Evidently then we have the order $BCN$ on the ray $BC$.

Theorem 87A. In a triangle $ABC$ if angle $B$ is acute, angle $C$ in greater than a right angle, $AB$ is not greater than a quadrant, and $E$ is a point on the ray $BC$ in the order $BCE$, then (1) angle $ACE$ is greater than angle $ABC$; (2) angle $BAC$ is acute.

Proof. If the theorem is not true, there exists on the interval $AC$ a point $a$ such that angle $aBC$ has the same measure as angle $ACE$. Let $O$ be that pole of $BC$ which is on the $A$ side of $BC$. By methods used in the proof of Theorem 87 we can prove the existence of a quadrant $OaH$ such that $H$ is a point of the ray $BC$ in the order $BCO$. Let $H$ be defined in the same way as the point $N$ in the proof of Theorem 87. The ray $AC$ falls within the angle $BAH$ which by Theorem 82 is acute. Hence by Theorems 87 and $83 \ BC < Ba \leq BA$. There exists a rotation $M$ about $O$ that transforms $C$ into $B$. By page 405 H. G. $M$ transforms the ray $CH$ into the ray $BC$. Hence it transforms the ray $CA$ into the ray $Ba$. Let $M(a) = b$. By Theorem 87 a $C$ is less than $aB$. Hence $b$ is a point of the interval $Ba$ and is distinct from $c_1$ since $M$ is not the identity motion. It is easily shown by preceding theorems that the set of points common to the line $Ba$ and to $I_{N_1}$ is the segment $ba$, and that this segment is also a subset of the interval $Ba$. By methods used in the proof of Theorems 66 and 72, and by Theorem 67, there exists a point $y$ on the segment $ba$ such that $Cy$ is tangent to $Ba$ at $y$. By Theorem 74 $Oy$ is perpendicular to $Ba$. In the order $Oyx$ on the line $Oy$ there exists a point $z$ such that $Oyz$ is a quadrant which is perpendicular to $BC$ at $z$, and that $z$ is a point of the ray $BC$. Since $y$ is a point on the interval $Ba$, $By$ is less than a quadrant. By Theorem 82 angle $Byz$ in then acute. Thus, if the theorem is false, angle $Byz$ is both an acute angle and a right angle.

Theorem 88. If $DBC$ is an acute angle and $H$ is a point on the ray $BC$ and $K$ is a great circle through $H$ and perpendicular to $BC$ then (1) $K$ contains exactly one point $A$ in common with the ray $BD$; (2) $AH$ is less than a quadrant; (3) if $HB$ is less than a quadrant, then so is $AB$, and if $HB$ is greater than a quadrant, so is $AB$.

Proof. It is evident by (3) Theorem 75, that $Q$, that pole of $BC$ which is on the $D$ side of $BC$, is not a point of the line $DBC$ and hence that $Q$ is on the near side of $DB$. By definition of $K$, $Q$ is a point of $K$. Hence it is easily shown that the quadrant $HQ$ contains a point $A$ of the ray $BD$. Let $Z$ be $GC_x$. If one of the intervals $AB$ and $BH$ is greater than a quadrant and the other is less than a quadrant, then the endpoints $A$ and $H$ of the interval $AH$ must be on opposite sides of $Z$ and $Z$ and the interval $AH$ contain in common a point $X$. The point $X$ is evidently neither $Q$ nor $Z$. But by Theorem 55 $Z$ contains $Q$ and $Z$. Hence $K$ and $Z$ contain in common three points and by Theorem 69 we are led to a contradiction. Hence we have proved (3).

Theorem 89. In a triangle $ABC$ the sum of the sides $AB$ and $AC$ is greater than the side $BC$.

Proof. Suppose that the theorem is not true and that there exists a triangle $ABC$ such that $BC$ is greater than, or equal to
the sum of $AB$ and $AC$. Then there exist on the interval $BC$
points $X$ and $x$ such that $Cx$ is congruent to $CA$ and $BX$ is congruent to $BA$. If $BC$ equals the sum of the other two sides then $X = x$. By Theorems 74 and 68 $C_x$ and $C_a$ are then tangent at $x$, and since these two circles have only one point in common it follows that $A = x$. Hence, $B$, $C$, and $A$ are collinear, contrary to hypothesis.

Hence, with the help of Theorem 65, we can prove that $Cx$ and $BX$ have no points in common. Then on $BC$ we have the order $BXC$. By the argument used above it follows that $C_x$ and $C_a$ are not tangent and since $C$ and $B$ are not poles the two circles are not identical (Theorem 62 A). Hence by Theorem 72, they have in common exactly two points. Let $X'$ be the point distinct from $X$ that is common to $C_x$ and the line $BC$. By Theorems 71 and 72 the segment $XAX'$ of $C_x$ is a subset of the $A$ side of $BC$.

Let $x_0$ be the point distinct from $x$ common to the circle with center $A$ and containing $A$. The sum of the intervals $C_x$ and $XB$ is less than $AB$ which is less than a semicircle. The intervals $C_{x_1}$ and $B_{x_1}$ are subsets of that interval $BC$ of the line $BC$ which is greater than a great semicircle, and hence the intervals $C_{x_1}$ and $B_{x_1}$ have no points in common. The points $X$ and $X'$ are thus both in $E_{x_1}$. (By Theorems 71 and 72 the only point common to $I_{x_1}$ and the great circle passing through $B$ and $C$ are the points of the interval $x_1; Cx_1$ of this great circle). It follows by Theorem 71 that the segment $XAX'$ of $C_x$ contains two points of $C_x$. By Theorem 77 we are led to a contradiction.

Let $OA$ be equal or less than a quadrant and let $OB$ be less than $OA$. Then by preceding theorem $ABOM$, $OM$, $ON$, $OP$, $CP$, $CP$, are each not greater than a quadrant. With the help of Theorem 79 we can show that if $H$ is the pole of $OM$ on the $A$ side of $OM$ then there exists in the interval $CH$ of the quadrant $HCP$ a point $E$ such that $AE$ is perpendicular to $HCP$ at $E$ and $EA$ is less than a quadrant. As in the proof of Theorem 84 we can show that by a semi rotation about $C$ the triangle $ECA$ goes into a triangle $FCB$, where $F$ is a point of the ray $CP$. The inequalities (reversed) of § 3 of Young follow without trouble. In § 4 Young applies the symmetry theorem and hence his argument is not valid for our purposes. We shall proceed as follows: In Figure 2 let $B$ be the same as $O$. Then $OC = \frac{1}{2} OA$, and by inequality (1) § 3, Young $CP < \frac{B}{OC} < \frac{A}{OA}$.

Now we can proceed exactly as in paragraph 5, Young, noting, however, that the inequalities of Young must be reversed. We have by the above and previous theorems that $CP$ is less than $AM$, which is less than $EP$, provided $OC$ is less than $OA$, and $OA$ is not greater than a quadrant. We have also by Theorem 82 that angle $OCP$ and hence angle $ECA$ is less than a right angle. Then in the right triangle $EAC$, since $AC$ is less than a quadrant, it follows that $EC$ is less than $OA$; hence (1) $\frac{C}{O} < \frac{A}{O} < \frac{ECA}{OC}$.

(2) $OA = OC + CA$. Hence, if we let $A$ approach $C$, we reach the conclusion at the end of § 7. In case $C$ approaches $A$, multiply the inequality (1) above by $\frac{C}{OC}$, and the same conclusion may be drawn.

We define and prove the existence of the sine of an angle $AOM$ as in § 2). We cannot prove at this stage that $\sin AOM = \sin MOA$.

The same problem will arise in connection with the other trigonometric functions of an angle; it will be proved, after we have developed the addition formulas, that $P(\angle AOM) = P(\angle MOA)$ where $P$ is any one of the six trigonometric functions. Concerning § 10, Young. Let $AOM$ be an acute angle and let $OA$ be equal or less than a quadrant. Use the notation of § 10. Let $F$ be that pole of $OM$ which is on the $A$ side of $OM$. By previous theorems the intervals $BN$, $CP$, $AM$ are each less than a quadrant and hence are subsets of the quadrants $FN$, $FP$, and $FM$ respectively. By Theorem 82 the angles $OCP$ and $OAM$ are acute. Hence by Theorem 79 there exists on the interval $CP$ a point $E$ such that the interval $BE$ is less than $OP$ and that $BE$ and $OP$ are perpendicular at $E$; similarly there exists on the interval $AM$ a point $H$ such that $CH$ is perpendicular to $AM$ at $H$, and $CH$, is less than a quadrant. Since $P$ is the midpoint of $NM$ there is a rotation about $F$ that carries $P$ into $N$ and $M$ into $F$. This rotation carries $C$ into a point $c$ of the quadrant $FN$ and $H$ into a point $h$ of the quadrant $FP$. By Theorem 79, $BN$ is less than $CP$ which is congruent to $NC$. Hence on the quadrant $FN$ we have

1) The sine ratio is never greater than unity. See Theorem 32.
the order $NBE$. Hence by Theorem 79 $BE$ is greater than $CH$ which is congruent to $CH$.

In triangle $BCE$ angle $E$ is a right angle. Hence angles $EBC$ and $BCE$ have a sum greater than a right angle. Angle $PCH$ the fourth angle of the tri-rectangle $PCHM$, is greater than a right angle, and hence the sum of angles $BCE$ and $ACH$ is less than a right angle. It follows that the angle $ACH$ is less than the angle $EBC$. Since $E$ is a point of the ray $CP$, the interior of the triangle $BEC$ is within the $M$ side of $OBA$. Similarly the interior of the triangle $CHA$ and all of this triangle except the side $AC$ are on the $M$ side of $OBA$. Let $Q$ be a pole of $OBA$. There exists a rotation about $Q$ that transforms $C$ into $B$. Let the transforms under this rotation of $A$ and $H$ be $A_1$ and $H_1$ respectively. By the formulas on page 405, H. G., $A_1$ will be a point of the ray $BC$. Since the angle $A_1BH_1$ is congruent to the angle $ACH$ which is less than the angle $CBE$, the ray $BH_1$ falls within the angle $CBE$. Also, since $BH_1$ is congruent to $CH$ which is less than $BE$, it follows by Theorem 78 (part 2) and Theorem 76 (part 3) that the interval $BH_1$ is $B$ is within the triangle $BEC$. Since the pole $T$ of $BE$ on the $C$ side of $BE$ is on the non-E side of $BC$ (by an argument we have used several times) there is a point $K$ common to the ray $BC$ and the interval $HT$. The ray $KHL$ will contain a point $L$ of the ray $BE$. By Theorem 76, parts (4) and (5) the segment $KL$ is a subset of the interior of the angle $CBE$ and contains $H$. If $K$ is a point of the ray $CA$, the interval $KL$ must then contain a point of the side $CE$ of the triangle $CBE$. But since the angles $KLE$ and $CEL$ are a right angles and $CE$ is less than a quadrant, such an intersection is evidently impossible. Also, evidently, $K$ is distinct from $B$. Hence $K$ is a point of the segment $BC$. Since $BK$ is less than $BC$ which is less than a quadrant it follows that $LK$ and hence $LH_1$ is less than a quadrant; also $BH_1$ is less than a quadrant. Hence from the right triangle $BKH$, the angle $BHL$ is acute. Hence the angle $BH_1K$ is greater than a right angle. Since the angles $BH_1K$ and $BH_1A_1$ have a ray $H_1B$ and a vertex $H_1$ in common either the interior of one is a subset of the interior of the other or their interiors have no points in common. By Theorem 76, part (4), the segment $KB$ is within the angle $BH_1K$, and the segment $BA_1$ is within the angle $BH_1A_1$; the first of these angles is greater than a right angle, and the second is a right angle, and the segments $BA_1$ and $BK$ are subsets of the ray $BC$. Hence $K$ is without the angle $BKH$ and $A_1$ is a point of the segment $BC$ and we have proved that $BC$ is greater than $BA_1$ which is congruent to $CA$.

If we reverse inequalities, §§ 7, 11, 12 and 14 follow with few or no modifications. § 13 will be considered later in connection with §§ 16–18.

Concerning § 15 of Young. The inequalities for the sine, cosine, and tangent given in § 15 of Young hold good here. Young's proofs for the case of the cosine and tangent hold here. We shall now consider the case of the sine.

Use the notation of Young in Figure 5. Assume that $OA$ is less than a quadrant. By Theorem 78 (2) $OM < OA' < OA$. Angle $AOA'$ is less than a right angle and hence by Theorem 79 and 82 there exists on the interval $OA$ a point $B$ such that $A'OB$ is a right triangle having a right angle at $B$ and having the properties of the triangle mentioned in Theorem 82. We have then from the right triangle $ABA'$ that $AB$ is less than $AA'$. We also have $OB < OA' < OA$. Hence $OA = OA' = eAA'$ where $e$ is a positive quantity less than unity. Also $A'M < OA'$. Hence

$$\frac{AM}{OA} = \frac{AA' + A'M}{OA'} = \frac{A'M}{OA'}.$$

From this inequality, if we let $OA$ approach zero, it follows easily that sin $(AOA')$ is not less than sin $(AOA)$.

Paragraphs 16, 17 and 18. Since the sine function and the tangent function are monotone increasing functions of the decreasing $OA$ (notation of § 10) it follows that the sines of acute angles are positive numbers, which, as pointed out by Young, are not greater than unity; and also that the tangents of acute angles are not zero. Refer now to Figure 7, §§ 17 and 18. Assuming that all the intervals mentioned are less than a quadrant, it follows that

$$\frac{OK}{OA} > \frac{AM}{OA}$$

and

$$\frac{KA}{OA} < \frac{OM}{OA}.$$

From the second of these inequalities it follows that if we let $OA$ approach zero, cos $(AOA)$ is greater than or equal to sin $(AOA)$, which is a positive number. Since the cosine function is less than unity when $OA$ is less than a quadrant and since it decreases
monotonely with $OA$, it follows that the cosine of an acute angle is between zero and one. It follows that the six trigonometric functions of an acute angle exist, and are all positive numbers. In Figure 7, let $OA$ have a definite value $r$ less than a quadrant. Let $OM$ have a length less than $e\cdot r$, where $e$ is a positive value which has been assigned in advance and which we will suppose is less than unity. Let $Q$ be the pole of $OM$ which is on the $A$ side of $OM$. The point $Q$ is without the circle $C_a$. The circle $C_a$ has only the point $A$ in common with the quadrant $QM$. Let $N$ be the pole of $OQ$ which is on the $M$ side of $OQ$. Then by Theorem 52 and later theorems, angle $QON$ is a right angle, and $M$ is a point of the quadrant $ON$. The ray $OA - Q$ is on the $Q$ side of $OM$. Also $A$ is on the $M$ side of $OQ$. Hence the ray $OA$ falls within the right angle $QOM = QON$ and contains a point $B$ of the segment $QN$ of the associated interval $QN$ of that angle. By Theorem 76, part (3) the segments $AQ$ and $BQ$ each contain one point of any ray which falls within the angle $QOB$. Let $y$ be a point belonging to the segment $AQ$ and $x$ a point common to the rays $BQ$ and $Oy$. $x$ is the difference between the measures of the angle $xON$ and the right angle $QON$. (By Theorems 71 and 72, applied at the point $x$, the intervals $Ox$ and $xN$ have only $x$ in common.) Let $Z$ be a point on the interval $Oy$ such that $OZ$ is equal to or less than $OA$. Let $ZN$ be the perpendicular to the ray $OM$ at $N$. By results we have established for the cosine ratio it follows that

$$\frac{ON}{OZ} < \frac{OM}{OA} < e.$$ (1)

Those inequalities hold for all $x$'s on the interval $QB$ and all $z$'s on $Oy$ such that $OZ$ is less than $OA=r$. Hence in the notation of § 21 Young, it follows that limiting value $f(x, z) = 0$ in the strict sense

$$\lim_{x \to \pm \infty} f(x) = 0.$$

where $f$ stands for the cosine ratio. In the same construction let $OM$ remain fixed, but let $B$ approach $Q$. $OBN$ and $OxQ$ are right angles, since by preceding theorems $O$ is a pole of the great circle $QN$.

The limit of $C_a$ as $AQ$ approaches zero is the cosine of angle $MQN$.

But the cosine function for fixed angles is a monotone decreasing function of a decreasing argument and is never less than cosine of the angle, which is positive number; hence, as $BQ$ approaches zero $AQ$ approaches zero; and from the right triangle $ABQ$ it follows that $AB$ also approaches zero. Since $OB$ and $QM$ are quadrants $\frac{1}{4}\pi = \frac{AQ}{OA}$ Whith the help of inequality (1) of the section concerning § 15 of Young it follows that if we define $x$, $y$, $Z$, $N$ as before we can find a position of $B$ such that if $e$ is a positive number

$$\frac{yM}{Oy} > \frac{AM}{OA} > 1 - e.$$ (2)

In the notation of § 21 we have proved that limiting value $f(x, z) = 1$, $\lim_{x \to \pm \infty} f(x) = 0$ where $f$ stands for the sine ratio, and where the angle $BON$ approaches the right angle $QON$ in the strict sense. (See note following Theorem 84). In inequality (1) let $OZ$ approach zero. Since the cosine function for a fixed angle decreases monotonely with the distance it follows that the cosine of angle $xOM$ is less than $e$. Similarly in inequality (2), since the sine function increases with the decreasing distance it follows that sin $xON$ is between 1 and $1 - e$. Hence we can make the statement made at the foot of page 361, Young, provided we say that the cosine has the limit zero and the sine has the limit unity. Finally, it is not necessary that the angle $AOM$ have the angle $QOM$ in the planar sense. For, if the point $B$ has both $Q$ and $Q$ as limiting positions it is evident that for a subsequence of the $B$'s which have $Q$ for a limiting position we can set up inequalities (1), and (2), while for that subsequence which has $Q$ as limiting position, we get similar inequalities. Adopt again the notation of Figure 7, Young. Since $\frac{KQ}{OQ} = \frac{AM}{OA}$ it follows from what we have just proved that as angle $AOK$ approaches a right angle and angle $AOM$ approaches zero and the distance $OA$ approaches zero, the sine function of angle $AOM$ and the distance $OM$ approaches zero. Also the cosine of angle $AOK$ is not less than the sine of $AOM$. Hence, from what we have proved before about the cosine it follows that as angle $AOM$ approaches zero its sine approaches zero.

From $\frac{AK}{OA} < \frac{OM}{OA}$, we can prove by methods analogous to those used above the statements in paragraphs 17 and 21 concerning the
convergence of the cosine to the value unity and the uniform convergence of the cosine ratio to this same value as the angle $AOM$ approaches zero and $OA$ approaches zero. We can now affirm the statements of $§$ 19 and accept the definitions there given.

We shall next establish some of the results given in $§§$ 20, 21, 23 and 24. See the construction in the section concerning $§$ 15.

From \[ \frac{AM}{OA} > \frac{A'M}{O'A'} \] it follows that \[ 0 < \frac{AM}{OA} - \frac{A'M}{O'A'} = \frac{AA' - A'B}{O'A'} = \frac{AA' - A'B}{O'A'}. \]

We wish to prove that if either the angle $AOM$ or the angle $A'OM$ is held fixed and has a measure $\alpha$, as the angle $AOA'$ approaches zero and the lengths $OA$ and $OA'$ approach zero, the right hand expression in the inequality above approaches zero. Since the angle $AOM$ approaches a fixed value $\alpha$, and the angle $OMA$ is a right angle, it follows from the statement in $§$ 1 on the lower half of page 251, Young, that the angle $OAM$ approaches the value $\frac{\pi}{2} - \alpha$. By the same reasoning it follows in the right triangle $ABA'$ that the angle $AA'B$ approaches the value $\alpha$.

Consider now a fixed angle $HIJ$ having a positive measure $\frac{\pi}{2} - \alpha - \epsilon$, where $\epsilon$ is a positive number, and let its symmetric angle be called $KIJ$. (See note at the end of Theorem 84). Let the variable angle $XIJ$ be congruent to the angle $AAB$ and further assume that the triangle $XIJ$ is congruent to the triangle $AAB$ in such a way that for each value of the variable $OA$ there is a motion that transforms $A$ into $I$, $A'$ into $X$, and $B$ into $J$. Let $OA$ be so small that angle $AAB$ is greater than $\frac{\pi}{2} - \alpha - \epsilon$. (By hypothesis $H$ and $K$ are on opposite sides of $IJ$. The ray $IJ$ is fixed and so is the point $I$ but the point $J$ is not fixed). Since $XIJ$ is greater than either of the fixed angles $HIJ$ and $KIJ$, if $X$ is on the $K$ side of $IJ$ the ray $IK$ must fall within the angle $XIJ$. If $X$ falls upon the $H$ side of $IJ$ the ray $IH$ will fall within the angle $XIJ$. We shall for the sake of simplicity consider the case where $IK$ is within the angle $XIJ$. In this case by Theorem 76 the ray $IK$ and the segment $XI$ contain a point $L$ in common.

Let $r$ be a definite value of $IL$ for which all the statements above are true, and let $k$ be the corresponding value of $JL$. Let $r'$ and $k'$ be defined similarly in terms of $IL$ and $JL$. From the properties of the sine ratio it follows that when $IL$ is less than $r$ the ratio $JL$ will be greater than $k$. But by inequality (1), in the section concerning $§$ 15, $JX > JL$. We can make a similar statement for the case when $X$ is on the $H$ side of $IJ$. Hence, since as $OA$ and $OA'$ approach zero, $AA'$ and $IX$ approach zero, it follows that $JX = \frac{1}{k} (AA')$ is always less than the smaller of the values $\frac{1}{k}$ and $\frac{1}{k'}$ and hence is bounded. $OM < OA < OA'$; therefore, as $OA$ approaches $O$, so do $OM$ and $OA'$.

The variable angle $AOM$ is less than some definite angle $\beta$ which is less than a right angle. From what we proved in the section concerning $§§$ 16 and 17 concerning the cosine function it follows that $OM$ is always greater than $\cos \beta$ which we have proved is a positive quantity. Hence, since $OM$ approaches zero with $OA'$, $OA$ will approach zero with $OA'$. Hence, we have proved that if one of the quantities $OA$, $OA'$ approaches zero the other does. But $\frac{A'B}{OA}$ approaches zero with decreasing $OA'$ and decreasing angle $A'OB$. Hence, if we keep either of the angles $AOM$ and $A'OM$ fixed and let the other one vary, the sine function of the variable one and a decreasing modulus approaches the sine of the fixed one as limit as angle $A'OB$ and the distance $OA$ and $OA'$ approach zero. Further, let $OA$ be so small that the difference between sine ratio and the sine of the fixed angle is less than a positive number $\frac{1}{\pi}$, where $\epsilon$ has been assigned in advance, and let $A'OA'$ and $OA'$ be so small that for all smaller values of $OA$ and $A'OA'$ the difference in inequality (1) will be less than $\frac{1}{\pi}$. Now keep both angles fixed and let $OA$ and $OA'$ approach zero. It follows that the sines of the two angles differ by less than $\epsilon$. Hence we have established the fact of the uniform convergence of the sine function and the continuity of the sine. (Cf. $§$ 21, Young).
Concerning the uniform convergence of the cosine ratio and the continuity of the cosine: It has been shown that $OA - O'A' = eA'A'$ where $0 < e < 1$. We have

$$0 < \frac{OM}{O'A'} = \frac{OM(OA - O'A')}{O'A' \cdot OA} \cdot \frac{OM \cdot AA'}{O'A' \cdot OA} \cdot e$$

is always less than unity. Hence from facts we have proved in the immediately preceding paragraphs and by methods analogous to those there used, we can prove the continuity of the cosine and the uniform convergence of the cosine ratio to the cosine. We have now proved properties that enable us to prove the results given in §§ 21, 20, 25, and 24. We are not yet ready for the developments in §§ 22, 25, and 26.

Concerning § 39 and following. Young's treatment of the existence of the length of a circle offers certain difficulties; his geometry is a hyperbolic geometry and so in § 27 he can use the fact that in that geometry the angle sum of a triangle is less than two right angles; in some of the following paragraphs he uses the symmetry relation. We can, of course, apply neither of these propositions. We shall change the order of the treatment he gives and prove first the addition formulas. Consider, then, (1) in Figure 12, paragraph 39, where $A$ and $N$ and the interiors of the angles $AOB$ and $BOM$ are on opposite sides of the ray $OB$ and $M$ belongs to the ray $ON$. Suppose $AOM$ is acute. Its measure is the sum of the measures of $AOB$ and $BON$. Let $OA$ be less than a quadrant. Let $Q$ be that pole of $ON$ which is on the $A$ side of $ON$. Then $A$ is on the quadrant $QM$. Let $P$ be that pole of $OQ$ which is on the $A$ side of $OQ$ and let $T$ be that pole of $QAM$ which is on the $B$ side of $QAM)$. By Theorem 52, $T$ and $P$ are points of $GC_O$ which by hypothesis contains $O$ and $M$. In the triangle $QXB$ the side $QX$ is congruent to $QB$ and both are greater than a quadrant. By Theorem 86 the angles $BXM$ and $XBN$ are both greater than a right angle. Both of these angles have right angles as limits as the sides of the quadrilateral $XBNM$ approach zero.

We shall prove that if $K$ is a point on the ray $AM$ such that $BKA$ is a right angle then the limit of $\frac{AK}{AB}$ as these intervals approach zero is the cosine of angle $BOM$. Let $a$ and $b$ be the measures of angles $AOB$ and $BOM$ respectively. For sufficiently small values of $OA$ it follows by propositions proved by Young that the measure of angle $OAM$ is as close as we please to $\frac{1}{2} \pi - (a + b)$ while that of angle $OAB$ is as close as we please to $\frac{1}{2} \pi - a$. Hence for sufficiently small values of $OA$, the measure of angle $BAK$ is as close as we please to $b$. The first of these angles is smaller than the second. It can be proved that $M$ and $B$ are on the same side of $OA$, the ray $AM$ falls within the angle $OAB$ and $O$ and $B$ are on opposite sides of $AM$. By Theorem 82 $AB$ is less than $OA$. It follows easily that angle $BAK$ is less than a right angle and hence from the triangle $BAK$ that $AK$ is less than a quadrant. Let $R$ be a rotation about $M$ that carries $T$ into $T$. Let $A_1, B_1, K_1$ be the transforms under $R$ of $A, B$ and $K$ respectively. By Theorem 52, $GC_O$ contains both $T$ and $Q$. By Theorem 62, $R$ carries $Q$ into $T$, the pole of $T$, and carries the quadrant $MQ$ into the quadrant $MT$. Hence the ray $AM$ is the ray $A_1M$. Also, since on the ray $OMN$ we have the order $OMT$, it follows that the ray $MA_1$ and the quadrant $MA_1$ contain $O$. Since $B$ and $T$ are on the same side of $AM$ it follows that $B_1$ and $Q$ are on the same side of $OT$ which is on $GC_O$. Let $S$ be a rotation about $Q$ which carries $A_1$ into $O$. Let the transforms under $S$ of the points $B_1$ and $K_1$ be $Z$ and $W$ respectively. It follows, (See H. G., pages 408 to 405), that $S$ transforms the ray $A_1M$ into the ray $OM$. Hence the motion $SB$ transforms $A_1$ into $O$, transforms $B_1$ into a point $Z$ which is on the $Q$ side of $OM$ and transforms $K_1$ into a point $W$ of the ray $OM$. Hence in the terms of the notation introduced at the end of Theorem 84.4, $OA$ approaches zero, the angle $BAK$ approaches the angle $BON$ in the strict sense, and, therefore, as we have proved in a preceding section, the limit of the ratio $\frac{AK}{AB}$ will be the cosine of the angle $BOM$. We can now prove with these additions to Young’s arguments, the formula at the foot of page 272:

$$\sin AOM = \cos BON \sin AOB + \sin BON \cos AOB.$$  

Consider now the formula for the cosine of the sum of two angles. We have on the interval $ON$, the order $OMN$. From the triangle $BKM$ we have $BK$ is less than $NM$. Let $Y$ be a point
of the interval $NM$ such that $MY$ is congruent to $BK$. Then

$$OM = ON - YN - MY, \text{ from which}$$

$$\frac{OM}{OA} = \frac{ON}{OB} \cdot \frac{KB}{AB} = \frac{NY}{BN} \cdot \frac{OB}{OA} = \frac{BN}{OB} \cdot \frac{OA}{OA}.$$  

With the help of results obtained above and methods used in § 39, Young, it follows

$$\cos AOM = \cos BON \cos AOB - \sin BON \sin AOB.$$  

The proof of these formulas for case (2), where $AON$ is a right angle is along the same lines as the proof for the case where it is acute. If now angles $BON$ and $AOB$ have the same measure they are congruent. If we indicate these congruent angles by $\alpha$, the angle $AOM$ will have a measure $2\alpha$; if we keep in mind the construction we made in proving the formulas (1) and (2) we have

$$\sin (2x) = 2 \sin x \cos x,$$

$$\cos (2x) = \cos^2 x - \sin^2 x.$$  

If we square and add we get

$$\cos^2(2x) + \sin^2(2x) = (\cos^2 x + \sin^2 x)^2.$$  

In the three formulas just proved substitute $\frac{1}{2}x$ in place of $x$, wherever

$$\cos^2 x + \sin^2 x = 1$$  

we can prove from formulas (5) and (4) that

$$\cos \left(\frac{1}{2}x\right) = \sqrt{\frac{1 + \cos x}{2}}$$

$$\sin \left(\frac{1}{2}x\right) = \sqrt{\frac{1 - \cos x}{2}}$$  

where the radicals stand for the positive square root.

We shall now prove the following theorem:

**Theorem 90.** If $AOB$ is an angle whose measure is $x$, then

(1) $\cos x = \cos x$ and $\sin x = \sin x$, where $\cos w$ and $\sin w$ are the cosine and sine of Euclidean geometry, and where the argument $w$ is expressed in radian measure (Euclidean); (2) $f(AOB) = f(BOA)$, where $AOB$ is an acute angle, and $\frac{x^2}{2}$ is any one of the six trigonometric functions.

**Proof.** When angle $AON$ is a right angle, the ratio $\frac{AM}{OA}$ is constant and equals unity, and the ratio $\frac{OM}{OA}$ is constant and equals zero. This agrees with our previous definitions of the sine and cosine of a right angle. If we let $x = \frac{1}{4} \pi$ it follows from formulas (6) and (7) that part (1) of Theorem 90 is true for $x$ equal to $\frac{1}{4} \pi$ or $\frac{3}{4} \pi$. But the formula (6) is true for $x$ equal to $\frac{1}{4} \pi$ and hence by (7) the theorem is true for $x = \frac{1}{4} \pi$. By mathematical induction, with the help of formulas (6) and (7), it follows that when $x = \frac{1}{4} \pi$, $\pi$ a positive integer, part (1) of the theorem is true. Finally we can prove by mathematical induction, with the help of formulas (1) and (2), that if $k$ is any positive integer not greater than $2\pi$, then for $x = k \frac{1}{4} \pi$, the theorem is true. But the numbers $x = k \frac{1}{4} \pi$ are everywhere dense in the number interval from $0$ to $\frac{1}{4} \pi$. Hence, from the continuity of the sine and cosine both in this (elliptic) space, and in the Euclidean space it follows that part (1) of the theorem holds for all values between zero and $\frac{1}{4} \pi$.

We have been assuming so far that we had the construction of Figure 12. However, the angle $AOB$ has the same measure as the angle $BOA$. From what we have just proved, it follows that the trigonometric functions of these two angles are the same. We have thus proved the theorem. The theorem we have just proved allows us to accept without question for this geometry the results of §§ 22, 25, 26, 39, to 45, Young. Also since the sine and cosine are the same functions respectively as the sine and cosine of angles of the same measure in Euclidean geometry, the formulas at the top of page 275 follow without further argument.

**Theorem 91.** See notation at end of Theorem 84. If $XYZ$ is a variable triangle such that the angle $Z$ approaches a right angle, and such that the measure of angle $X$ approaches the measure of the acute angle $BOA$, as $g$, the greatest side of the triangle $XYZ$, approaches zero, then $\frac{XY}{XZ}$ approaches the sine of $BOA$ which is the same as the sine of $AOB$, and the ratio $\frac{YZ}{XY}$ approaches the cosine of $BOA$ which is the same as the cosine of angle $AOB$. The

1) See note at bottom of page 254 of Young.
approach need not be strict in the sense of the note following Theorem 84.

Proof. We shall assume that $g$ is sufficiently small so that angle $Y$ is acute. We shall assume that $Z$ is constant and is a right angle. In the notation of the note at the end of Theorem 84, for a certain subsequence of values of $g$ the ray $Ox$ is on the $A$ side of $OB$ and the angle $xOB$ approaches the angle $A0B$ in the strict sense; we have proved before that the sine ratios and the cosine ratios approach the sine and cosine of $A0B$ (over this sequence). For a sequence where $x$ is on the $A_1$ side of $OB$ we get in the same way the sine of $A_10B$ and the cosine of $A_1OB$. But $A0B$ and $A_10B$ have the same measure. Hence by Theorem 90 the first case follows.

In case angle $Z$ is not a right angle, let $W$ be a point on the ray $YZ$ such that $ZWX$ is a right angle. We may assume that $YX$ is less than a quadrant, that angle $X$ is acute, and that $g$, the greatest side of either of the triangles $XYZ$ and $YWX$ is less than a quadrant. As $g$ approaches zero, $XZY$ and $XZW$ approaches right angles and hence, since angle $YWX$ is a right angle, it follows from the triangle $ZXW$ that angle $XZW$ approaches zero $XZ = XZ - XW = XW - XY$. By the first case it follows that the first expression on the right of the equality has unity as limit, while the second approaches $A0B$. Since $Z$ and $W$ are both points of the ray $YZ$ it follows that either $Z$ is a point of the interval $YW$ or that $W$ is a point of the interval $YZ$. Hence $ZY$ is either the sum or the difference of $YW$ and $ZW$. We shall consider the second case. $XY = XW - XY$. It can easily be proved by case one and previous theorems that this ratio has a limit $A0B$.

Theorem 92. If $ABC$ is an acute angle then there exists on the ray $BC$ a point $D$ such that the measures of the angles $DAB$ and $ADB$ have the same numerical value.

We leave the proof of this theorem to the reader.

We are now ready to treat the questions of the length of circles and related topics. Let $A0A'$ be an acute angle, and let $OA$ and $OA'$ be congruent and less than quartant. By Theorems 88 and 76 there exists on the ray $OA$ a point $B$ and on the ray $OA'$ a point $B'$ such that $BA'O$ and $B'A0$ are right angles, such that all the sides of the triangles $BA'O$ and $B'A0$ are less than a quadrant, and that the segments $BA'$ and $AB'$ are within the angle $A0A'$. By Theorem 82 both $OB$ and $OB'$ are greater than $OA$. It follows easily that the segments $AB'$ and $A'B$ have in common exactly one point $D$. Let $E$ be any point of $OA$, which is within the angle $A0A'$. By Theorem 76 the segment $OE$ contains exactly one point $F$ on the segment $AA'$, one point $G$ on the segment $A'B'$, and one point $J$ on $A'B$. By Theorems 78 and 83, $OE$ is greater than $OF$ but is less than the smaller of $OG$ and $OJ$. Let $K$ be a great circle which is perpendicular to $OE$ at $E$. By the method we used in proving that $AA'$ and $A'B'$ intersect in a point $D$ we can prove, (since $BOE$ and $EOA'$ are acute angles), that $K$ contains a point $L$ of the segment $AG$ and a point $L$ of the segment $JA'$ of the interval $JA'$. We wish to prove next that $L$ is a point of the segment $A'D$. Suppose that it is not. The point $L$ cannot be $A'$, since $K$ is tangent to $C_{OA}$ at $E$. Hence, since the intervals $JA'$ and $DA'$ are both subsets of the interval $A'B$, $L$ is either $D$ or a point of $A'J$ in the order $A'DL$. By Theorem 78 $OJ$ is less than $OB$ which is less than a quadrant. Since $LJOA'$ is acute, it follows by Theorem 82 that $JA'$ and hence $JD$ is less than a quadrant and that $OJA'$ is acute. By the same theorem it follows that $EL$ is less than a quadrant. Since $E$ and $L$ are within the angle $A0A'$, the interval $EL$ contains no point of $OA$ or for $OA'$. Since $OF$ is less than $OE$ which is less than $OG$ it follows that $E$ is a point of the segment $FG$. By Theorem 76 $E$ is then within the angle $GAF$ which is the same as the angle $DA'A'$. In the same way we can prove that $E$ is within the angle $DAA'$ but from this it follows that $E$ is on the $A$ side of $A'D$ and on the $A'$ side of $AD$, and hence is within the angle $DA'A'$ and hence within the triangle $DA'A'$. If $L$ is a point distinct from $D$ in the order $A'DL$ it follows that $L$ is on the opposite side of $AD$ from $E$. Hence the interval $EL$ contains a point $Z$ of the great circle passing through $A$ and $D$. Since $EL$ is entirely within the angle $A0A'$, and since by Theorem 76, the interval $AA'$ is the only part of the great circle passing through $A$ and $D$ which is a subset of the lines $AOA'$, $Z$ is a point of the interval $A'B'$. Hence whether $D$ is $L$ of whether $D$ is a point in the order $A'DL$ the interval $EL$ contains a point $Z$ of the interval $AB'$ and also contains the point $L$ of the interval $A'B$. Let $L$ and $E$ be the poles of $L$ and $E$ respecti-
vely. The point $E$ is without the triangle $A\Delta A'$. (Theorem 60)
Hence the semicircle $E \ell E$ contains a point $V$ of the triangle.
But the point $F$ of the segment $AA'$ is within $C_{\Delta A}$ ($OF$ is less
than $OA$ by Theorem 85); by Theorem 78, since $OE$ is perpendicular
to $K$ at $E$, if $Q$ is any point of $K$ distinct from $E$, $OQ$ is
greater than $OE$. Hence the side $AA'$ contains no point in common
with $K$. Hence $V$ must be a point of one of the sides $AD$ and $A'D$.
But then one of the intervals $AB'$ and $A'B$ must contain two points
of $K$. This leads to a contradiction. Hence $L$ is a point of the
segment $A'D$ and similarly $I$ is a point of the segment $AD$.

By Theorem 86 angle $AA'O$ is acute. Since angle $AOA'$ is
also acute it is easily proved that there exists on the rays $OA'$
and $A'O$ (and hence on the interval $OA'$) a point $W$ such that $AWO$
is a right angle and that $AW$ is less than a quadrant and is also
less than $AA'$. Also, in the right triangle $DAB$, $A'B$ and $BD$ are
both less than a quadrant, and hence by Theorem 82 $AD$ is less
than $DB$. With the help of Theorem 89 we can now prove the following
inequality

\[ AW < AA' < AE + E' \ell I + I'L + LA' < AD + DA' < A'B. \]

From inequality (1) we get in the notation of Young, Figure 10,
(assuming $AK$ and $A'K$ to be tangent to $C_{\Delta A}$)

\[ A'A < AB + A'B < AK + A'K < AT < \frac{\pi}{n}. \]

Hence we can draw the conclusion of § 30, since by the inequality (1)
above it follows that the inscribed polygons have an upper bound,
and that the circumscribed polygons have a lower bound which is
not less than the upper bound of the inscribed polygons. Also if
we add new vertices to any inscribed polygon we increase the
sum, while if we add vertices to a circumscribed polygon we decrease
the sum.

From inequality (1) above it follows that

\[ 1 > \frac{n (A'A)}{n (A'D + AD)} > \frac{(AA')}{(A'B)}. \]

Angles $AA'O$ and $A'AO$ are acute. Hence, from the triangle
$AOA'$ it follows that when angle $AOA'$ approaches zero the
angles $AA'O$, $A'A'B$, $A'AO$, $A'AB$, must approach right angles.

Also, the associated interval of angle $AOA'$ approaches zero with
this angle; by Theorem 79 it follows that $BA'$ is less than this
associated interval also angle $BA'B'$ is a right angle. Hence angle
$AA'B$ approaches zero and we have by Theorem 91 that $AA'$
has cosine zero, that is unity, for a limit as angle $AOA'$ approaches
zero. Now we can follow §§ 33, 34, 35 and the definition of length
of circles in § 36.

From inequality (1) we can prove that

\[ AW < \text{arc} AA' < A'B. \]

But when $OA'$ approaches zero $OB$ approaches zero, given that
angle $AOA'$ is fixed. Hence if we divide, the above inequality (3)
by $OA$, and denote by $L$ the limit $\frac{\text{arc} AA'}{OA}$ as $OA$
approaches zero it follows that if this limit exists and we let $x$ be the measure
of angle $AOA'$, then

\[ \sin x \leq L \leq \tan x. \]

Even if this limit does not exists, if $L$ is any number between
the upper and lower limits of $\frac{\text{arc} AA'}{OA}$ as $OA$ approaches zero,
inequality (4) will still be true. Let $C$ and $C'$ be points on the
rays $OA$ and $OA'$ respectively such that $OC$ and $OC'$ are
quadrant. Then the interval $CC'$ is the associated interval of the
angle $AOA'$. Let $C''$ be the midpoint of this interval. Then by
previous theorems $C''$ and the ray $OC$ will be within angle $AOA'$
and this ray contains exactly one point $A''$ of $C_{\Delta A}$. There is a
rotation about $O$ that transforms $C$ into $C''$ and transforms $C''$ into $C$.
Hence angle $AOA'$ is congruent to $AA''OA'$. It is easily shown
that each of these angles has a measure $\frac{1}{4} x$. Also (see paragraph 37,
Young), the lengths of the arcs $AA''$ and $AA'''$ of $C_{\Delta A}$ are the
same and are each equal to a half of the length of the arc $AA''$ of
this circle. Hence by an argument like that used in the proof
of inequality (4) we can prove that $2 \sin \frac{1}{4} x \leq L \leq 2 \tan \frac{1}{4} x$. By
a repetition of this argument it follows that $4 \sin \frac{1}{2} x \leq L \leq 4 \tan \frac{1}{2} x$.
With the use of mathematical induction we can prove that

\[ \text{We are referring to those arcs which are within the angle } AOA'. \]
But it follows from what we have proved, including inequality (2), that \( n \cdot KY \) approaches zero and similarly that \( n \cdot YH \) has the same limit. It follows from inequality (1) that

\[
\lim_{\text{as } n \to \infty} S(B, n) = L_y. \tag{3}
\]

With the help of this result and from the continuity of the sine ratio of an angle, we can prove, by arguments largely similar to those used by Young in \( \S 48 \), the continuity of \( f(r) \). Using the notation of \( \S 49 \), we can show by the argument he there uses, the following inequality:

\[
S(C, n) - S(B, n) > S(A, n) - S(C, n). \tag{4}
\]

If we now proceed to the limit, we can make the statements which are given on page 276, Young, provided we reverse the inequality signs throughout. From inequality (1) in our treatment of the lengths of circles it follows that the length of that part of \( C \alpha \) which is within the angle \( BON \) is less than the interval \( BN \) (assuming that \( OB \) is less than a quadrant and that angle \( BON \) is acute, see Figure 13). From this it is easily proved

\[
\lim_{\tau \to 0} f(r) = 0. \quad \text{Let } f(0) = 0.
\]

From this it follows by \( \S 37 \) and by definition of \( f(r) \), that

\[
f'(0) = \lim_{\tau \to 0} \frac{f(r) - f(0)}{r - 0} = 1.
\]

But the argument we have given in the preceding sections to establish inequality (4) is valid if \( B \) is the point \( O \). Since in this treatment the inequalities are reversed it follows from the argument given in Young that the incremental ratio of \( f(r) \) is never greater than unity or less than zero for \( r \) less than or equal to \( \frac{1}{4} \alpha \). It follows from his argument that the derivative of \( f(r) \) exists for all values of \( r \) and is never greater than unity or less than zero, for \( r \) from \( 0 \) to \( \frac{1}{4} \alpha \) inclusive.

\( \S \S 52-55 \). In the figure on page 277, Young let \( ABC \) be a right triangle that satisfies the hypothesis of Theorem 82. Let \( P \)
be that pole of $BC$ which is on the $A$ side of $BC$. Then by arguments we have used before it follows that all the triangle $ABC$ plus its interior with the exception of the side $BC$ is on the $A$ side and $P$ side of $BC$, and every quadrant from $P$ to a point of $BC$ contains a point of the side $BA$. By hypothesis, the point $C_1$ is the midpoint of the side $BC$. Hence there is a rotation about $P$ that transforms $B$ into $C_1$ and transforms $C_1$ into $C$, and that transforms the interval $BC_1$ into the interval $C_1C$. By Theorem 64 this rotation transforms $C_1$, the midpoint of the interval $BC_1$, into $D_1$, the midpoint of the interval $C_1C$. Hence it transforms the quadrant $PA_1C_1$ into the quadrant $PQ_1D_1$, the point $A_1$ into the point $Q_1$, hence the triangles $A_4C_1B$ and $Q_1D_1C_1$ are congruent triangles and have the same angle sum. Since $D_1$ is the midpoint of $C_1D_1C$, we can prove in the same way that there is a rotation about $P$ which transforms $C_1$ into $D_1$, which transforms $D_1$ into $C$, and which transforms $Q_1$ into a point $E_1$ on the quadrant $PAC$. Let $d_1$ be the angle excess of each of the three congruent triangles $A_4B_1C_1$, $Q_1D_1C_1$, and $E_1D_1C$. Then we can write inequalities (1) and (2), page 277, Young, as Young does, provided we replace $\tan P_1BC$ by $\frac{P_1C}{BC}$. By the method we have just used it is easily proved that we can construct $2^{n-1}$ triangles between $C_1$ and $C$ which are congruent to the triangle $A_4BC$. Hence we can write as Young does

$$2^{n-1} d_n < d_1,'$$

$$\frac{P_1C}{BC} < \frac{A_4C_1}{(2^{n-1} + 1)BC_1}.$$

From the second of these inequalities it follows from the argument given by Young that $\frac{P_1C}{BC}$ approaches zero. Let $H_1K_1$ be the associated interval of the angle $P_1BC$. From the monotone character of the tangent ratio it follows that $\frac{P_1C}{BC}$ is greater than $H_1K_1$, divided by $4\pi$. Hence the interval $H_1K_1$, and the angle of which it is the measure approaches zero as $n$ approaches infinity.

---

Concerning the triangle with equal base angles.

Let $AA'$ be an interval less than a quadrant and let $M$ and $N$ be points on the same side of $AA'$ such that angle $MAA'$ has the same measure as $NAA'$ and let both angles be acute. Let $Q$ be that pole of $AA'$ which is on the same side of $AA'$ as $M$ and $N$. With the help of Theorem 76, parts (3) and (4) it follows that the rays $AM$ and $AN$ intersect in a point $P$ such that all of the triangle $PAAA'$ except the side $AA'$ is in the interior of the rectangular triangle $QAA'$. Let $PA = x$, $PA' = y$, and $AA' = a$. Let the measure of the angles $PAA'$ and $PA'A$ be $X_A$ and let the measure of $APA'$ be $Y$. The measure of angle $QA'A$ is $\alpha$, since $AA'$ is the associated interval of this angle. The angle sum of triangle $AQA'$
is greater than that of \(APA'\) since the second triangle is contained within the first. Hence

\[
\pi + \alpha > Y + 2X > \pi \\
\pi + \alpha - 2X > Y > \pi - 2X.
\]

Let \(X_1\) denote a number such that

\[
0 < X_1 < X.
\]

There exists a point \(P_1\) such that each of the angles \(P_1AA'\) and \(P_1A'A\) is of measure \(X_1\), and the triangle \(P_1AA'\), except for the interval \(AA'\), is within the triangle \(APA'\). From inequality (1) we have

\[
\pi + \alpha - 2X_1 > \pi + \alpha - 2X > Y.
\]

**Theorem 93.** Consider a variable \(w\) and for each value \(w\) of \(w\) a pair of fixed points \(A\) and \(A'\), and a set \([APA']_w\) of triangles such that (1) the interiors of all the triangles of the set \([APA']_w\) are on the same side of \(AA'\); (2) for each triangle \(APA'\) of the set \([APA']_w\), the angles \(PA'A\) and \(PA'A\) have the same measure \(X\); (3) \(AA'\) is less than a quadrant; (4) the upper bound, for the set of triangles \([APA']_w\), of \(X\) is \(\frac{1}{2}\pi\) and the lower bound is a number \(X_0\); (5) \(\lim X_0 = \frac{1}{2}\pi\) and \(\lim AA' = 0\). Then, if \(\varepsilon\) is a positive number, there exists a positive number \(\delta\) such that for any triangle \(APA'\) in the set \([APA']_w\),

\[
\frac{|PA - PA'|}{AA'} < \varepsilon, \text{ for all } w \text{'s such that } |w - \varepsilon| < \delta.
\]

**Proof:** The variable triangle \(APA'\) is a function of the variable \(X\); the range of \(X\) is determined by the variable \(w\); \(X_0\) and \(AA'\) are single-valued functions of \(w\). Consider, during that part of this argument which precedes inequality (5), a fixed value of \(w\) for which \(\frac{1}{2}\pi > \pi + \alpha - 2X_0\), and the corresponding set \([APA']_w\) of triangles. Let the measure of angle \(APA'\) be \(Y\), and let the lengths of the intervals \(AP, AP', AA'\) and \(AA'\) respectively be \(x, y, a\). Let \(Q\) be that pole of \(AA'\) which is on the \(P_1\) side of \(AA'\), where \(P_1\) is the vertex \(P\) of that triangle \(APA'\) of the set \([APA']_w\), whose base angle \(PA'A\) and \(PA'A\) have the measure \(X_0\). For each triangle \(APA'\) let \(H\) denote a point on the ray \(PA\) and \(K\) a point on the ray \(PA\) such that \(PAH\) and \(PAK\) be right angles. It follows from the statement of the theorem and the paragraph preceding the statement of the theorem that when \(X\) is greater than \(X_0\) the ray \(A'P\) falls without the angle \(PA\) and hence the ray \(A'H\) falls within the angle \(A'\). Also, the ray \(A'P\) falls without the angle \(PA\) and within the angle \(QQ'\). Since the angles \(PAH\) and \(PAH\) are right angles, it follows that the ray \(AH\) falls within the angle \(A'H\). It can easily be shown that the point \(P\) is within the triangle \(A'H\) and hence that we have on the ray \(A'H\) and on the segment \(AH\) a point \(Q\), such that the interval \(A'H\) is less than the interval \(A'Q\). From the triangle \(A'H\), with angle \(A'Q_0\) greater than a right angle, it can be shown by Theorem 87 that angle \(A'Q_0A\) is an acute angle and angle \(A'Q_0H\) is greater than a right angle. By inequality (3) preceding the statement of the theorem and the choice of \(w\) we have made, \(Y_0\) is less than a right angle. By Theorem 83 \(AH\) and \(A'P\) are each less than a quadrant. By Theorem 88 triangle \(PAH\) satisfies the hypothesis of Theorem 82. By Theorem 87 \(A'Q_0 < A'H\) and hence

\[
A'H < A'H, \text{ and similarly } AK < AK_0.
\]

It is easily seen from the preceding inequalities and the argument that follows that the case where \(A'P\) is less than \(A\) can be handled in the same way as the case where \(AP\) is less than \(A'P\); so we shall consider only the first of these two cases. We have then

\[
0 \leq |x - y| = AP - A'P < PH - A'P = A'H = A'H_0.
\]

Divide every element in (5) by \(AA'\) and let \(w\) approach \(k\). Since \(X_0\) approaches \(\frac{1}{2}\pi\), angle \(A'\) approaches zero, and angle \(A'H\) approaches a right angle. Since \(AA'\) also approaches zero, we can show by methods we have used before, (involving in particular Theorem 91), that \(A'H\) and \(A'H\) approach zero. Then by Theorem 91 \(A'H\) approaches the tangent of zero, that is zero. We have thus established the conclusion of the theorem.

We shall now derive a differential equation involving the sides and angles of the triangles discussed above when the interval \(AA'\) is kept fixed.

For a fixed value \(w\) of the variable \(w\) discussed in Theorem 93 the triangles \(APA'\) of the set \([APA']_w\) are functions of \(X\) alone; where \(X_0\), as indicated before, is a variable whose range \([X_0]\) is
determined by \( \bar{w} \). Consider then, during the argument preceding Theorem 94, the range \([X]_1\) and the set \([APA']_1\) for the fixed \( \bar{w} \) just mentioned. Let \( APA' \) and \( AP_1 A' \) be two triangles of the set \([APA']_1\); such that angle \( PAA' \) is greater than \( P_1A_1A' \); that is, \( X \) is greater than \( X_1 \); also let \( \pi + a - 2X_1 < \frac{\pi}{2} \). On the ray \( AP \) let \( E \) be a point such that \( AE = AP_1 = x_1 \), and on the ray \( A'P \) let \( A'F = A'P_1 = y_1 \). In the order \( A'P, P, T \) let \( T \) be a point common to the ray \( A'P_1 \) and the interval \( AP \). By Theorem 87 \( AATA' \) is an acute angle and \( Y_1 > \angle ATA' > Y_1 \). By Theorem 87 \( AP_1 < \frac{\pi}{2} AP \). Similarly \( A'P_1 < A'P \).

On the ray \( FP_1 \), there exists in the order \( FP_1 G \) a point \( G \) that belongs also to the ray \( FA \). It follows by the statement made at the foot of page 201, Young, above the footnote, that as \( X \to X_1 \), approaches zero the angle \( Y \to Y_1 \) approaches zero. By Theorem 86 the following angles will under the same circumstances approach right angles: \( AEp_1, PEP_1, EPP_1, A'P_1, F, A'FP_1, P, FP \). From the triangle \( PFG \) it follows that the angle \( PFG \) approaches the value \( \frac{\pi}{2} - Y_1 \) and from the triangle \( GP_1E \) it follows that the angle \( GP_1E \) approaches the value \( Y_1 \).

By § 57, Young, we have that except for infinitesimals of order higher than the second, \( P_1F = (X - X_1) \cdot f(y_1) \) and \( P_1E = (X - X_1) \cdot f(x_1) \).

Hence

\[
\lim_{x \to x_1} \frac{P_1F}{P_1E} = \frac{f(y_1)}{f(x_1)}.
\]

Also as \( X \to X_1 \), \( EP_1 \) and \( FP_1 \) approach zero. With the help of Theorem 91 we can show from the triangle \( EGP_1 \), that \( GP_1 \) and \( EG \) also approach zero. It follows that \( GF = GP_1 + \frac{\pi}{2} P_1F \) approaches zero. By an argument like that just given, and involving the triangle \( GPP \) we can show that the intervals \( GP \) and \( PF \) also approach zero. It follows by Theorem 91 that \( \frac{GP_1}{EP_1} \) approaches \( \frac{\pi}{2} \), and \( \frac{GP_1}{EP_1} \) approaches \( \sec (Y_1) \).

But

\[
\frac{GF}{FP} = \frac{GP_1 + P_1F}{FP} = \frac{P_1F}{FP} \left( \frac{GP_1}{EP_1} \right) = \frac{\frac{GP_1}{EP_1}}{\frac{P_1F}{FP}} \left( \frac{GP_1}{EP_1} \right).
\]

From this we get that except for infinitesimals of order higher than the first

\[
\frac{P_1F}{FP} = \frac{X - X_1}{y - y_1} \cdot f(y_1) \quad \frac{GF}{FP} = \frac{X - X_1}{y - y_1} \cdot \frac{f(y_1)}{1 + \frac{GP_1}{EP_1}} \cdot \frac{EP_1}{P_1F}.
\]

Now let one of the angles \( X - X_1 \) remain fixed, and let \( X - X_1 \) approach zero. Dropping subscripts, we get

\[
\frac{dy}{dX} = f(y) + \sec Y \cdot f(x)
\]

Similarly

\[
\frac{dx}{dX} = f(x) + \sec Y \cdot f(y).
\]

Subtracting the foregoing equations and simplifying, we get

\[
\frac{d}{dX} (y - x) = (f(x) - f(y)) \cdot \tan \frac{Y}{2}.
\]

**Theorem 94.** Hypothesis of Theorem 93. If \( X_1 \) and \( AA' \) are of the same order, the limit

\[
\lim_{x \to x_1} \left\{ \frac{[APA]_1 - [APA']_1}{[AA]_1\pi^2} \right\} = 0.
\]

**Proof.** Differential equation \( (2) \) shows that, for \( w \) fixed, \( x \) and \( y \) are continuous functions of \( X \) for \( X_1 \leq X \leq \frac{\pi}{2} \). Whenever \( \pi + a - 2X_1 < \frac{\pi}{2} \), we have then by differential equation \( (3) \), that if we keep \( w \) fixed and allow \( X \) to vary

\[
y_1 - y = \int_{x_1}^{\pi} \left( f(x) - f(x) \right) \cdot \tan \frac{Y}{2} \cdot dX.
\]

For values of \( X \) such that \( x = y \), the integrand is zero. For values of \( X \) such that \( x = \frac{\pi}{2} \), we can write the integrand in the preceding equation as follows:

\[
\int_{x_1}^{\pi} \left( f(x) - f(x) \right) \cdot \left( \frac{y - x}{a} \right) \cdot \left( \tan \frac{Y}{2} \right) \cdot a \cdot dX.
\]

But the incremental ratio of \( f(x) \) is never greater than unity. If \( \varepsilon \) is a positive number, there exists by Theorem 93 a positive
δ, such that whenever |w − k| < δ, then for any triangle APA' in the set \( \{APA'\} \), 
\[
\frac{|AP - A'P|}{AA'} = \frac{|x - y|}{a} < \epsilon.
\] It follows from the proof preceding this theorem that \( Y \leq Y_q \). Since \( Y_q \) and \( AA' = a \) are of the same order, it follows that \( \tan \frac{Y}{a} \) has an upper bound \( M \).

Hence we have the following:
\[
\frac{|x_q - y_q|}{a^2} < \epsilon \cdot M \cdot (\frac{1}{4}a - X_q).
\]

But \( M \) is fixed, and \( \epsilon \) and \( \frac{1}{4}a - X_q \) may be made as small as we please. Hence the conclusion of the theorem follows.

§ 59. Young. Make the following modifications is Young's treatment. Let \( Q' \) be a point on the ray \( OP' \) such that angle \( OPQ' \) and angle \( OQP \) have the same measure \( \chi \) and let \( Q \) be a point on the ray \( OP \) such that angles \( OP Q \) and \( OQP' \) have the same measure. Let \( OAP \) be a triangle satisfying the hypothesis of Theorem B2.

Let \( K \) be a point on the ray \( OQ' \) such that \( OK \) is congruent to \( OP \), and let \( L \) be a point on the ray \( OP \) such that \( OP' \) is congruent to \( OL \). By Theorems 86 and 84 the angles \( OQP' \), \( OQP \), \( OKP \), and \( OPK \) approach a right angle as \( PP' \) and \( P'OP \) approach zero. Then angle \( Q'PK \) approaches zero, and we have by Theorem 91 that \( Q'P \) \( KP \) approaches unity. It follows by Young's argument that \( KP \) is of the same order as \( \Delta \Phi \). Hence we have by Theorem 94, \( Q'K \) is of order higher than the second with respect to \( Q'P \). But \( \Delta Y = KP' \) which differs from \( Q'P \) by \( Q'K \). Similarly, \( \Delta Y \) differs from \( PQ \) by infinitesimals of order higher than the second.

Hence if we replace \( \Delta Y \) by \( Q'P \) or by \( PQ \), the result, in the limit, in the difference equation set up by Young, will be the same. Also we may replace \( QP \) by \( \Delta \Phi - f(y) \), and \( P'Q \) by \( \Delta \Phi - f(y + \Delta y) \). With these modifications we can proceed as at the foot of page 283. We can proceed as Young does in solving Killing's equation on pages 284 and 285. His argument at the top of page 286 shows that \( f(x) = k \cdot \sin \frac{z}{k} \). These formulas are sufficient to establish the categoricity of the space \( S \).

Independence examples.

We give here a set of independence examples similar to those given in L. R. H. H., and shall make the following changes 1) in the examples there given. Let \( S_1 \) and \( S_1 \) be the surface of a sphere of radius 10. Let \( S_1 \) be the set of rational points belonging to a circle and the regions for this space be segments of this set. All the spaces \( S_1 \) to \( S_{12} \) are the same as \( S_1 \). In the case of \( S_{10} \) let \( P \) and \( P' \) be a pair of opposite points and let \( C \) be the great circle having these centers. Let the symbol \( y \) indicate the ordinate of the point \( Y \) of \( S_{10} \); that is, \( y = 0 \) if \( Y \) is a point of \( C \); if \( Y \) is not on \( C \), \( y \) is numerically equal to the length of the interval \( YH \), not greater than a quadrant, of a great circle containing \( Y \) and perpendicular to \( C \); further, \( y \) is positive or negative according as \( Y \) is on the \( P \) or the \( P' \) side of \( C \). Let \( M \) be a continuous, single-valued transformation of \( S_{10} \) into itself such that \( y' = \frac{y}{25\pi^2} \), where \( y' \) is the ordinate of \( \bar{M}(Y) \) and such that \( \bar{M}(H) = H \). In \( E_13 \) let a motion be a one to one continuous transformation of \( S_{14} \) into itself that transforms great circles into great circles an preserves distances.


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