# Homotopy 2-regular mappings whose inverses are open 3-cells *

by

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**Abstract.** We prove that if $f$ is an open homotopy 2-regular mapping of a complete metric space $X$ onto some Euclidean space $E^n$ such that $f$ has fiber $E^3$, then $(X, f, E^n, E^3)$ is a trivial fiber space.

**Introduction.** Let $f$ be a mapping from a space $X$ onto a space $Y$ with fiber $K$. Restrictions have been specified by Dyer and Hamstrom [4], Hamstrom [8], [9]. Hall [7], Seidman [11], and others, under which $(X, f, Y, K)$ is a locally trivial fiber space. Dyer and Hamstrom consider cases in which $K$ is compact and $f$ homotopy $n$-regular ($n \leqslant 2$) or completely regular (Seidman) or totally regular (Hall).

The principal result of this paper is the following. If $f$ is an open homotopy 2-regular mapping of a complete metric space $X$ onto some Euclidean space $E^n$ such that $f$ has fiber $E^3$, then $(X, f, E^n, E^3)$ is a trivial fiber space. A similar result for the case where $f$ is a homotopy $k$-regular mapping ($k \leqslant 1$) with fiber $E^{k+1}$ has been proved by Fendrich [6].

**1. Definitions and results from other papers.** Continuous functions will be referred to as *mappings*. If $A$ is a subspace of a metric space $B$ and $g$ is a mapping from $A$ into $B$ which moves no point as much as $\delta > 0$, $g$ will be called a $\delta$-*mapping*. If $g$ is a mapping from $A$ onto $B$ and there is a space $K$ such that for each $b \in B, f^{-1}(b)$ is homeomorphic to $K$, we call $K$ the *fiber* of the mapping $g$.

Let $f$ be a mapping from $X$ onto $Y$ with fiber $K$. We say that $(X, f, Y, K)$ is a *locally trivial fiber space* if for each $y$ in $Y$ there is an open set $U_y$ such that $y \in U_y \subset Y$, and a homeomorphism $h_y$ from $f^{-1}(U_y)$ onto $K \times U_y$ such that $\pi \circ h_y = f | f^{-1}(U_y)$, where $\pi$ denotes the projection from $K \times U_y$ onto $U_y$. In the sequel, when the domain and range of a projection are clear from the context, it will be denoted by the Greek letter $\pi$. We say that $(X, f, Y, K)$ is a *trivial fiber space* if there is a homeomorphism $h$ from $X$ onto $Y \times K$ such that $\pi \circ h = f$.

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In this paper, an *$n$-manifold* $M$ is a separable Hausdorff space with the property that each point has an open neighborhood which is homeomorphic to $E^n$ or to $H^n$, where $H^n$ denotes the half space $\{(x_1, \ldots, x_n) \in E^n\colon x_n \geqslant 0\}$. The set of points of $M$ which are not contained in open sets homeomorphic to $E^n$ is called the *boundary* of $M$ and is denoted by $\operatorname{Bdry} M$. The set of non-boundary points of $M$ is called the *interior* of $M$ and is denoted by $\operatorname{Int} M$.

A continuous mapping $g$ from a metric space $X$ onto a metric space $Y$ is *homotopy $m$-regular* (*$h$-$m$-regular*) provided that for every $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that each mapping of a $k$-sphere ($k \leqslant m$) into $S(x, \delta) \cap f^{-1}(y)$, $y \in Y$, is homotopic to zero in $S(x, \varepsilon) \cap f^{-1}(y)$. The symbol $S(x, \delta)$ denotes the open sphere about $x$ with radius $\delta$. The mapping $g$ is *completely regular* provided that for every $\varepsilon > 0$ and $y \in Y$, there is some $\delta > 0$ such that whenever $d(y, y') < \delta$, there is a homeomorphism from $g^{-1}(y)$ onto $g^{-1}(y')$ which moves no point as much as $\varepsilon$.

If $g$ is a homotopy $m$-regular (completely regular) mapping from $X$ onto $Y$ and $Y = \{y_0, y_1, \ldots\}$, where the sequence $(y_i)$ converges to $y_0$, then the sequence $g^{-1}(y_i))$ is said to converge *homotopy $m$-regularly* (*completely regularly*) to $g^{-1}(y_0)$.

THEOREM A [4, Theorem 5, p. 110 and the remark following it, p. 113]. *If $f$ is a completely regular mapping of a complete metric space $X$ onto an $n$-cell, $Y$, such that each point inverse under $f$ is homeomorphic to a point set $K$, where $K$ is a compact metric space such that the space of homeomorphisms of $K$ onto itself is locally connected, then there is a homeomorphism $h$ of $X$ onto $Y \times K$ such that $\pi \circ h = f$.*

THEOREM B [4, p. 106]. *Suppose $K$, $X$, and $Y$ are metric spaces, $K$ compact $X$ complete and $Y$ finite dimensional, and $f$ is a completely regular mapping of $X$ onto $Y$ such that* (1) *for each point $p$ of $Y$ there is a homeomorphism $f_p$ of $C(K)$ onto $f^{-1}(p)$, where $C(K)$ denotes the cone on $K$, and* (2) *there is a homeomorphism $h$ of $\bigcup f_p(K)$ ($p \in Y$) onto the direct product $Y \times K$ such that $\pi \circ h = f \mid \bigcup f_p(K)$ ($p \in Y$). Then there is a homeomorphism $h^*$ of $X$ onto the direct product $Y \times C(K)$ which is an extension of $h$ and is such that $\pi \circ h^* = f$.*

THEOREM C [8, p. 420]. *If $f$ is a homotopy $2$-regular mapping of a metric space $X$ onto a metric space $Y$ such that each inverse under $f$ is a compact $3$-manifold with boundary which is imbeddable in $E^3$ and the boundaries of the inverses under $f$ are mutually homeomorphic, then $f$ is completely regular.*

THEOREM D [8, p. 422]. *If $M$ is a compact $3$-manifold with boundary, then for each positive number $\varepsilon$ there is a positive number $\delta$ such that every $\delta$-homeomorphism of $\operatorname{Bdry} M$ onto itself can be extended to an $\varepsilon$-homeomorphism of $M$ onto itself.*

The next theorem is taken from Section 4 of [9]. Essentially, it is Theorem 4.16 (p. 27) and the corollary following it (p. 28). Its setting is as follows: $(M_i)$ is a sequence of compact connected $3$-manifolds with boundary converging $h$-$2$-regularly to $M_0$, $\bigcup M_i$ is a compact metric space, the boundaries, $K_0, K_1, K_2, \ldots$ of these $3$-manifolds are mutually homeomorphic. The space $\bigcup M_i$ can be, and is considered to be, imbedded in some Euclidean space $E^m$ in such a way that $M_0$ is a polyhedral subset of $E^3$. The $3$-manifold $M_i$ has a triangulation $T_i$. In each $M_i$, polyhedra, piecewise linear mappings, and so forth, are defined relative to $T_i$ and its refinements. No relationships among the $T_i$ are assumed. Distances are the ordinary distances in $E^m$.



THEOREM E [9]. *If $S_0$ is a polyhedral $2$-sphere in $\operatorname{Int} M_0$, then there is a sequence $(S_i)$, $S_i \subset M_i$, of polyhedral $2$-spheres which converges completely regularly to $S_0$.*

The setting of the next theorem is similar to that of Theorem E except that each $M_i$ is a $3$-cell, and $\bigcup M_i$, $i \geqslant 0$, is embedded in $E^m$ in such a way that

$$M_0 = \{(x_1, \ldots, x_m) \mid x_1^2 + x_2^2 + x_3^2 \leqslant 1 \text{ and } x_i = 0 \text{ for } 4 \leqslant i \leqslant m\}.$$

The theorem remains valid if each $M_i$ is simply a $3$-manifold with boundary such that $M_i$ is imbeddable in $E^3$.

THEOREM F [8, p. 415]. *If $R_0$ is a polyhedral $3$-cell in $M_0$ bounded by the (polyhedral) $2$-sphere $S_0$ and $S_1, S_2, \ldots$ is a sequence of polyhedral $2$-spheres converging $0$-regularly to $S_0$ such that for each $i$, $S_i$ bounds the polyhedral $3$-cell $R_i$ in $M_i$ (and $(R_i)$ converges to $R_0$), then the sequence $(R_i)$ converges $h$-$2$-regularly to $R_0$.*

THEOREM G [9, p. 33]. *If $M$ is a compact $3$-manifold with boundary, then the space of homeomorphisms of $M$ onto itself leaving the boundary of $M$ pointwise fixed is $\mathrm{LC}^m$ for each $m$.*

THEOREM H [9, p. 38]. *The space of homeomorphisms of a compact $3$-manifold with boundary onto itself is $\mathrm{LC}^n$ for each $n$.*

THEOREM J [9, p. 33]. *Suppose that $C_1$ is a $3$-cell and $C_2$ is a $3$-cell in $C_1$ such that $C_2 \cap \operatorname{Bdry} C_1$ is a $2$-manifold with boundary. Denote by $H_1$ the space of homeomorphisms of $C_1$ onto itself leaving $\operatorname{Bdry} C_1$ pointwise fixed, by $i_1$ its identity, and by $S_1(i_1, \varepsilon)$ the (open) $\varepsilon$-neighborhood of $i_1$ in $H_1$. Denote by $H_2$ the space of homeomorphisms of $C_2$ into $C_1$ leaving $C_2 \cap \operatorname{Bdry} C_1$ pointwise fixed and carrying $C_2 - (C_2 \cap \operatorname{Bdry} C_1)$ into $\operatorname{Int} C_1$, by $i_2$ its identity, and by $S_2(i_2, \varepsilon)$ the (open) $\varepsilon$-neighborhood of $i_2$ in $H_2$. Denote by $K$ the closure of $C_1 - C_2$, by $H$ the space of homeomorphisms of $K$ into itself leaving $\operatorname{Bdry} K$ pointwise fixed, by $i$ its identity, and by $S(i, \varepsilon)$ the $\varepsilon$-neighborhood of $i$ in $H$. The space $H_2$ should be further restricted to consist only of those homeomorphisms $f$ for which $\mathrm{Cl}(C_1 - f(C_2))$ is a $3$-manifold with boundary.*

If $n$ is an integer and $\varepsilon > 0$, then there is a positive number $\delta$ such that (i) if $f$ is a mapping of $B^n$ into $S_2(i_2, \delta)$, then there is a mapping $F$ of $B^n$ into $S_1(i_1, \varepsilon)$ such that for each $x$ in $B^n$, $F(x) \mid C_2 = f(x)$ and (ii) if $f$ is a mapping of $S^n$ into $S_2(i_2, \delta)$, then there is a mapping $F$ of $S^n$ into $S_1(i_1, \varepsilon)$ such that for each $x$ in $S^n$, $F(x) \mid C_2 = f(x)$.

We shall use this theorem in the case where $C_2$ is a $3$-cell in the interior of a $3$-cell $C_1$. Then, very roughly, the theorem states that each $n$-sphere of "small" homeomorphisms of $C_2$ into $\operatorname{Int} C_1$ extends to an $n$-sphere of "small" homeomorphisms of $C_1$ onto itself which leave $\operatorname{Bdry} C_1$ pointwise fixed. For $n = 0$, this simply means that each "small" homeomorphism of $C_2$ into $\operatorname{Int} C_1$ extends to a "small" homeomorphism of $C_1$ onto itself that leaves $\operatorname{Bdry} C_1$ pointwise fixed.

**2. Preliminary lemmas.** Let $f$ denote an open homotopy 2-regular mapping of a complete metric space $X$ onto some Euclidean space $E^n$ such that each point inverse under $f$ is homeomorphic to $E^3$.

Suppose that $(y_i)$ converges to $y_0$ in $E^n$. For each $i$, let $T_i$ be a triangulation of $f^{-1}(y_i)$. In each $f^{-1}(y_i)$, polyhedra will be defined relative to $T_i$. No relationships among the $T_i$'s will be assumed. For any subset $Q$ of $f^{-1}(y_i)$, $\operatorname{Int} Q$ ($\operatorname{Ext} Q$) will denote the interior (exterior) of $Q$ with respect to $f^{-1}(y_i)$.

DEFINITION. (a) A 3-cell $C$ (2-sphere $S$) is *tame* in a homeomorphism $Z$ of $E^3$ if there is a homeomorphism of $Z$ onto $E^3$ which sends $C$ ($S$) onto a standard 3-cell (2-sphere). (b) A 3-cell $C$ contained in the interior of a 3-cell $C'$ is *tame* in $C'$ if there is a homeomorphism of $C'$ onto the standard 3-cell of radius 1 which sends $C$ onto the standard 3-cell of radius $\frac{1}{2}$.

2.1. LEMMA. *If $B_0$ is a tame 3-cell in $f^{-1}(y_0)$, then there is a sequence $(B_{\alpha(i)})$ of tame 3-cells, $B_{\alpha(i)} \subset f^{-1}(y_{\alpha(i)})$ such that $(B_{\alpha(i)})$ converges completely regularly to $B_0$.*

Proof. For 3-cell in 3-space, the concepts of tame and polyhedral with respect to some triangulation are equivalent. Let $T_0$ denote some triangulation of $f^{-1}(y_0)$ under which $B_0$ is polyhedral. Let $B_0'$ denote a polyhedral 3-cell in $f^{-1}(y_0)$ which contains $B_0$ in its interior. From the openness and $h$-2-regularity of $f$ and the compactness of $B_0'$, it follows easily that given $\varepsilon > 0$, for each of $k = 0, 1$, and 2, there is some $\zeta_k > 0$ such that if $g$ is a mapping of $S^k$ (the $k$-sphere) into $f^{-1}(y_i)$ with the diameter of $g(S^k)$ less than $\zeta_k$ and the distance between $g(S^k)$ and $B_0$ less than $\zeta_k$ (the distance between two sets is defined to be the infimum of the distances between pairs of points, one from each set), then $g$ is nullhomotopic on a subset of $f^{-1}(y_i)$ of diameter less than $\varepsilon$.

By an argument similar to the proof of Lemma 2.6 of [8], it follows that there exists a subsequence $(y_{\alpha(i)})$ of $(y_i)$ and a sequence $(M_{\alpha(i)})$ of polyhedra, $M_{\alpha(i)} \subset f^{-1}(y_{\alpha(i)})$, such that $(M_{\alpha(i)})$ converges to $M_0 = B_0'$. Let $M = M_0 \cup M_j$ $(j \geq 1)$. The space $M$ is of finite dimension, so it can be imbedded in some Euclidean $m$-space. Then by a theorem of Klee's [10, p. 36], $M$ can be imbedded in $E^3 \times E^m$ such that $M_0$ is a polyhedral subset of $E^3$.

For each polyhedral 2-sphere $S_0$ in $\operatorname{Int} M_0$, there are polyhedral 2-spheres $S_k$, $S_k \subset M_k$, such that $(S_k)$ converges completely regularly to $S_0$ (Theorem E). From simple homotopy arguments and the 0-regularity of $f$, it follows that the sequences $(\operatorname{Cl}(\operatorname{Ext} S_k))$ and $(\operatorname{Cl}(\operatorname{Int} S_k))$ converge to $\operatorname{Cl}(\operatorname{Ext} S_0)$ and $\operatorname{Cl}(\operatorname{Int} S_0)$, respectively. Then Theorems F and C imply that $(\operatorname{Cl}(\operatorname{Int} S_k))$ converges completely regularly to $\operatorname{Cl}(\operatorname{Int} S_0)$. Hence, if we take $S_0$ to be the 2-sphere that is the boundary of $B_0$, the conclusion follows.

2.2. COROLLARY. (i) *If $y_0 \in E^n$ and $B_0$ is a tame 3-cell in $f^{-1}(y_0)$, then given $\varepsilon > 0$, there is some $\zeta > 0$ such that whenever $d(y, y_0) < \zeta$, there is a 3-cell $B_y$ contained in $f^{-1}(y)$ and an $\varepsilon$-homeomorphism $h_y$ from $B_0$ onto $B_y$.*



(ii) *For each positive integer $i$, let $S_i$ denote a 2-sphere (tame 2-sphere) in $f^{-1}(y_i)$, and let $(S_i)$ converge completely regularly to a tame 2-sphere $S_0$ in $f^{-1}(y_0)$. Then the sequence $(\operatorname{Cl}(\operatorname{Int} S_i))$ converges (converges completely regularly) to $\operatorname{Cl}(\operatorname{Int} S_0)$, and the sequence $(\operatorname{Cl}(\operatorname{Ext} S_0))$ converges to $\operatorname{Cl}(\operatorname{Ext} S_0)$.*

**3. $X$ is an $(n+3)$-manifold.** In this section, as in Section 2, $f$ will denote a continuous, open, homotopy 2-regular mapping of a complete metric space $X$ onto some Euclidean space $E^n$ such that each point inverse is homeomorphic to $E^3$.

Since (i) $f$ is open, and (ii) $E^n$ and each point inverse is separable, we have the following lemma.

3.1. LEMMA. *$X$ is separable.*

The remainder of Section 3 is devoted to establishing Lemma 3.2.

3.2. LEMMA. *Each point $p$ of $X$ is contained in some open subset of $X$ that is an open $(n+3)$-cell whose closure in $X$ is a closed $(n+3)$-cell.*

We will use the following selection theorem of E. A. Michael, as stated in [4, p. 104].

THEOREM M. *If $A$ and $B$ are metric spaces such that $A$ is complete and the covering dimension of $B \leq n+1$, $Z$ is a closed subset of $B$, $F$ is a mapping of $A$ onto $B$ such that the collection of inverses under $F$ is lower semi-continuous* (see Lemma 3.11 below) *and equi-*$LC^n$ (see Definition 3.5 below) *and $g$ is a mapping of $Z$ into $A$ such that for $z \in Z$, $g(z) \in F^{-1}(z)$, then there is a neighborhood $U$ of $Z$ in $B$ such that $g$ can be extended to a mapping $g^*$ of $U$ into $A$ such that for $b \in U$, $g^*(b) \in F^{-1}(b)$. If each inverse under $F$ has the property that its homotopy groups of order $\leq n$ vanish, then $U$ may be taken to be the entire space $B$.*

DEFINITION. An embedding $h_y$ of the standard 2-sphere $S^2$ into $f^{-1}(y)$ is *tame* if it extends to a homeomorphism from $E^3$ onto $f^{-1}(y)$.

For each $y$ in $E^n$, let $G_y$ denote the space of tame imbeddings of $S^2$ into $f^{-1}(y)$, topologized by $d(h_y, h_y') = \sup\{d(h_y(x), h_y'(x)),\ x \in S^2\}$. Let $\bar{G}_y$ denote the closure of $G_y$ in the space of all mappings from $S^2$ into $f^{-1}(y)$ with the topology induced by the "sup" metric. Let $G = \{G_y, y \in E^n\}$, let $G^* = \bigcup G_y$, and let $\bar{G}^* = \bigcup \bar{G}_y$.

3.3. LEMMA. *For each positive integer $i$, let $g_i$ be an imbedding (not necessarily tame) of $S^2$ into $f^{-1}(y_i)$ for some $y_i$ in $E^n$. Suppose that $(g_i)$ converges to a tame imbedding $g$ of $S^2$ into $f^{-1}(y)$ $y \in E^n$. Then $(\operatorname{Int} g_i(S^2))$ converges to $\operatorname{Cl}(\operatorname{Int} g(S^2))$ and $(\operatorname{Ext} g_i(S^2))$ converges to $\operatorname{Cl}(\operatorname{Ext} g_i(S^2))$.*

Proof. Clearly, $(g_i(S^2))$ converges completely regularly to $g(S^2)$. The result follows from Corollary 2.2.

3.4. LEMMA. *$G^*$ is complete under some metric $d^*$.*

Proof. (a) The space $\bar{G}^*$ is a closed subspace of the space of all mappings of $S^2$ into $X$ and thus is complete.

With the aid of Corollary 2.2, the proof of (a) is similar to the first part of the proof of Lemma 2 of [4].

(b) The space $G^*$ is a $G_\delta$ in $\bar{G}^*$, and therefore is complete under some metric $d^*$.

Proof of (b). For each positive integer $m$, let $H_m$ denote the collection of all mappings $g$ in $\bar{G}^*$ such that $\sup\{\operatorname{diam} g^{-1}(x),\ x \in S^2\} \geqslant 1/m$. As in the second part of the proof of Lemma 2 of [4], $H_m$ is a closed subspace of $\bar{G}^*$.

A theorem of Bing's [1, p. 109] characterizes tame 2-spheres in $E^3$ as those 2-spheres which can be approximated arbitrarily closely on both sides by 2-spheres (a 2-sphere $S'$ in $\operatorname{Int} S$ or in $\operatorname{Ext} S$ is within $\varepsilon$ of $S$ if there exists an $\varepsilon$-homeomorphism from $S$ onto $S'$). Let $K_m$ denote the collection of all mappings $g$ in $G^*$ such that $g$ is an imbedding of $S^2$ into some $f^{-1}(y)$ but $g(S^2)$ cannot be approximated closer than $1/m$ on some one side. We will show that $G^* = \bar{G}^* - [\bigcup(\bar{K}_m \cup H_m),\ m \geqslant 1]$.

Let $g$ be an element of $\bar{K}_m$. If $g$ is not one-to-one, then $g$ is in some $H_r$. Thus suppose that $g$ is 1-1. We will show that $g$ is in $\bigcup K_m$, $m \geqslant 1$. For some $y_0$ in $E^n$, $g$ is an imbedding of $S^2$ into $f^{-1}(y_0)$. There is a sequence $(g_i)$, $g_i \in K_m$, which converges to $g$. From the definition of $K_m$, for each $i$, $g_i(S^2)$ cannot be approximated closer than $1/m$ on some one side. Without loss of generality, we can assume that the approximation fails either in the interior of all the $g_i(S^2)$'s or the exterior of all the $g_i(S^2)$'s. Here we assume the former. The proof for the later case is similar.

Suppose that $g$ is in $G^*$, i.e., $g$ is tame. From Bing's theorem, there exists a 2-sphere $S'$ contained in $\operatorname{Int} g(S^2)$ and a $1/2m$ homeomorphism $\xi'$ from $g(S^2)$ onto $S'$. Let $B$ be a tame 3-cell in $f^{-1}(y_0)$ that contains $g(S^2)$ in its interior. From Corollary 2.2, there is a sequence $(h_i)$ of $\varepsilon_i$-homeomorphisms, $h_i\colon B \to f^{-1}(y_i)$, such that $(\varepsilon_i) \to 0$. From Lemma 3.3 and the fact that $(g_i) \to g$, if $i$ is large enough, $h_i(S')$ is contained in $\operatorname{Int} g_i(S^2)$. Hence for large $i$, $h_i \circ \xi' \circ g \circ g_i^{-1}$ is a homeomorphism of $g_i(S^2)$ onto $h_i(S')$ that moves no point as much as $1/m$. This contradicts the assumption that $g_i(S^2)$ cannot be approximated within $1/m$ from the interior.

3.5. DEFINITION (i) A space $W$ is *$m$-locally connected* ($\mathrm{LC}^m$) provided that for each point $w$ of $W$ and each positive number $\varepsilon$ there is a positive number $\delta$ such that every mapping of a $k$-sphere, $k \leqslant m$, onto a subset of $S(y, \delta)$ is homotopic to a constant (nullhomotopic) on a subset of $S(y, \varepsilon)$.

(ii) A collection $G$ of closed point sets filling up a metric space is said to be *equi*-$\mathrm{LC}^m$ provided that it is true that if $y$ is a point of an element $g_0$ of $G$ and $\varepsilon$ is a positive number, there is a positive number $\delta$ such that if $g$ is an element of $G$, then any mapping of a $k$-sphere, $k \leqslant m$, onto a subset of $g \cap S(y, \delta)$ is nullhomotopic on a subset of $g \cap S(y, \varepsilon)$.

Remark. These properties are clearly topological and hence in order to prove that each $G_y$ is $\mathrm{LC}^m$ and that $G$ is equi-$\mathrm{LC}^m$ under the metric $d^*$, it suffices to establish them for the metric $d$.

3.6. LEMMA. *Suppose that $(g_i)$ is a sequence of tame imbeddings of $S^2$ (the standard 2-sphere) into $E^3$ which converges to the identity mapping $g_0$. Let $B_i$ denote the 3-cell bounded by $S_i = g_i(S^2)$ and let $B$ denote the (standard) closed 3-cell in $E^3$ that is bounded by $S^2$. Then given $\varepsilon > 0$, there is a number $N$ such that whenever $i > N$, there is a homeomorphism $G_i$ of $B$ onto $B_i$ such that* (i) $G_i|S^2 = g_i$, *and* (ii) $d(\mathrm{id}_B, G_i) < \varepsilon$.



Proof. Let $y_0 = 0$, $y_1 = 1$, $y_2 = \frac{1}{2}, \ldots$ Let $Z = E^3 \times \{y_i, i > 0\}$. Metrize $Z$ by $d[(x, y), (x', y')] = |x - x'| + |y - y'|$. For each $i$, let $g_i'$ be the imbedding of $S^2 \times \{0\}$ into $E^3 \times y_i$ defined by $g_i'(x) = (g_i(x), y_i)$. It follows from Lemma 3.3 that $(\operatorname{Int} g_i'(S^2))_{i \geqslant 1}$ converges to $\operatorname{Cl}(\operatorname{Int} g_0'(S^2))$. Since $(g_i)_{i \geqslant 1}$ converges to $g_0'$, the sequence $(g_i'(S^2))_{i \geqslant 1}$ converges completely regularly to $g_0'(S^2)$. For each $i$, let $B_i = \operatorname{Cl}(\operatorname{Int} g_i'(S^2))$ and let $S_i = g_i'(S^2)$. From Theorems F and C, the sequence $(B_i)$ converges completely regularly to $B_0'$. Therefore there is a sequence $(\eta_i)$, $\eta_i > 0$, which converges to zero, and a sequence $(h_i)$ of $\eta_i$-homeomorphisms from $B_i'$ onto $B_0'$.

For each $i$, $h_i \circ g_i'$ is a homeomorphism from $S_0'$ onto itself. If $i$ is large enough, $h_i \circ g_i'$ extends to a small homeomorphism $H_i$ from $B_0'$ onto itself (Theorem D). Then $h_i^{-1} \circ H_i$ is a small homeomorphism from $B_0'$ onto $B_i'$ that extends $g_i'$. Hence $\pi \circ h_i^{-1} \circ H_i$ is a small homeomorphism of $B$ onto $B_i$ that extends $g_i$, where $\pi$ denotes the projection from $E^3 \times \{y_i\}$ onto $E^3$. ∎

Let $B$ and $B'$ denote the standard 3-cells of radius $\frac{1}{2}$ and 1, respectively. Let $S^2 = \operatorname{Bdry} B$. In the space of tame imbeddings of $S^2$ into $\operatorname{Int} B'$, let $S'(\mathrm{id}, r)$ denote the ball with center the identity mapping and radius $r$. In the space of tame imbeddings of $B$ into $\operatorname{Int} B'$, let $S''(\mathrm{id}, r)$ denote the ball with center the identity mapping and radius $r$. Let $S(\mathrm{id}, r)$ denote the similar ball in the space of homeomorphisms of $B'$ onto itself leaving $\operatorname{Bdry} B'$ pointwise fixed.

3.7. LEMMA. *If $m$ is a nonnegative integer and $\varepsilon > 0$, there is a number $\delta > 0$ such that if $g$ is a mapping of $S^m$ into $S'(\mathrm{id}, \delta)$, then there is a mapping $G$ of $S^m$ into $S''(\mathrm{id}, \varepsilon)$ such that for each $x$ in $S^m$, $G(x)|S^2 = g(x)$.*

Proof by induction on $m$. For $m = 0$, denial of the theorem easily leads to a contradiction of Lemma 3.6. Assume that the lemma is valid for $m \leqslant k$. Let $\varepsilon > 0$ and let $g$ be a mapping of $S^{k+1}$ into $S'(\mathrm{id}, \delta)$. The proof of Theorem J applies here, with $B'$, $\operatorname{Cl}(B' - B)$, and $B$ taking the place of $C_1$, $C_2$, and $K$, respectively. For the hypotheses of Theorem J to be satisfied, $g$ should be a mapping from $S^{k+1}$ into the space of tame imbeddings of $C_2$ into $C_1$, whereas $g$ only maps $S^{k+1}$ into the space of tame imbeddings of $[\operatorname{Bdry} C_2 - (C_2 \cap \operatorname{Bdry} C_1)]$ into $C_1$. However, in the proof of Theorem J, only the restrictions to $[\operatorname{Bdry} C_2 - (C_2 \cap \operatorname{Bdry} C_1)]$ of the imbeddings from $C_2$ into $C_1$ are used.

3.8. LEMMA. *Given a nonnegative integer $m$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $g$ is a mapping of $S^m$ into $S'(\mathrm{id}, \delta)$, then there is a mapping $G$ of $S^m$ into $S(\mathrm{id}, \varepsilon)$ such that for each $x$ in $S^m$, $G(x)|S^2 = g(x)$.*

Proof. Use part (ii) of Theorem J, and Lemma 3.7.

3.9. THEOREM. *The space of tame imbeddings of $S^2$ into $E^3$ is $\mathrm{LC}^m$ for all $m$. Hence, for each $y$ in $E^n$, $G_y$ is $\mathrm{LC}^m$ for all $m$.*

Proof. From Theorem [G] (or Černavskii [3], or Edwards and Kirby [5]), the space of homeomorphisms of $B'$ onto $B'$ which leave the boundary pointwise fixed is $\mathrm{LC}^k$ for all $k$. This fact and Lemma 3.8 imply the theorem.

3.10. LEMMA. *$G$ is equi-$\mathrm{LC}^m$ for each nonnegative integer $m$.*

Proof. Suppose $p \in E^n$, $s \in G_p$, $m$ is a nonnegative integer, and $\varepsilon > 0$. Let $B_p$ denote a tame 3-cell in $f^{-1}(p)$ which contains $s(S^2)$ in its interior. From Corollary 2.2, given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(y, p) < \delta$, $y \in E^n$, there is a tame 3-cell $B_y$ contained in $f^{-1}(y)$ and an $\frac{1}{2}\varepsilon$-homeomorphism $h_y$ from $B_y$ onto $B_p$. The remainder of this proof of Lemma 3.10 is similar to the proof of Lemma 3 of [4].

3.11. LEMMA. *The space $G$ is lower semi-continuous in the sense that if $(p_i)$ converges to $p$ in $E^n$, then $G_p$ is in the closure of $\bigcup G_{p_i}$.*

The proof of this lemma is similar to the proof of Lemma 4 of [4].

Let $F$ denote the mapping from $G^*$ onto $E^n$ that carries $G_y$ onto $y$ for each $y$ in $E^n$. Recall that $p$ denotes some element of $X$. Let $f(p) = y_p$. Define a function $g$ from the singleton $\{y_p\}$ into $G^*$ by letting $g(y_p)$ denote some element of $G_{y_p}$ for which $p$ is contained in the interior of $[g(y_p)]\ (S^2)$. The hypotheses of Theorem M are satisfied by $G^*$, $E^n$, $F$, $\{y_p\}$, and $g$. Therefore, there is an open set $U$ containing $y_p$ and a mapping $H$ from $U$ into $G^*$ such that (i) $H|\{y_p\} = g$, and (ii) $F \circ H$ is the inclusion mapping of $U$ into $E^n$.

Let $K$ denote some closed $n$-cell in $U$ which contains $y_p$ in its interior. Define a function $\gamma$ from $K \times S^2$ into $f^{-1}(K)$ by $\gamma(y, z) = H(y)(z)$. Clearly, $\gamma$ is continuous and 1-1, and hence, $\gamma$ is an imbedding.

Let $N = \bigcup \mathrm{Cl}[\mathrm{Int}\gamma(y \times S^2)]$, $y \in K$, where the interior of $\gamma(y \times S^2)$ is taken with respect to $f^{-1}(y)$. The function $f|N$ from $N$ onto $K$ is continuous, and for each $y$ in $K$, $(f|N)^{-1}(y)$ is a 3-cell in $f^{-1}(y)$ which we denote by $B_y$. Let $S_y$ denote the 2-sphere boundary of $B_y$.

3.12. LEMMA. *If $(y_i)$ converges to $y_0$ in $K$, then $(B_{y_i})$ converges completely regularly to $B_{y_0}$.*

Proof. Let $B'_{y_0}$ denote a tame 3-cell in $f^{-1}(y_0)$ that contains $B_{y_0}$ in its interior. From Lemma 2.1, there exists a sequence $(B_{y_i})$ of 3-cells, $B'_{y_i} \subset f^{-1}(y_i)$, which converges completely regularly to $B'_{y_0}$. For large $i$, $S_{y_i}$ is contained in $B'_{y_i}$ (Corollary 2.2). Since the sequence $(H(y_i))$ of homeomorphisms converges to $H(y_0)$, $(S_{y_i})$ converges completely regularly to $S_{y_0}$. The lemma follows from Theorems F and C.

3.13. COROLLARY. *$f|N$ is completely regular.*

A simple argument yields the following lemma.

3.14. LEMMA. *$N$ is complete.*

It is now established that $f|N$ is a completely regular mapping from the complete metric space $N$ onto the $n$-cell $K$ with point-inverses homeomorphic to the 3-cell $B$. From Theorem A, $N$ is homeomorphic to the $(n+3)$-cell $B \times K$ under a homeomorphism $h$ such that $\pi \circ h = f|N$.

We assert that the open $(n+3)$-cell $\mathrm{Int}N$ is an open subset of $X$ containing $p$. It is clear that $p$ is in $\mathrm{Int}N$. Suppose that $\mathrm{Int}N$ is not open in $X$. Let $x_0$ be a point of $\mathrm{Int}N$ and let $(x_i)$ be a sequence in $X - \mathrm{Int}N$ that converges to $x_0$. For each $i$, let $y_i = f(x_i)$. Since $x_0$ is in the interior (with respect to $f^{-1}(y_0)$) of the 3-cell $(f|N)^{-1}(y_0)$, almost all of the $y_i$'s are different from $y_0$. Without loss of generality, we may assume that the $y_i$'s, $i \neq 0$, are distinct, and that none is equal to $y_0$. Furthermore, since $(y_i)$ converges to $y_0$, and $y_0$ is in $\mathrm{Int}K$, we may assume that all the $y_i$'s are in $K$. From Lemma 3.12, the sequence $(B_i = (f|N)^{-1}(y_i))$ converges completely regularly to $B_0 = (f|N)^{-1}(y_0)$. From Corollary 2.2, the sequence $(\mathrm{Cl}(\mathrm{Ext}B_i))$ converges to $\mathrm{Cl}(\mathrm{Ext}B_0)$. Then, since for each $i$, $x_i \in \mathrm{Cl}(\mathrm{Ext}B_i)$, it follows that $x_0 \in \mathrm{Cl}(\mathrm{Ext}B_0)$. But this is a contradiction, since $x_0$ was taken to be in $\mathrm{Int}B_0$. This completes the proof of Lemma 3.2.



3.15. COROLLARY. *$X$ is locally compact.*

3.16. COROLLARY. *$X$ is an $(n+3)$-manifold.*

Proof. Lemmas 3.1 and 3.2.

## 4. Proof of the Main Theorem.

4.1. LEMMA. *Let $X$ be a complete metric space. Let $f$ be a homotopy 2-regular open mapping of $X$ onto $I^n$ ($I = [0, 1]$) such that for each $y$ in $I^n$, $f^{-1}(y)$ is homeomorphic to $E^3$. Suppose that $A$ is a compact subset of $X$ and $f(A)$ is contained in $\mathrm{Int}I^n$. Then there is an open $(n+3)$-cell in $X$ which contains $A$ and whose closure $M$ is homeomorphic to the $(n+3)$-cell $I^n \times B^3$ under a homeomorphism $\xi$ such that* (i) $\pi \circ \xi = f|M$, *and* (ii) *for each $y$ in $I^n$, $\xi^{-1}\ (B^3 \times y)$ is a tame 3-cell in $f^{-1}(y)$.*

Proof. The lemma is trivially valid for $n = 0$. Assume that it is valid for $n \leqslant k$. Let $n = k+1$ and suppose that $X$, $I^k \times I$, $f$: $X \to I^k \times I$, and $A$ satisfy the hypotheses of the lemma. For each $(y, z)$ in $I^k \times I$, let $G_{(y,z)}$ denote the set of tame imbeddings $\gamma$ of $S^2$ into $f^{-1}(y, z)$ such that $\gamma(S^2)$ contains $A \cap f^{-1}(y, z)$ in its interior. Let $G$ denote the collection of all $G_{(y,z)}$. Let $G^* = \bigcup G_{(y,z)}$ and topologize $G^*$ by the metric given by the distance function

$$d(\gamma, \gamma') = \sup\{d(\gamma(x), \gamma'(x)),\ x \in S^2\}.$$

For each $y_\alpha$ in $I$, let $X_\alpha = f^{-1}(I^k \times y_\alpha)$. Each $X_\alpha$, as a closed subspace of the complete space $X$, is itself complete. Let $f_\alpha = f|X_\alpha$: $X_\alpha \to I^k \times y_\alpha$. Clearly, $f_\alpha$ is continuous, open, and homotopy 2-regular. The set $A_\alpha = A \cap f^{-1}(I^k \times y_\alpha)$ is compact. Since (i) $f(A)$ is contained in $\mathrm{Int}(I^k \times I)$, and (ii) $\mathrm{Bdy}(I^k \times y_\alpha)$ is contained in $\mathrm{Bdy}(I^k \times I)$, $f(A_\alpha)$ is contained in the interior of $I^k \times y_\alpha$. From the induction hypothesis, there is a closed neighborhood $M_\alpha$ of $A_\alpha$ in $X_\alpha$ and a homeomorphism $\xi_\alpha$ from $M_\alpha$ onto $B \times I^k \times \{y_\alpha\}$ such that (i) $\pi \circ \xi_\alpha = f_\alpha$, and (ii) for each $y$ in $I^k$, the 3-cell $\xi_\alpha^{-1}(B \times y \times y_\alpha)$ is tame in $f^{-1}(y, y_\alpha)$.

For each $y_\alpha$ in $I$, define a mapping $g_\alpha$ from $I^k \times y_\alpha$ into $G^*$ as follows. For each $(y, y_\alpha)$ in $I^k \times y_\alpha$, let $(g_\alpha(y, y_\alpha))(s) = \xi_\alpha^{-1}(s, y, y_\alpha)$ for all $s$ in $S^2$. For each $y$ in $I^k$, $g_\alpha(y, y_\alpha)$ is an imbedding of $S^2$ into $f^{-1}(y, y_\alpha)$. Moreover, since the induction hypothesis implies that the 3-cell $\xi_\alpha^{-1}(B \times y \times y_\alpha)$ bounded by $g_\alpha(y, y_\alpha)(S^2)$ is tame in $f^{-1}(y, y_\alpha)$, this imbedding is tame.

Denote by $F$ the mapping from $G^*$ onto $I^k \times I$ that carries $G_{(y,z)}$ onto $(y, z)$ for each $(y, z)$ in $I^k \times I$. As in Section 3, $F$ is continuous. For each $y_\alpha$ in $I$, $F \circ g_\alpha$ is the injection from $I^k \times y_\alpha$ into $I^k \times I$.

As in Section 3, $G^*$ is complete, $G$ is equi-$LC^m$ for each $m$, and $G$ is lower semi-continuous (Lemmas 3.4, 3.10, and 3.11). Hence for each $y_\alpha$ in $I$, Theorem M guarantees the existence of a neighborhood $U$ of $I^k\times y_\alpha$ in $I^k\times I$ with the property that $g_\alpha$ can be extended to a mapping $H_\alpha$ of $U$ into $G^*$ such that if $z\in U$, $H_\alpha(z)\in F^{-1}(z)$. Since $I^k\times y_\alpha$ is compact, there is some interval $(a_\alpha, b_\alpha)$ for which $I^k\times y_\alpha\subset I^k\times(a_\alpha, b_\alpha)\subset U$. Then $F\circ H_\alpha$ is the injection of $I^k\times(a_\alpha, b_\alpha)$ into $I^k\times I$.

Since $I$ is compact, there is a finite subset $\{a_i, 1\leqslant i\leqslant r\}$ of $I$, and mappings $\{H_i, 1\leqslant i\leqslant r\}$, such that (i) $0 = a_0<a_1<a_2<\ldots<a_r = 1$, and (ii) for each $i$, $1\leqslant i\leqslant r$, $F\circ H_i$ is the injection of $I^k\times[a_{i-1}, a_i]$ into $I^k\times I$.

As in Section 3, define

$$\delta_i\colon I^k\times[a_{i-1}, a_i]\times S^2\to f^{-1}(I^k\times[a_{i-1}, a_i])$$

by

$$\delta_i(w, y, z) = \big(H_i(w, y)\big)(z)\,.$$

Let

$$N_i = \bigcup \mathrm{Cl}(\mathrm{Int}\,\delta_i(w, y, S^2)),\qquad (w, y)\in I^k\times[a_{i-1}, a_i]\,.$$

The arguments used in Section 3 prove that there is a homeomorphism $h_i$ from $N_i$ onto $B\times I^k\times[a_{i-1}, a_i]$ such that

(a) $\pi\circ h_i = f|N_i$, and

(b) $\mathrm{Int}\,N_i\supset A\cap f^{-1}(I^k\times(a_{i-1}, a_i))$.

The set $N_i$ is compact and therefore closed in $X$.

The remainder of the proof of Lemma 4.1 is devoted to modifying the $N_i$'s and fitting them together into a single $(k+4)$-cell that satisfies the conclusion of the lemma.

Let $A'$ denote the compact set $\bigcup N_i$, $1\leqslant i\leqslant r$. For each $i$, $1\leqslant i\leqslant r-1$, let $X_i = f^{-1}(I^k\times a_i)$, let $f_i' = f|X_i$, and let $A_i = A'\cap f^{-1}(X_i)$. Clearly, each $f_i$ is a homotopy 2-regular open mapping of the complete metric space $X_i$ onto the $k$-cell $I^k\times a_i$ such that for each $(y, a_i)$ in $I^k\times a_i$, the inverse of $(y, a_i)$ under $f_i'$ is homeomorphic to $E^3$. Also, $f_i'(A_i)$ is contained in $I^k\times a_i$. Hence from the induction hypothesis, there is a closed neighborhood $M_i'$ of $A_i$ in $X_i$ and a homeomorphism $\xi_i'$ from $M_i'$ onto $I^k\times B$ such that $\pi\circ\xi_i' = f_i'$.

For each $(y, z)$ in $I^k\times I$, let $G'_{(y,z)}$ denote the set of tame imbeddings $\gamma'_{(y,z)}$ of $S^2$ into $f^{-1}(y, z)$ such that $\gamma'_{(y,z)}(S^2)$ contains $A'\cap f^{-1}(y, z)$ in its interior. Let $G^* = \bigcup G'_{(y,z)}$ and topologize $G^{*\prime}$ via the metric given by

$$d(\gamma', \gamma'') = \sup\{d(\gamma'(x), \gamma''(x)),\ x\in S^2\}\,.$$

For each $i$, $1\leqslant i\leqslant r$, define a mapping $g_i'$ from $I^k\times a_i$ into $G^{*\prime}$ as follows. For each $(y, a_i)$ in $I^k\times a_i$, let $(g_i(y, a_i))(s) = \xi_i'^{-1}(y, s)$ for all $s$ in $S^2$.

For each $i$, $1\leqslant i\leqslant r-1$, there are numbers $c_i$ and $d_i$, $c_i<a_i<d_i$, and a mapping $H_i'$ from $I^k\times[c_i, d_i]$ into $G^{*\prime}$ such that (i) $H_i'$ extends $g_i'$, and (ii) $F\circ H_i'$ is the injection of $I^k\times[c_i, d_i]$ into $I^k\times I$. The $c_i$'s and $d_i$'s may be taken so that

$$0 = a_0<c_1<a_1<d_1<c_2<a_2<d_2<\ldots<c_{r-1}<a_{r-1}<d_{r-1}<a_r = 1\,.$$



As in Section 3, define

$$\zeta_i'\colon I^k\times[c_i, d_i]\times S^2\to f^{-1}(I^k\times[c_i, d_i])$$

by

$$\zeta_i'(w, y, z) = \big(H_i'(w, y)\big)(z)\,.$$

Let

$$N_i' = \bigcup \mathrm{Cl}(\mathrm{Int}\,\zeta_i'(w, y, S^2)),\qquad (w, y)\in I^k\times[c_i, d_i]\,,$$

where interiors are taken relative to the point inverses $f^{-1}(w, y)$. The arguments used in Section 3 apply here to prove that (a) there is a homeomorphism $h_i'$ from $N_i'$ onto $B'\times I^k\times[c_i, d_i]$ (where $B'$ denotes a standard 3-cell) such that $\pi\circ h_i' = f|N_i'$, and (b) the interior of $N_i'$ contains $A'\cap f^{-1}(I^k\times(c_i, d_i))$.

The set $N_i'$ is compact and hence closed in $X$. Redefine $N_i$, $1\leqslant i\leqslant r$ as follows.

$$N_1 = h_1^{-1}(B\times I^k\times[0, c_1]),\qquad N_r = h_r^{-1}(B\times I^k\times[d_{r-1}, 1])\,,$$

and for $2\leqslant i\leqslant r-1$,

$$N_i = h_i^{-1}(B\times I^k\times[d_{i-1}, c_i])\,.$$

Let

$$T_i = h_i^{-1}(B\times I^k\times c_i) = (f|N_i)^{-1}(I^k\times c_i)$$

and let

$$T_i' = h_i'^{-1}(B'\times I^k\times c_i) = (f|N_i')^{-1}(I^k\times c_i)\,.$$

By construction, $T_i\subset T_i'$. Let $\alpha_i$ denote the inclusion mapping of $T_i$ into $T_i'$. Let $\bar f = f|\mathrm{Cl}(T_i'-T_i)$. The mapping $h_i'|\mathrm{Cl}(T_i'-T_i)$ takes $\mathrm{Cl}(T_i'-T_i)$ onto

$$\mathrm{Cl}\,[B'\times I^k\times c_i-(h_i'\circ\alpha_i\circ h_i^{-1})(B\times I^k\times c_i)]\,,$$

and $\pi\circ h_i'|\mathrm{Cl}(T_i'-T_i) = \bar f$.

For each $(y, c_i)$ in $I^k\times c_i$, $\bar f^{-1}(y, c_i)$ is homeomorphic to the shell between two 3-cells. Let $L'$ $(L)$ denote a standard 3-cell with radius $1$ $(\frac12)$. Let $K = \mathrm{Cl}(L'-L)$. Then $\bar f$ is a mapping of $\mathrm{Cl}(T_i'-T_i)$ onto $I^k\times c_i$ such that each point inverse is homeomorphic to $K$.

The mapping $\bar f$ is completely regular. For suppose that $(y_k)$ converges to $y_0$ in $I^k\times c_i$. The mappings $f|T_i'$ and $f|T_i$ are completely regular, since (a) The sets $T_i'$ and $T_i$ are compact, and (b) $\pi\circ h_i'|T_i' = f|T_i'$ and $\pi\circ h_i|T_i = f|T_i$.

For $k = 1, 2, \ldots$, let $M_k$ denote the 3-cell $(f|T_i')^{-1}(y_k)$, and let $Q_k$ denote the 3-cell $(f|T_i)^{-1}(y_k)$. The sequences $(M_k)$ and $(Q_k)$ converge completely regularly to $M_0$ and $Q_0$, respectively. Let $S_k$ denote the 2-sphere boundary of $Q_k$. The sequence $(S_k)$ converges homotopy 2-regularly to $S_0$. Let $R_k = \mathrm{Cl}(M_k-Q_k) = \bar f^{-1}(y_k)$. The proof of Theorem F applies to prove that $(R_k)$ converges homotopy 2-regularly to $R_0$. Then from Theorem C, $\bar f$ is completely regular.

Theorem H implies that the space of homeomorphisms from $K$ onto itself is locally connected. Hence from Theorem A, there is a homeomorphism $H_i$ from $\mathrm{Cl}(T_i'-T_i)$ onto $K\times I^k\times c_i$ such that $\pi\circ H_i = \bar f$.

We extend $H_i$ to a homeomorphism of $T_i$ onto $L\times I^k\times c_i$ as follows. The hypotheses of Theorem B are satisfied. In fact,

(a) $f|T_i$: $T_i\to I^k\times c_i$ is completely regular,

(b) for each $(y, c_i)$ in $I^k\times c_i$, $(f|T_i)^{-1}(y, c_i)$ is homeomorphic to the 3-cell $L$, which is the cone on the 2-sphere $S^2$, and

(c) $H_i|H_i^{-1}(S^2\times I^k\times c_i)$ is a homeomorphism of $H_i^{-1}(S^2\times I^k\times c_i)$ onto $S^2\times I^k\times c_i$ such that

$$\pi\circ H_i|H_i^{-1}(S^2\times I^k\times c_i) = f|H_i^{-1}(S^2\times I^k\times c_i)\,.$$

Hence there is a homeomorphism $\overline{H}_i$ from $T_i$ onto $L\times I^k\times c_i$ which extends $H_i|H_i^{-1}(S^2\times I^k\times c_i)$, such that $\pi\circ\overline{H}_i = f|T_i$. Define the homeomorphism $H_i^*$ from $T_i'$ onto $L'\times I^k\times c_i$ by $H_i^*|\mathrm{Cl}(T_i'-T_i) = H_i$ and $H_i^*|T_i = \overline{H}_i$.

Next, we extend $H_i^*$ to a homeomorphism $\Theta_i$ from $N_i'$ onto $L'\times I^k\times[c_i, d_i]$ such that $\pi\circ\Theta_i = f|N_i'$. The mapping $H_i^*\circ h_i'^{-1}(B'\times I^k\times c_i)$ takes $B'\times I^k\times c_i$ homeomorphically onto $L'\times I^k\times c_i$, or, if the coordinate $c_i$ is suppressed, onto $L'\times I^k$. If for each $e_i$, $c_i\leqslant e_i\leqslant d_i$, we refer to $B'\times I^k\times e_i$ as the "$e_i$th level" of $B'\times I^k\times[c_i, d_i]$, then the mapping $\beta_i$ of $B'\times I^k\times[c_i, d_i]$ onto $L'\times I^k\times[c_i, d_i]$ defined by

$$\beta_i(w, y, z) = [(H_i^*\circ h'^{-1}|\ B'\times I^k\times c_i)(w, y, c_i), z]$$

is a homeomorphism that preserves levels, and acts on each level as $H_i^*\circ h_i'^{-1}|\ B'\times I^k\times c_i$ does on the $c_i$th level. Then $\Theta_i = \beta_i\circ h_i'$ is the desired extension of $H_i^*$.

4.2. LEMMA. *For each positive integer $k$ there is an isotopy $\{\Phi_t, 0\leqslant t\leqslant 1\}$ from $L\times I^k$ into $L'\times I^k$ such that*

(i) *$\Phi_0$ is the inclusion mapping,*

(ii) *$\Phi_1$ is a homeomorphism from $L\times I^k$ onto $L'\times I^k$,*

(iii) *if $t'>t$, then $\Phi_{t'}(L\times I^k)$ contains $\Phi_t(L\times I^k)$,*

(iv) *each $\Phi_t$ is level preserving, that is, $p\circ\Phi_t = q$ (where $p$ and $q$ are the apparent projections), and*

(v) *for each $t$, $0\leqslant t\leqslant 1$, and each $z$ in $I^k$, $\Phi_t(L\times\{z\})$ is tame in $L'\times\{z\}$.*

Proof. Use radial extensions of the inclusion mapping of $L\times I^k$ into $L'\times I^k$.

In the space $T_i'$, $A\cap T_i'\subset\mathrm{Int}\,T_i$. Therefore, in the space $L'\times I^k\times c_i$, $\Theta_i(A)\cap L'\times I^k\times c_i\subset\mathrm{Int}(L\times I^k\times c_i)$. This and the compactness of $\Theta_i(A)$ imply that there is some $\varepsilon_i$, $0<\varepsilon_i<(d_i-c_i)$, such that for each $z$ in $[c_i, c_i+\varepsilon_i]$,

$$\Theta_i(A)\cap(L'\times I^k\times z)\subset\mathrm{Int}(L\times I^k\times z)\,.$$

Let

$$\psi_i\colon L\times I^k\times[c_i, c_i+\varepsilon_i]\to L'\times I^k\times[c_i, c_i+\varepsilon_i]$$

be defined by

$$\psi_i(x, y, z) = (\Phi_{(z-c_i)/\varepsilon_i}(x, y), z)\,.$$



This mapping is a level preserving homeomorphism such that (a) $\psi_i|L\times I^k\times c_i$ is the inclusion mapping of $L\times I^k\times c_i$ into $L'\times I^k\times c_i$, and (b) $\psi_i|L\times I^k\times(c_i+\varepsilon_i)$ is a homeomorphism onto $L'\times I^k\times(c_i+\varepsilon_i)$. Let

$$N_i'' = (\Theta_i^{-1}\circ\psi_i)(L\times I^k\times[c_i, c_i+\varepsilon_i]\,.$$

Then $N_i''$ is a $(k+4)$-cell contained in $h_i'^{-1}(B'\times I^k\times[c_i, c_i+\varepsilon_i])$ which in turn is a subset of $N_i'$. The set $N_i\cup N_i''$ is a $(k+4)$-cell which extends $N_i$.

In a manner similar to the derivation of $N_i''$, we can define a $(k+4)$-cell $M_i''$ such that (a) for some number $\eta_i$ for which $c_i<c_i+\varepsilon_i<c_i+\eta_i<d_i$, the set $M_i''$ is a subset of $h_i'^{-1}(B'\times I^k\times[c_i+\eta_i, d_i])$, which in turn is a subset of $N_i'$, and (b) $N_{i+1}\cup M_i''$ is a $(k+4)$-cell that extends $N_{i+1}$.

Let

$$N_A = N_r\cup\bigcup_{j=1}^{r-1}[N_i\cup N_i''\cup h_i'^{-1}(B'\times I^k\times[c_i+\varepsilon_i, c_i+\eta_i])\cup M_i'']\,.$$

We assert that $N_A$ is the desired $(k+4)$-cell of the induction step of Lemma 4.1. It must be shown that

(i) $A\subset\mathrm{Int}\,N_A$ and $N_A$ is closed in $X$,

(ii) for each $(y, z)$ in $I^k\times I$, the 3-cell $N_A\cap f^{-1}(y, z)$ is tame in $f^{-1}(y, z)$, and

(iii) there is a homeomorphism $H$ from $N_A$ onto $B''\times I^k\times I$ (where $B''$ denotes a standard 3-cell) such that $\pi\circ H = f|N_A$.

In Lemma 4.2, for each $t$ in $[0, 1]$ and each $y$ in $I^k$, the 3-cell $\Phi_t(L\times y)$ is tame in $L'\times y$. Therefore, for each $y$ in $I^k$ and each $z$ in $[c_i, c_i+\varepsilon_i]$, the 3-cell $\psi_i(L\times y\times z)$ is tame in $L'\times y\times z$. Hence the 3-cell $(\Theta_i^{-1}\circ\psi_i)(L\times y\times z)$ is tame in $\Theta_i^{-1}(L'\times y\times z)$. Also, $\Theta_i^{-1}(L'\times y\times z)$ is tame in $f^{-1}(y, z)$ since its boundary is the image of a tame imbedding of $S^2$ into $j^{-1}(y, z)$.

Let $z$ be a point either in $(c_i, c_i+\varepsilon_i)$ or in $(c_i+\eta_i, d_i)$. Let $C$ and $C'$ denote $(\Theta_i^{-1}\circ\psi_i)(L\times y\times z)$ and $\Theta_i^{-1}(L'\times y\times z)$, respectively. Since $C'$ is tame in $f^{-1}(y, z)$, there is a homeomorphism $g$ of $f^{-1}(y, z)$ onto $E^3$ such that $g(C')$ is a standard 3-cell in $E^3$. Since $C$ is tame in $C'$, there is a homeomorphism $\xi$ from $C'$ onto $B_1'$, the standard 3-cell of radius 1 in some Euclidean 3-space $E'^3$, which sends $C$ onto $B_{\frac12}'$, the standard 3-cell in $E'^3$ of radius $\frac12$. Then $\xi\circ g^{-1}|g(C')$ is a homeomorphism from $g(C')$ onto $B_1'$, and

$$(\xi\circ g^{-1}|g(C'))(g(C)) = \xi(C) = B_{\frac12}'\,.$$

Extend $\xi\circ g^{-1}$ radially to a homeomorphism $H$ of $E^3$ onto $E'^3$. Then $H\circ g$ is a homeomorphism from $f^{-1}(y, z)$ onto $E'^3$ that carries $C$ onto $B_{\frac12}'$. Hence

$$C = (\Theta_i^{-1}\circ\psi_i)(L\times y\times z) = N_A\cap f^{-1}(y, z)$$

is tame in $f^{-1}(y, z)$.

If $z = c_i+\varepsilon_i$ or $z = c_i+\eta_i$, then

$$(\Theta_i^{-1}\circ\psi_i)(L\times y\times z) = N_A\cap f^{-1}(y, z) = \Theta_i^{-1}(L'\times y\times z),$$

which, as stated above, is tame in $f^{-1}(y, z)$.

For the points $z$ in $I$ that are not contained in any of the $[c_i, c_i+\varepsilon_i]$ or $[c_i+\eta_i, d_i]$, there is some $i$, $1\leqslant i\leqslant r$, such that $N_A\cap f^{-1}(y, z)$ is equal to either $N_i\cap f^{-1}(y, z)$ or $N_i'\cap f^{-1}(y, z)$, each of which, from the definitions of $N_i$ and $N_i'$, is tame in $f^{-1}(y, z)$.

Let $s = 4(r-1)+1$. Let

$$R_1 = N_1,\quad R_2 = N_1'',\quad R_3 = h_1'^{-1}(B'\times I^k\times[c_1+\varepsilon_1, c_1+\eta_1]),\quad R_4 = M_1'',$$
$$R_5 = N_2,\ \ldots,\ R_{s-1} = M_{r-1}'',\quad R_s = N_r.$$

For each $R_j$, $1\leqslant j\leqslant s$, there is a homeomorphism $\gamma_j$ from $R_j$ onto $B''\times I^k\times[\alpha_{j-1}\alpha_j]$ such that $\pi\circ\gamma_j = f|R_j$, where $\alpha_0 = 0$, $\alpha_1 = c_1$, $\alpha_2 = c_1+\varepsilon_1$, $\alpha_3 = c_1+\eta_1$, $\alpha_4 = d_1$, ..., $\alpha_{s-4} = c_{r-1}$, $\alpha_{s-3} = c_{r-1}+\varepsilon_{r-1}$, $\alpha_{s-2} = c_{r-1}+\eta_{r-1}$, $\alpha_{s-1} = d_{r-1}$, $\alpha_s = 1$.

Let $H_1 = \gamma_1$. Then $\gamma_1\circ\gamma_2^{-1}|B''\times I^k\times\alpha_1$ is a level preserving homeomorphism of $B''\times I^k\times\alpha_1$ onto itself. Define the homeomorphism $T_2$ from $B''\times I^k\times[\alpha_1, \alpha_2]$ onto itself by

$$T_2(w, y, z) = [(\gamma_1\circ\gamma_2^{-1}|B''\times I^k\times\alpha_1)(w, y, \alpha_1), z].$$

Let $H_2 = T_2\circ\gamma_2$. Continue in this fashion, and define $H_j$ for $1\leqslant j\leqslant s$. Now define the mapping $H$ from $N_A$ onto $B''\times I^k\times[0, 1]$ by $H|R_j = H_j$, $1\leqslant j\leqslant s$. Clearly, (a) $H$ is a homeomorphism onto, and (b) $\pi\circ H = f|N_A$.

By the construction of $N_A$, for each $y$ in $I^k\times[0, 1]$, $f^{-1}(y)\cap N_A$ is a 3-cell that contains $f^{-1}(y)\cap A$ in its interior. Then, as in the proof in Section 3 that $p$ is in $\operatorname{Int}N$, it follows that $A\subset\operatorname{Int}N_A$.

The compact set $N_A$ is closed in $X$. This completes the induction step of the proof of Lemma 4.1.

4.3. LEMMA. *If $A$ is a compact subset of $X$, there is some open superset $U$ of $A$ that is homeomorphic to $E^{n+3}$ and whose closure is homeomorphic to $B^{n+3}$.*

Proof. Since $f(A)$ is compact, it is contained in some $n$-cell $K = [0, m]^n$. Let $X' = f^{-1}(K)$. From Lemma 4.1, there is a closed subset $M$ of $X'$ that is a closed $(n+3)$-cell such that $A$ is contained in the open $(n+3)$-cell $\operatorname{Int}M$. From Corollary 3.16, $X'$ is an $(n+3)$-manifold. Therefore, by the Brouwer domain invariance theorem, $\operatorname{Int}M$ is open in $X'$. Let $U = \operatorname{Int}M$.

4.4. PROPOSITION. *Let $X$ be a complete metric space and $f$ a continuous, open, homotopy 2-regular mapping of $X$ onto Euclidean $n$-space $E^n$, $n$ a nonnegative integer, such that for each $y$ in $E^n$, $f^{-1}(y)$ is homeomorphic to $E^3$. Then $X$ is homeomorphic to $E^{n+3}$.*



Proof. We use the following characterization of $E^n$ due to Brown [2, p. 812]. Let $X$ be a topological space which is the union of a sequence $V_1\subset V_2\subset\ldots$ of open subsets, where each $V_i$ is homeomorphic to $E^n$. Then $X$ is homeomorphic to $E^n$.

Since $X$ is a locally compact separable metric space (Lemmas 3.1 and 3.15), it has a countable open basis $\{U_i\}$ of relatively compact sets. From Lemma 4.3, there is some open subset $V_1$ of $X$ such that $V_1$ is homeomorphic to $E^{n+3}$, $\operatorname{Cl}(V_1)$ is homeomorphic to $B^{n+3}$, and $\operatorname{Cl}(U_1)$ is contained in $V_1$. Let $W_2 = \operatorname{Cl}(U_2)\cup\operatorname{Cl}(V_1)$. There is an open subset $V_2$ of $X$ such that $V_2$ is homeomorphic to $E^{n+3}$, $\operatorname{Cl}(V_2)$ is homeomorphic to $B^{n+3}$, and $W_2\subset V_2$. Let $W_3 = \operatorname{Cl}(U_3)\cup\operatorname{Cl}(V_2)$ and repeat this process. Continuing in this fashion, we obtain a sequence $V_1\subset V_2\subset\ldots$ of open subsets of $X$ that satisfies the hypothesis of Brown's Theorem.

4.5. THEOREM. *Under the hypotheses of Proposition 4.4, $(X, f, E^n, E^3)$ is a trivial fiber space, that is, there is a homeomorphism $\Phi$ from $X$ onto $E^{n+3} = E^n\times E^3$ such that $f\leqslant\pi\circ\Phi$, where $\pi$ denotes the projection from $E^n\times E^3$ onto $E^n$.*

Proof. We will show that $(X, f, E^n, E^3)$ is locally trivial. Then, since $E^n$ is contractible, locally compact, and separable, $(X, f, E^n, E^3)$ is trivial ([12], p. 53). Let $y$ in $E^n$ be given. Let $t>0$ be such that $y$ is contained in the interior of $J^n = [-t, t]^n$. For $i = 1, 2, \ldots$, let $L_i = \{\alpha x|\ x\in E^3,\ |x| = 1, 0\leqslant\alpha\leqslant i\}$, i.e., the closed 3-ball in $E^3$ of radius $i$. The space $f^{-1}(J^n)$ is a locally compact metric space. Hence, it has a countable basis, say $\{U_n\}$, consisting of relatively compact open subsets.

From Lemma 4.1, there is a closed neighborhood $K_1$ of $\overline{U}_1$ in $f^{-1}(J^n)$ and a homeomorphism $h_1$ from $K_1$ onto $L_1\times J^n$ such that $\pi\circ h_1 = f|K_1$. Again, from Lemma 4.1 there is a closed neighborhood $K_2$ of the compact set $K_1\cup\overline{U}_2$ in $f^{-1}(J^n)$ and a homeomorphism $h_2$ from $K_2$ onto $L_2\times J^n$ such that $\pi\circ h_2 = f|K_2$. In this fashion, we define sets $K_i$ and homeomorphisms $h_i$ such that (a) $f^{-1}(J^n) = \bigcup K_i$, (b) for each $i$, $K_{i+1}$ is a closed neighborhood of $K_i$ in $f^{-1}(J^n)$, and (c) for each $i$, $\pi\circ h_i = f|K_i$.

In the proof of Lemma 4.1, we considered subsets $T_i'$ and $T_i$ of $X$, such that $T_i'$ ($T_i$) is homeomorphic to $B'\times I^n$ ($B\times I^n$) under a fiber preserving homeomorphism $h_i'$ ($h_i$). We derived a fiber preserving homeomorphism $H_i^*$ from $T_i'$ onto $L'\times I^n$ such that $H_i^*|T_i$ is a homeomorphism from $T_i$ onto $L\times I^n$, where $L'$ ($L$) is a standard 3-cell of radius 1 ($\frac{1}{2}$). In the present case, we argue similarly to show the existence of homeomorphisms $H_i^*$ from $K_i$ onto $L_i\times J^n$ such that (a) $\pi\circ H_i^* = f|K_i$, and (b) $H_i^*|K_{i-1}$ is a homeomorphism from $K_{i-1}$ onto $L_{i-1}\times J^n$.

Let $\delta_1 = h_1$. For each $y$ in $J^n$, $\delta_1\circ H_2^{*-1}|\operatorname{Bdry}L_1\times\{y\}$ is a homeomorphism from $\operatorname{Bdry}L_1\times\{y\}$ onto itself. Let $\varrho_y$ denote its radial extension to a homeomorphism of $\operatorname{Cl}(L_2-L_1)\times\{y\}$ onto itself. Let $g_2$ be the homeomorphism of $\operatorname{Cl}(L_2-L_1)\times J^n$ onto itself defined by $g_2(x, y) = (\varrho_y(x), y)$ for each $(x, y)$ in $\operatorname{Cl}(L_2-L_1)\times J^n$.

Define $\delta_2\colon K_2\to L_2\times J^n$ by

$$\delta_2|K_1 = \delta_1,\qquad \delta_2|\operatorname{Cl}(K_2-K_1) = g_2\circ H_2^*.$$

Similarly, define $\delta_3, \delta_4, \ldots$ such that for each $i$, $\delta_i$ is a homeomorphism of $K_i$ onto $L_i\times J^n$ for which (a) $\pi\circ\delta_i = f|K_i$, and (b) $\delta_i|K_{i-1} = \delta_{i-1}$.

Finally, define the homeomorphism $T$ from $f^{-1}(J^n) = \bigcup K_i$ onto $E^3 \times J^n$ by $T|K_i = \delta_i$. Clearly $\pi \circ T = f|f^{-1}(J^n)$.

Remark. Theorem 4.5 does not imply that $f$ is completely regular. An example is given by Seidman [11, p. 465] of a metric for $E^1 \times E^1$ that yields the product topology, but that with respect to this metric, the projection mapping onto the first factor is not completely regular. However, if the usual metric on $E^3 \times E^n$ is imposed upon $X$ under some homeomorphism which satisfies the conclusion of theorem, then $f$ is completely regular.

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# Neighborhoods of compacta in euclidean space

by

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**Abstract.** In this paper the question of when a compact subset of Euclidean $n$-space has arbitrarily small piecewise linear neighborhoods with $k$-dimensional spines is considered. A theory is developed which completely answers the question in terms of the fundamental dimension of the compactum and an embedding condition which is a weak form of the cellularity criterion. The theory is the shape theoretical analogue of the demension theory of M. A. Štańko.

**1. Introduction.** Suppose $X$ is a compact subset of Euclidean $n$-space $E^n$. It is, of course, well-known that $X$ has arbitrarily small piecewise linear (PL) neighborhoods with $(n-1)$-dimensional spines. We want to determine the smallest value of $k$ such that $X$ has arbitrarily small PL neighborhoods with $k$-dimensional spines. This problem leads naturally to shape theory, since if $X$ has arbitrarily small PL neighborhoods with $k$-dimensional spines, then $X$ has the shape of the inverse limit of an inverse sequence of $k$-dimensional polyhedra and thus has fundamental dimension $\leqslant k$. Hence we immediately see that the fundamental dimension of $X$ is a lower bound for the possible values of $k$.

In this paper we present a theory which tells exactly when that lower bound is achieved and what the smallest value of $k$ is otherwise. Our theory is the shape theory analogue of the demension theory of M. A. Štańko [13]. Štańko looks for neighborhoods which not only have $k$-dimensional spines, but also have small retractions onto the spines. (A precise statement of Štańko's results is given below). In our theory the fundamental dimension plays the role of the covering dimension in Štańko's theory and a weak form of McMillan's cellularity criterion (the inessential loops condition) plays the role of the 1-ULC property. Our theory unifies the various proofs of finite dimensional complement theorems which have appeared in [8], [3], [5], [6], [14] and [11] since the main step in each of those proofs involves finding small neighborhoods of compacta with $k$-dimensional spines where $2k+2 \leqslant n$. Before stating our main result (Theorem 1.4) we define the terms used.

Definition 1.1 [1, p. 227]. The *fundamental dimension* of a compactum $X$ is defined by $\mathrm{Fd}(X) = \min\{\dim Y | \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)\}$.

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