Les ensembles linéaires qui sont des images continues des ensembles CA (linéaires) coïncidant avec les projections orthogonales des ensembles CA plans [1]), il résulte tout de suite de notre théorème qu'*un produit d'une infinité dénombrable d'ensembles linéaires, dont chacun est une projection d'un ensemble plan CA, est un ensemble de même nature.* Cela résoût le problème 41 posé dans le vol. VIII de ce journal (p. 376).

Remarquons encore qu'en utilisant les propriétés connues des ensembles projectifs, on pourrait appliquer notre démonstration pour prouver la proposition suivante:

Un produit d'une infinité dénombrable d'ensembles P_n est un ensemble P_n (pour $n = 1, 2, 3, \ldots$) [2]).

Il en résulte sans peine que *le résultat d'un nombre fini ou d'une infinité dénombrable d'additions, de soustractions et de multiplications d'ensembles, effectuées (dans un ordre quelconque) en partant des ensembles projectifs de classe n, est toujours un ensemble projectif de classe $n+1$ (à la fois P_{n+1} et C_{n+1}).*

En effet, désignons par $\mathscr{K}_n$ la plus petite famille $\mathscr{K}$ d'ensembles jouissant de deux propriétés suivantes:

1) Tout ensemble projectif de classe n appartient à $\mathscr{K}$.

2) Si les ensembles $E_1, E_2, E_3, \ldots$ appartiennent à $\mathscr{K}$, leur somme $E_1 + E_2 + E_3 + \ldots$ et leur produit $E_1 E_2 E_3 \ldots$ appartient à $\mathscr{K}$.

Les ensembles complémentaires des ensembles de classe n étant des ensembles de classe n, il résulte sans peine de la définition de la classe $\mathscr{K}_n$ qu'elle jouit encore de la propriété suivante:

3) La différence de deux ensembles appartenant à $\mathscr{K}_n$ appartient à $\mathscr{K}_n$ [3]).

La famille $\mathscr{P}_{n+1}$ de tous les ensembles P_{n+1} étant une de familles $\mathscr{K}$ satisfaisant aux conditions 1) et 2), nous avons $\mathscr{K}_n \subset \mathscr{P}_{n+1}$. De même on prouve que $\mathscr{K}_n \subset \mathscr{C}_{n+1}$.

Remarquons enfin qu'on pourrait démontrer que *le résultat d'une opération A effectuée sur les ensembles P_n est toujours un P_n.*

[1]) Voir: *Fund. Math.* t. VII, p. 239.

[2]) Pour la définition des ensembles P_n, voir ce volume, p. 121.

[3]) Cf. la démonstration de la propriété analogue de la classe K_4: *Fund. Math.* t. VIII, p. 367 et 368.

A characterization of continuous curves by a property of their open subsets [1]).

By

R. L. Wilder (Ann Arbor, Michigan, U. S. A).

Definition. If K and N are subsets of a point set M, then K and N are *separated in M in the weak sense* provided M contains no connected subset which contains points of both K and N. The sets K and N are *separated in M in the strong sense* provided M is the sum of two mutually separated [2]) sets X and Y which contain K and N, respectively.

Example. Let M_n $(n = 1, 2, 3, \ldots)$ be the set of all points on the straight line interval connecting the points $(1/n, 0)$, $(1/n, 1)$ of the plane. Let A, B and C be, respectively the points $(0, 0)$, $(0, 1)$ and $(1, 1)$. Let

$$M = A + B + \sum_1^\infty M_n,$$
$$K = A + B,$$
$$N = C.$$

Here M is the sum of two mutually separated sets $X = M_1$ and $Y = M - M_1$, so that K and N are separated in M in the strong sense. However, if $K = A$ and $N = B + C$, M contains no connected set which contains points of both K and N, nor does there exist a separation of M into mutually separated sets containing, respectively, K and N. In this case K and N are separated in M in the weak sense, but not in the strong sense

It is the purpose of this note to show that continuous curves *(lignes de Jordan)* may be characterized by the following property

[1]) Presented to the American Mathematical Society, April 2, 1926.

[2]) Two sets are *mutually separated* if they have no point in common and neither contains a limit point of the other.

of their open subsets[1]): If two subsets of an open subset N of a continuous curve M are separated in N in the weak sense, they are also separated in N in the strong sense. Although the proof is given for the plane, it will be observed that it extends, by suitable change of terminology, to any number of dimensions.

Lemma 1. *If K is a collection of maximal connected subsets of a regular*[2]) *point set N, and H the set of all points contained in sets of the collection K, then, if $N-H$ is not vacuous, H and $N-H$ are mutually separated sets.*

Lemma 2. *If N is a regular point set, and K and H are subsets of N which are separated in N in the weak sense, then K and H are separated in N in the strong sense*[3]).

Lemma 1 is an obvious result of the regularity of N. To prove Lemma 2 it is necessary only to notice that the set of all points contained in sets of the collection of all maximal connected subsets of M that contain points of K, contains no points of H. The result then follows at once from Lemma 1.

Corollary. *If N is a regular point set, and A and B are points separated in N in the weak sense, then A and B are separated in N in the strong sense.*

Corollary. *If N is any point set, A and B are points that*

[1]) If K is a closed subset of a continuous curve M, then $M-K$ is an open subset of M.

[2]) A point set is called *regular* at a point P provided that if ε is any positive number there exists a positive number δ such that all points of M at a distance from P less than δ lie, with P, in a connected subset of M of diameter less than ε. This is an obvious extension, to non-closed sets, of the Hahn definition of *connectedness im kleinen* (H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wien. Ak. Sitz., CXXIII Band Abt. IIa (1914), pp. 2433—2489) or the equivalent Mazurkiewicz definition of a *point of the first kind* (S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math. vol. 1 (1920), pp. 166—209). A point set is itself called regular if it is regular at all of its points.

[3]) Lemma 2 is the analogue, for regular point sets, of a theorem proved by Miss Anna M. Mullikin for closed and bounded point sets. Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, Trans. Amer. Math. Soc. vol. 24 (1922), pp 144—162, Theorem 2. In presenting the abstract of the present note to the American Mathematical Society (see Bull Amer. Math. Soc., vol. 32 (1926), p. 217), I included a theorem (Theorem 1) wich, I have since observed, is a special case of Miss Mullikin's theorem. It should be noted that Miss Mullikin's theorem holds for any number of dimensions, although stated for the plane.

are separated in N in the weak sense, and N is regular at every point of $N(A)$, where $N(A)$ is the maximal connected subset of N determined by A, then A and B are separated in N in the strong sense, and, moreover, $N(A)$ and $N-N(A)$ are mutually separated sets containing A and B, respectively.

Theorem. *In order that a bounded continuum M should be a continuous curve, it is necessary and sufficient that every two subsets A and B which lie in an open subset N of M and are separated in N in the weak sense should be separated in N in the strong sense.*

Proof. That the condition stated in the theorem is necessary follows from Lemma 2, since $M-N$ is regular.

To prove the condition sufficient, I shall use a property established by R. L. Moore[1]) for continua that are not continuous curves. Suppose M is not a continuous curve. Then there exist, according to Moore's result, two concentric circles, J and K, and a sequence of subcontinua of M, viz., M', $M_1, M_2, M_3, \ldots$, such that (1) each of these subcontinua contains at least one point on J and K, respectively, but no points exterior to J or interior to K, (2) no two of these subcontinua have a point in common, and no two of them contain points of any connected subset of M which lies wholly in the set $J+K+I$, where I is the annular domain bounded by J and K, (3) M' is the sequential limiting set of the sequence $M_1, M_2, M_3, \ldots$ For every n, let R_n be a connected subset of M_n which contains no points of J or K, but which has limit points on both J and K[2]). The sequential limiting set, R', of the sequence $R_1, R_2, R_3, \ldots$, is a subset of M'. Let H be a connected subset of R' which contains no points of either J or K, but has limit points on both J and K. Let R be the maximal connected subset of M which contains H and lies wholly in I.

Let A and B be two distinct points of H. Let C_1 and C_2 be circles with centers at A and B, respectively, such that (1) C_1 and C_2 have no point in common, nor have their interiors any point in common, (2) C_1 and C_2 lie wholly in I and do not enclose K. Denote the set of all points of R interior to C_i $(i=1,2)$ by D_i. Let

$$F = R-(D_1+D_2).$$

[1]) *A Characterization of Jordan regions by properties having no reference to their boundaries*, Proc. Nat. Acad. Sci., vol. 4 (1918), pp. 364—370.

[2]) Cf. Anna M. Mullikin, *loc. cit.*, Theorem 1.

Then F is closed in I; i. e., the only limit points of F that do not belong to F lie on J and K. Let the set of all points of M on J and K be denoted by G. Then $F+G$ is a closed subset of M. Hence $M-(F+G)=N$ is an open subset of M.

The points A and B are separated in N in the weak sense. For suppose there exists a connected subset, U, of N, which contains both A and B. The set U must lie wholly in I, since N contains no points of either J or K. Then, as U contains A, and hence a point of H, it is a subset of R. Also, since U contains a point A interior to C_1, and a point, B, exterior to C_1, it contains a point, say P, of C_1. The point P is, then, a point common to C_1, R and N. But this is impossible, since those points of R which N contains lie wholly interior to either C_1 or C_2. Thus no such set as U can exist, and A and B are separated in N in the weak sense.

The points A and B are not separated in N in the strong sense. For suppose N_1 and N_2 are mutually separated sets whose sum is N. Then, since every R_n is a subset of N, at least one of the sets N_1, N_2, say N_1, contains infinitely many sets of the sequence $R_1, R_2, R_3, \ldots$ Call the totality of such sets G. As R' is a sequential limiting set of the sequence $R_1, R_2, R_3, \ldots$ it is also the sequential limiting set of the sets of G. Hence both A and B are limit points of the set of points G', composed of all points contained in sets of the collection G. Therefore both A and B belong to N_1 Thus any separation of N into mutually separated sets implies that A and B are both contained in one of these sets. It follows that A and B are not separated in N in the strong sense.

The supposition that M is not a continuous curve leads, then, to a contradiction of the condition stated in the theorem.

It is clear that the above proof also establishes the following theorem, which is stronger than the above theorem from the standpoint of the sufficiency of the conditions involved:

Theorem: *In order that a bounded continuum M should be a continuous curve, it is necessary and sufficient that every two points A and B which lie in an open subset N of M and are separated in N in the weak sense should be separated in N in the strong sense.*

Corollary. *If N is an open subset of a continuous curve M and A and B are two points of N which are not separated in N in the strong sense, then N contains a simple continuous arc whose end-points are A and B.*

This corollary is a direct consequence of the above theorem and a result due to R. L. Moore[1]) to the effect that if two points lie in a connected subset of an open subset of a continuous curve, then they are the end-points of an arc which lies wholly in that open subset

[1]) *Concerning continuous curves in the plane*, Math. Zeit., vol 15 (1922), pp. 254—260.