On a problem of W. L. Ayres.

By

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In the paper *Concerning continuous curves of certain types*, (this volume) W. L. Ayres raises the question as to whether or not every connected subset of a continuous curve M having the property that for any positive number ε, M contains not more than a finite number of simple closed curves of diameter greater than ε, is arcwise connected. It is the purpose of the present paper to show that the answer to this question is affirmative.

In establishing the theorem below use will be made of the decomposition of a continuous curve into its *cyclic elements*, an extensive theory of which has been given by the author in a number of recent articles [1]). In particular the properties of the *maximal cyclic curves* of a continuous curve will be used.

Definitions. A continuous curve M is said to be *cyclicly connected* provided that every two points of M lie together on some simple closed curve which is a subset of M. The continuous curve C is said to be a *maximal cyclic curve* of a continuous curve M provided that C is cyclicly connected and is a subset of M but is not a proper subset of any cyclicly connected continuous curve which is also a subset of M. These definitions will be found in my papers just cited.

[1]) G. T. Whyburn, *Cyclicly connected continuous curves*. Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31—38; *Some properties of continuous curves*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 305—308; *Concerning the structure of a continuous curve*, presented to the American Mathematical Society December 31, 1926, offered to the American Journal of Mathematics, but not yet published.

In my paper *Concerning the structure of a continuous curve* I proved the following two propositions which are essential in the proof given below for Theorem I.

Theorem A. *If H is any connected subset of a continuous curve M and C is any maximal cyclic curve of M, then $H.C$ is either vacuous or connected.*

Theorem B. *In order that every connected subset of a continuous curve M should be arcwise connected, it is necessary and sufficient that every connected subset of each maximal cyclic curve of M should be arcwise connected.*

The proof that the condition of Theorem B is sufficient may be summarized as follows. Let H be any connected subset of a continuous curve M which satisfies the condition, and let A and B be any two points of H. Let t be any arc in M from A to B and K the set of all those points of M which separate [1]) A and B in M. By a theorem of the author's [2]), the set of points $K+A+B$ is closed.

Clearly $K+A+B$ is a subset both of t and of H. Hence $t-(K+A+B)$ is the sum of a countable number of maximal segments $S_1, S_2, S_3, \ldots$ For each positive integer i, M contains [3]) a maximal cyclic curve C_i containing S_i. Let A_i and B_i denote the endpoints of S_i. Now by Theorem A, for each i, the set of points $H.C_i$ is connected; and by hypothesis it follows that for each i, $H.C_i$ is arcwise connected. Then since, for each i, $H.C_i$ contains A_i and B_i, therefore $H\,C_i$ contains an arc t_i from A_i to B_i. Then the set of points

$$T=K+A+B+\sum_{i=1,2,3,\ldots} t_i$$

is a subset of H and, using the properties of the maximal cyclic curves of a continuous curve established in my paper *Cyclicly connected continuous curves* (loc. cit), it is not difficult to show that T is a simple continuous arc from A to B. Hence H is arcwise connected.

Theorem 1. *If for every positive number ε, the continuous curve M contains not more than a finite number of simple closed curves of diameter greater than ε, then every connected subset of M is arcwise connected.*

[1]) The point X of M is said to separate the two points A and B in M provided that $M-X$ is the sum of two mutually separated point sets containing A and B respectively.

[2]) *Some properties of continuous curves*, loc. cit., Theorem 1.

[3]) G. T. Whyburn, loc. cit., Theorem 2.

Proof. Let C be any maximal cyclic curve of M, and A and B any two points of C. Since C is cyclicly connected, therefore it contains a simple closed curve J containing both A and B. The set of points $C-J$ consists of not more than a finite number of maximal connected subsets. For suppose, on the contrary, that there are infinitely many maximal connected subsets $R_1, R_2, R_3, \ldots$ of $C-J$. Then J contains a point P which is a limit point of the set of points $P_1+P_2+P_3+\ldots$, where for each integer $i>0$, P_i is a point belonging to R_i. Let X be a point of $J-P$ and let $2d$ be the distance from X to P. Then since, by a theorem of W. L. Ayres [1]), there are not more than a finite number of the sets R_i of diameter greater than any preassigned positive number, it readily follows that there exists an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$, such that for each i, every point of the set R_{n_i} is at a distance less than d from P. By a theorem of the author's [2]). for each i, J contains at least two limit points of R_{n_i}. It easily follows with the aid of a theorem of R. L Moore's [3]) that for each i, $\overline{R}_{n_i}$ contains an arc t_i whose endpoints A_i and B_i belong to J, but such that $t_i-(A_i+B_i)$ belongs to R_{n_i}. Then for every positive integer i, the simple closed curve J_i formed by the arc t_i plus the arc A_iXB_i of J belongs to C and is of diameter greater than d. But this is impossible since no two of the curves $[J_i]$ are identical and by hypothesis M cannot contain more than a finite number of simple closed curves of diameter greater than d. Thus it follows that $C-J$ consists of not more than a finite number of maximal connected subsets. If there are any such sets, let them be ordered $R_1, R_2, R_3, \ldots, R_{n_1}$.

Let i be any positive integer $\leqslant n_1$. Then J contains only a finite number of limit points of R_i. For suppose it contains infinitely many such points. Then if AXB and AYB denote the two arcs of J from A to B, one of them, say AXB, must contain an infinite set D of limit points of R_i. Let E be a point of D. Then

1) W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, vol. 28 (1927), pp. 396—418, Theorem 1.

2) *Cyclicly connected continuous curves*, loc. cit., Theorem 9.

3) R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254—260, Theorem 1.

with the aid of an accessibility theorem of R. L. Wilder's [1]) is readily seen that $\overline{R}_i$ contains an infinite sequence of arcs $[EX_nF_n]$, $n=1,2,3,\ldots$, no two of which are identical and where for every positive integer n, the point F_n belongs to D but $EX_nF_n-(E+F_n)$ belongs to R_i. Then for each positive integer n, the simple closed curve J_n formed by the arc $EAYBF_n$ of J plus the arc EX_nF_n belongs to C and is of diameter $\geq$ the distance from A to B. But this is impossible, since no two of the curves $[J_n]$ are identical and by hypothesis M contains not more than a finite number of simple closed curves of diameter greater than any preassigned positive number. Thus the supposition that J contains infinitely many limit points of R_i leads to a contradiction.

Now since there are just a finite number of the sets $[R_i]$, and since J contains just a finite number of limit points of each of these sets, it follows that there exists an arc UOV which is a subset of J and which contains no limit point of $R_1+R_2+R_3+\ldots+R_{n_1}$.

For each positive integer $i\leq n_1$, it was seen above that J contains at least two limit points of R_i. Then with the aid of R. L. Moore's theorem cited above it readily follows that for each $i\leq n_1$, $\overline{R}_i$ contains an arc t_{1i} whose endpoints A_{1i} and B_{1i} are on J but such that $t_{1i}-(A_{1i}+B_{1i})$ belongs to R_i. Let K_1 denote the set of points $J+\sum_{i=1}^{n_1} t_{1i}$. Then just as in the case of $C-J$, it follows that $C-K_1$ is the sum of at most a finite number of maximal connected subsets $S_1, S_2, S_3, \ldots, S_{n_2}$; and just as above it is shown that for every $i\leq n_2$, $\overline{S}_i$ contains an arc t_{2i} whose endpoints A_{2i} and B_{2i} are on K_1 but such that $t_{2i}-(A_{2i}+B_{2i})$ belongs to S_i. Let K_2 denote the set of points $K_1+\sum_{i=1}^{n_2} t_{2i}$. Then $M-K_2$ is the sum of at most a finite number of maximal connected subsets $T_1, T_2, T_3, \ldots, T_{n_3}$. And each set T_i contains an arc t_{3i} whose endpoints A_{3i} and B_{3i} belong to K_2 but such that $t_{3i}-(A_{3i}+B_{3i})$ belongs to T_i. Let this process be continued. This process must terminate after a finite number of steps. For suppose it does not. Then $\sum_{n=1}^{\infty}\sum_{i=1}^{n_n} t_{ni}$ contains infinitely many different arcs $[t_{ni}]$. But each arc t_{ni} plus the set K_{n-1}

1) R. L. Wilder, *Concerning continuous curves*, this journal, vol. 7, (1925), pp. 340—377, Theorem 1.

(where $K_0 = J$) contains a simple closed curve J_{ni} containing both t_{ni} and the arc UOV. And if t_{ab} and t_{xy} are distinct arcs of the collection $[t_{ni}]$, $a < x$, then since J_{ab} is a subset of $K_{a-1} + t_{ab}$ and J_{xy} is not a subset of this set of points (since $a < x$), therefore J_{ab} and J_{xy} are different simple closed curves. But each of simple closed curves J_{ni} belongs to C and contains UOV. This is contrary to the hypothesis that M contains at most a finite number of simple closed curves of diameter greater than any preassigned positive number. Hence it follows that the above process must terminate after a finite number of steps, i. e., there exists a positive integer k such that

$$J + \sum_{n=1}^{k} \sum_{i=1}^{n_n} t_{ni} = C.$$

Let K denote the set of points $A + B + \sum_{n=1}^{k} \sum_{i=1}^{n_n} (A_{ni} + B_{ni})$.

Then $C - K$ consists of a finite number of its maximal connected subsets $s_1, s_2, s_3, \ldots, s_n$. It is easy to see that each of these sets s_i is a segment of an arc t_i whose endpoints belong to K, i. e., t_i is the segment s_i plus its two endpoints. Now if N is any simple closed curve belonging to C, then for each positive integer $i < n$, N either contains all of the segment s_i or else it contains no point whatever of s_i. Hence every simple closed curve which C contains is the sum of a certain number (two or more) of the arcs $t_1, t_2, t_3, \ldots, t_n$; and since there are only a finite number of these arcs, it follows that C can contain only a finite number of simple closed curves. Then by a theorem due to W. L. Ayres [1]) it follows that every connected subset of C is arcwise connected.

Now since, as I have just shown, every connected subset of each maximal cyclic curve of M is arcwise connected, it follows by Theorem B that every connected subset of M is arcwise connected. This completes the proof of Theorem 1.

The following propositions either follow directly from the proof of Theorem 1, or else they follow by similar methods and from the results of the paper of Ayres, using the decomposition of a continuous curve into its cyclic elements and the properties of those elements previously established by the author. For definitions of the terms „simply cyclic“, „almost simply cyclic“, and „simply joined“ see the paper of Ayres.

a) *Every maximal cyclic curve of an almost simply cyclic continuous curve is simply cyclic.*

b) *Every maximal cyclic curve of an almost simply cyclic continuous curve is simply joined.*

c) *For cyclicly connected continuous curves, the properties „simply cyclic“, „almost simply cyclic“, and „simply joined“ are equivalent.*

d) *If the cyclicly connected continuous curve M is simply cyclic, and K is any closed and connected subset of M, then $M - K$ is the sum of a finite number of its maximal connected subsets each of which has only a finite number of limit points in K. Furthermore M is the sum of a finite number of simple continuous arcs no two of which have an interior point of either in common.*

The University of Texas, August 4, 1927.

[1]) *Concerning continuous curves of certain types*, loc. cit., Theorem 8 I have just shown that C is „simply cyclic“ in the sense of Ayres, i e., that it contains only a finite number of simple closed curves