Concerning continuous curves of certain types 1).

By

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A number of authors 2) have discussed continuous curves 3) which contain no simple closed curve and have shown that they possess a number of interesting properties. It is the purpose of this paper to show that many of these properties remain true in more general types of continuous curves.

We shall consider the following five types of continuous curves $M$: (1) $M$ contains at most a finite number of simple closed curves, (2) if $e$ is any positive number, $M$ contains at most a finite number of simple closed curves of diameter greater than $e$, (3) $M$ contains only a finite number of arcs between any two points of $M$, (4) every connected subset of $M$ is arc-wise connected, (5) every closed and connected subset of $M$ is a continuous curve. A continuous curve which satisfies the first condition will be said to be simply cyclic, one which satisfies the second condition will be said to be almost simply cyclic and one which satisfies the third condition will be said to be simply joined.

Of the twenty possible relations between the five types I have been able to settle all except two. The questions as to whether, or not, (2) implies (4) and that (5) implies (4) are not settled in this paper 1).

Excluding these two cases every implication between the five types is given in this paper. In every other case an example may be found to show that there is no implication.

**Theorem 1.** If every connected subset of a continuous curve is arc-wise connected, then every closed and connected subset is a continuous curve 3).

**Proof.** Let $K$ denote any closed and connected subset of the continuous curve $M$ and let $T$ denote any open subset of $K$. If $x$ and $y$ are any two points which lie in a connected subset of $T$, by hypothesis there exists an arc $xy$ which lies wholly in $T$. Therefore $K$ is a continuous curve by a theorem due to R. L. Wilder 4).

**Theorem 2.** Every closed and connected subset of an almost simply cyclic continuous curve is a continuous curve.

**Proof.** Let $K$ be any closed and connected subset of an almost simply cyclic continuous curve $M$. Let us suppose that $K$ is not a continuous curve. Then by the Moore-Wilder Lemma 5) there exist two concentric circles $C_1$ and $C_2$ and an infinite sequence of continua, $K_0, K_1, K_2, \ldots$, all belonging to $K$ such that (1) each of these continua contains a point on $C_1$ and a point on $C_2$ and lies entirely in $L = C_1 + C_2 + I$, where $I$ denotes the annular domain bounded by $C_1$ and $C_2$, (2) no two of the continua have a point in common and no one of them except possibly $K_0$ is a proper subset of any connected point set which is common to $K$ and $L$, (3) the set $K_0$ is the sequential limiting set of the sequence of sets $K_1, K_2, K_3, \ldots$.

1) B. Knaster and C. Kuratowski have given an example which satisfies condition (5) but not condition (4) and thus show that (5) does not imply (4).

2) See A connected and connected im kleinen point set which contains no perfect subset, Bull. Amer. Math. Soc., vol 33 (1927), pp. 106—9. The question as to whether (2) implies (4) remains as an unsettled question.


4) Presented to the American Mathematical Society May 1, 1926.

Let $K = K_0 + K_1 + K_2 + \ldots$. The sets $K_1$ and $K_2 - K_1$ are both closed and have no common points. Then for each point of $K_1$ there exists a circle whose interior contains this point but no point of $K_2 - K_1$. The set of all such interiors of circles for all points of $K_1$ forms a connected domain containing $K_1$ but no points of $K_2 - K_1$. The set of points of $M$ lying in this domain is an open subset of $M$. The continuum $K_1$ contains a point $A_1$ on $C_1$ and a point $B_1$ on $C_2$. Then $M$ contains an arc $A_1 B_1$ which lies wholly in the open subset of $M$) and thus contains no point of $K_2 - K_1$. The arc $A_1 B_1$ has a last point $x_1$ on $C_1$ and the subarc $x_2 B_1$ of $A_1 B_1$ has a first point $y_1$ on $C_2$.

Let us continue this process. In general, the sets $K_n$ and $K_1 = K - (K_0 + K_1 + \ldots + K_n) + x_1 y_1 + x_2 y_2 + \ldots + x_{n-1} y_{n-1}$ are closed and have no common points. Then for each point of $P$ of $K_n$, there exists a circle whose interior contains $P$ but no point of $K_n$. The set of all such interiors for all points of $P$ of $K_n$ forms a connected domain $D_n$ containing $K_n$ but no point of $K_n$. The set $M \cap D_n$, the intersection of $M$ and $D_n$, is an open subset of $M$. The continuum $K_n$ contains a point $A_n$ on $C_1$ and a point $B_n$ on $C_2$ and thus $M$ contains an arc $A_n B_n$ which lies wholly in the open subset $M \cap D_n$. The arc $A_n B_n$ contains a last point $x_n$ on $C_1$ and the subarc $x_{n+1} B_n$ of $A_n B_n$ contains a first point $y_n$ on $C_2$. The arc $x_n y_n$ lies, except for its end-points, wholly in the set $I$.

Let $N = x_n$. Then $M$ contains an infinite sequence of arcs $N_1, N_2, N_3, \ldots$ such that no two of the arcs have a point in common and for every value of $i$, $N_i$ contains a point $x_i$ on $C_1$ and a point $y_i$ on $C_2$ and except for these sets, $N_i$ wholly lies in $I$. There exists a sequence of positive integers $n_1, n_2, n_3, \ldots$ and two points $X$ and $Y$ such that (1) $X$ lies on $C_1$ and is the sequential limit point of the sequence $x_1, x_2, x_3, \ldots$ and $Y$ lies on $C_2$ and is the sequential limit point of the sequence $y_1, y_2, y_3, \ldots$. (2) All of the points $x_n$ lie on one of the two arcs of $C_1$ from $x_n$ to $X$ and in the order $x_n, x_{n+1}, x_{n+2}, \ldots$. $X$ and all of the points $y_n$ lie on one of the two arcs of $C_2$ from $y_n$ to $Y$ and in the order $y_n, y_{n+1}, y_{n+2}, \ldots$.

The limiting set $N_0$ of the sequence $N_1, N_2, N_3, \ldots$ is connected and contains $X$ and $Y$. It thus contains a point on every circle concentric with and lying between $C_1$ and $C_2$. Let the radius of $C_1$ be denoted by $r_1$ and suppose $\eta$ is a number such that

$$r_1 - r > 10 \eta > 0 \quad \text{and} \quad r_2 > \eta.$$

Let $C_1$ and $C_2$ be circles concentric with $C_1$ and with radii $r_1 + \eta$ and $r_2 - \eta$ respectively. The set $N_0$ contains a point $Z_1$ on $C_2$ and a point $Z_2$ on $C_2$. Since $M$ is connected, there is a point in $M$ between $Z_1$ and $Z_2$, which is in the distance $\eta$ of either $Z_1$ or $Z_2$ as the case may be. There exists an integer $n_0$ such that $N_0$ contains two points $p_1$ and $p_2$ such that

$$d(p_1, Z_1) < \eta \quad \text{and} \quad d(p_2, Z_2) < \eta.$$

Then $p_1$ can be joined to $Z_1$ by an arc $U_1$ of $M$ and $p_2$ can be joined to $Z_2$ by an arc $U_2$ of $M$ such that every point of $U_1$ is within a distance $\eta$ of $Z_1$ and every point of $U_2$ is within a distance $\eta$ of $Z_2$.

The arc $U_i (i = 1, 2)$ has at least one point in common with $N_0$ for every $j \geq k$. The arc $U_i$ has a last point $L_i$ in common with $N_0$. The subarc $L_i Z_i$ of $U_i$ has a first point $F_i$ in common with $N_0$. Then $L_i Z_i$ has a last point $L_i$ in common with $N_0$. Continuing we have, for each $i$ and $j$ such that $i = 1, 2, 3, \ldots$, the subarc $L_i Z_i Z_j$, of $U_i$ has a first point $F_i$ in common with $N_0$ and the subarc $F_i Z_i$, of $U_i$ has a last point $L_i$ in common with $N_0$. For every $j$, the arc $F_j L_{j-1}$, has one end-point on $N_0$ and the other on $N_0$ and no other point in common with either arc. Then the arcs $F_n L_{n-1}$ of $U_1, L_{n-1} Z_{n-1}$ of $N_0$, $L_{n-1} F_{n-1}$ of $U_2$ and $F_{n-1} L_n$ of $N_0$ form a simple closed curve $J$, which is of diameter greater than $6 \eta$ since $U_i$ lies in the interior of a circle of radius $r_1 + 2 \eta$ and $U_i$ lies in the exterior of a concentric circle of radius $r_1 - 2 \eta > r_2 + \eta$.

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$x_n, x_{n+1}, x_{n+2}, \ldots$. $X$ and all of the points $y_n$ lie on one of the two arcs of $C_2$ from $y_n$ to $Y$ and in the order $y_n, y_{n+1}, y_{n+2}, \ldots$. $Y$.

The limiting set $N_0$ of the sequence $N_1, N_2, N_3, \ldots$ is connected and contains $X$ and $Y$. It thus contains a point on every circle concentric with and lying between $C_1$ and $C_2$. Let the radius of $C_1$ be denoted by $r_1$ and suppose $\eta$ is a number such that

$$r_1 - r > 10 \eta > 0 \quad \text{and} \quad r_2 > \eta.$$
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Case I. Suppose that \([J_n]\) contains an infinite subsequence \(J_{n_1}, J_{n_2}, J_{n_3}, \ldots\) (if \(i > j\), \(J_i = J_{n_i}\) and \(J_j = J_{n_j}\), then \(k > m\) such that each curve \(J_n\) contains both \(A\) and \(B\). By hypothesis \(M\) contains only a finite number of arcs \(C_1, C_2, \ldots, C_s\) from \(A\) to \(B\) such that no two have any points in common except \(A\) and \(B\).

Let \(L_n\) denote the set \(\sum \overline{C_i}\). The set \(L_n\) contains only a finite number of simple closed curves so there exists an integer \(r_i\) such that for \(i \geq r_i\), \(J_i\) contains at least one point not in \(L_n\). Then \(J_i = J_{n_i}\) contains a point \(p_i\) not in \(L_n\). On the arc \(A\overline{p_i}B\) of \(J_{n_i}\), let \(x_i\) be the first point of \(L_n\) on \(A\overline{p_i}\) and \(y_i\) the first point of \(L_n\) on \(\overline{Bp_i}\) on \(B\). Suppose \(y_i\) belongs to \(C_{n_i}\) and \(y_i\) to \(C_{n_s}\) \(s_i\) and \(t_i\) being not necessarily different but if \(s_i = t_i\), then we will suppose the order \(Ax_i y_i B\).

Let \(C_{n_{i+1}}\) be the arc formed of \(A x_i\), \(C_{n_i}, \overline{p_i}, y_i, B\) of \(C_{n_i}\) and \(y_i\) of \(C_{n_s}\), and \(y_i\) of \(C_{n_s}\). Continue this process with \(L_{n_{i+1}}\) in place of \(L_n\). In general the set \(L_{n_{i+1}}\) contains only a finite number of simple closed curves. Thus there exists a number \(r_i\) such that if \(i \geq r_i\) then \(J_i\) contains at least one point not in \(L_{n_{i+1}}\). Then \(J_i = J_{n_i}\) contains a point \(p_i\) not in \(L_{n_{i+1}}\). On the arc \(A\overline{p_i}B\) of \(J_{n_i}\), let \(x_i\) be the first point of \(L_{n_{i+1}}\) on the arc \(A\overline{p_i}\) and \(y_i\) be the first point of \(L_{n_{i+1}}\) on \(B\). Suppose \(x_i\) belongs to \(C_{n_i}\) and \(y_i\) to \(C_{n_s}\). If \(s_i = t_i\) then we will suppose the order \(Ax_i y_i B\) on \(C_{n_i}\). Let \(C_{n_{i+1}}\) be the arc of \(M\) from \(A\) to \(B\) composed of \(Ax_i\), \(C_{n_i}, p_i, y_i, C_{n_s}, y_i, B\) of \(C_{n_s}\).

The set \(L_{n_{i+1}}\) contains at least \(i\) arcs from \(A\) to \(B\) in \(M\) and, since we may continue the process indefinitely, the hypothesis that \(M\) is simply joined is contradicted.

Case II. Suppose that \([J_n]\) contains an infinite subsequence \(J_{n_1}, J_{n_2}, J_{n_3}, \ldots\) such that each curve \(J_n\) contains \(A\) but does not contain \(B\). Let \(K_1\) be a circle with center at \(B\) and radius \(\frac{1}{2}\varepsilon\). The exterior of \(K_1\) contains \(A\). Since \(M\) is connected in \(B\), at \(B\) there exists a circle \(K_1\) with center at \(B\) such that every point of \(M\) in \(K_1\) can be joined to \(B\) by an arc of \(M\) every point of which lies in the interior of \(K_1\). As \(B\) is the sequential limit point of the sequence \([J_n]\) there exists an integer \(r_i\) such that \(J_{n_i}\) contains a point \(q_i\) in the interior of \(K_1\). Let \(C_i\) be an arc of \(M\) from \(q_i\) to \(B\) lying entirely in the interior of \(K_1\). Let \(p_i\) be the first point of \(C_i\) in common with \(J_{n_i}\) in the order from \(B\) to \(q_i\). Let \(C_i\) be the arc from \(B\) to \(A\) which consists of \(Bp_i\) of \(C_i\) together with either of the arcs of \(J_{n_i}\) from \(p_i\) to \(A\).

\footnote{Cf. R. L. Moore, Concerning continuous curves in the plane, loc. cit.}
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Case IV. Suppose that \([J_n]\) contains an infinite subsequence \(J_{i_1}, J_{i_2}, J_{i_3}\), such that each curve \(J_{i_n}\) contains \(A\) but not \(B\).

Cases III and IV may be proved impossible by methods similar to those of Case II. Thus all four cases are impossible. But if \(M\) is not almost simply cyclic we must have one of the four cases. Therefore the continuous curve \(M\) is almost simply cyclic.

Theorem 6. A simply cyclic continuous curve is almost simply cyclic.

This is an obvious consequence of the definitions.

Theorem 7. A simply cyclic continuous curve is simply joined.

This result may be proved by methods very similar to those used in Case I of the proof of Theorem 5.

Theorem 8. Every connected subset of a simply cyclic continuous curve is arc-wise connected.

This theorem is a consequence of Theorems 3 and 7.

Theorem 9. Every closed and connected subset of a simply cyclic continuous curve is a continuous curve.

Theorem 9 follows from Theorems 1 and 8.

Theorem 10. Every boundary point of an \(S\)-domain \(^1\) of a simply joined continuous curve is accessible from the domain.

Proof. Let \(P\) be a boundary point of an \(S\)-domain \(D\) of a continuous curve \(S\) which is simply joined. Then \(D \cup P\) is connected and is therefore arc-wise connected by Theorem 3.

Theorem 11. If \(e\) is any positive number, then a simply joined continuous curve contains at most a finite number of mutually exclusive connected sets of diameter greater than \(e\).

Suppose that a simply joined continuous curve \(M\) contains an infinite set, \(K_1, K_2, K_3, \ldots\), of mutually exclusive connected subsets each of diameter greater than some positive number \(e\). Each set \(K_i\) contains two points \(x_i, y_i\) which are at a distance apart greater

\(^1\) A connected subset \(D\) of a continuous curve \(S\) is said to be a \(S\)-domain if for every point \(P\) of \(D\) there exists a circle \(K\) with center at \(P\) such that the set of all points of \(S\) which (1) lie interior to \(K\), and (2) lie with \(P\) in a connected subset of \(S\) that lies wholly interior to \(K\), is a subset of \(D\). Cf. R. L. Wilder, loc. cit., p. 841. A point \(P\) is said to be a boundary point of a \(S\)-domain \(D\) if \(P\) is a limit point of \(D\) but does not belong to \(D\). A boundary point \(P\) of a \(S\)-domain \(D\) is said to be accessible from the domain if for every point \(Q\) of the domain there exists a line \(PQ\) which lies except for the point \(P\) entirely in the domain \(D\).
Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist.

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1. Unter einer abgeschlossenen Menge wird im Folgenden ein beliebiger kompakter metrisierbarer topologischer Raum 1) verstanden. Eine zusammenhängende abgeschlossene Menge heisst ein Kontinuum.

Bekanntlich ist ein Kontinuum dann und nur dann stetiges Bild der Einheitsstrecke \( 0 \leq t \leq 1 \), wenn es lokal (oder im Kleinen) zusammenhängend ist 2); im letzteren Satze ist auch die Bedeutung des Begriffes des lokalen Zusammenhanges enthalten.

Der Dimensionsbegriff wird im allgemein üblichen Urysohn-Menger'schen Sinne verstanden 3).

Endlich heisst ein topologischer Raum \( R^n \) in einem anderen

