Flipping properties and supercompact cardinals *

by

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Abstract. We give a characterization of supercompact cardinals in terms of a combinatorial property. Weakening this property we obtain a second combinatorial property which characterizes strongly compact cardinals. We prove that a more uniform version of this second property is again equivalent to supercompactness.

In a paper entitled "Flipping properties and large cardinals", Abramson, Harrington, Kleinberg and Zwicker [1] study a certain type of combinatorial property, the so called flipping properties, which link, in a uniform way, the different notions used to define various large cardinals.

In this paper we continue the same line, introduce new flipping type properties and discuss their relationship with λ-ineffability, strong-compactness and supercompactness.

Let $\varkappa$ be an uncountable cardinal and $\lambda \geqslant \varkappa$ be an ordinal. $P_\varkappa(\lambda)$ is the sets of subsets of λ of cardinality less than $\varkappa$. The definition of super compact cardinals was introduced in [4].

$\varkappa$ is *λ-supercompact* if there is an ultrafilter U on $P_\varkappa(\lambda)$ such that

(i) For all $P \in P_\varkappa(\lambda)$, $\hat{P} = \{Q \in P_\varkappa(\lambda) |\ P \subseteq Q\} \in U$ (U is "fine").

(ii) For every collection $\{A_\alpha |\ \alpha < \delta < \varkappa\}$ such that $A_\alpha \in U$ for all $\alpha < \delta$, $\bigcap_{\alpha<\delta} A_\alpha \in U$ (U is "$\varkappa$-complete").

(iii) If $\{A_\alpha |\ \alpha < \lambda\}$ is a λ-sequence of element of U then the diagonal intersection of the A_α's, $\triangle_{\alpha<\lambda} A \in U$ where $\triangle_{\alpha<\lambda} A_\alpha = \{P \in P_\varkappa(\lambda) |\ P \in A \text{ for each } \alpha \in P\}$ (U is "normal").

$\varkappa$ is *supercompact* if it is λ-supercompact for all $\lambda \geqslant \varkappa$.

Standard notation is used. If A is a set $\mathscr{P}(A)$ denotes A's power set, and $|A|$ is A's cardinality. We will switch freely between discussing and ultrafilter U and its associated measure μ: $\mathscr{P}(P_\varkappa(\lambda)) \to 2$. If A and B are sets, ${}^B A$ is the set of functions with domain B and range contained in A. Lower case Greek letters name ordinals. $\varkappa^\lambda$ is the cardinality of ${}^\lambda\varkappa$ and $\lambda^{\underset{\smile}{\varkappa}} = |\bigcup_{\alpha<\varkappa} \lambda^\alpha|$.

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Jech [2] defined closed, unbounded, and stationary subsets of $P_\varkappa(\lambda)$ and showed that any member of a normal $\varkappa$-complete fine ultrafilter is stationary. Magidor used the definition to generalize the notion of ineffability as follows:

Let $\varkappa$ be a cardinal and $\lambda \geqslant \varkappa$ an ordinal. $\varkappa$ is called *λ-ineffable* if for any function $f\colon P_\varkappa(\lambda) \to P_\varkappa(\lambda)$ such that $f(P) \subseteq P$ for all $P \in P_\varkappa(\lambda)$, there is a subset A of λ such that the set $\{P \in P_\varkappa(\lambda) |\ A \cap P = f(P)\}$ is stationary.

Following the ideas of [1], if $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$, we call t' a *flip* of t ($t' \sim t$) f $t'\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ and for all $\alpha < \lambda$, $t'(\alpha) = t(\alpha)$ or $t'(\alpha) = P_\varkappa(\lambda) - t(\alpha)$.

THEOREM 1. *The following are equivalent:*

(i) *$\varkappa$ is supercompact.*

(ii) *$\varkappa$ is λ-ineffable for all $\lambda \geqslant \varkappa$.*

(iii) *For all $\lambda \geqslant \varkappa$, if $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ then there is a $t' \sim t$ such that $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary.*

Proof. The proof will follow from the next three lemmas. Lemma 1 establishes the equivalence of (ii) and (iii) "level by level", that is to say for each $\lambda \geqslant \varkappa$, $\varkappa$ is λ-ineffable if and only if for each λ-sequence of subsets of $P_\varkappa(\lambda)$ there is a flip of the sequence which has stationary diagonal intersection. In Lemma 2 we obtain the same flipping property from the assumption that $\varkappa$ is λ-supercompact. Lemma 3 shows how to build fine $\varkappa$-additive normal ultrafilters by using (iii).

The equivalence of (i) and (ii) was first proved by Magidor in [3] using some embedding properties of supercompact cardinals. The combinatorial proof that follows seems to us a much simpler one. (Stan Wagon has informed us that he and J. Baumgartner also obtained a combinatorial proof of Magidor's theorem).

LEMMA 1. *A cardinal $\varkappa$ is λ-ineffable if and only if for all $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ there is a t' such that $t' \sim t$ and $\underset{\alpha<\lambda}{\triangle} t'(\alpha) = \{P \in P_\varkappa(\lambda) |\ P \in t'(\alpha)$ for all $\alpha \in P\}$ is stationary. That is to say, $\varkappa$ is λ-ineffable if and only if for any λ-sequence of subsets of $P_\varkappa(\lambda)$ there is a flip of the sequence with stationary diagonal intersection.*

Proof. Suppose $\varkappa$ is λ-ineffable, and let $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$. Define $f\colon P_\varkappa(\lambda) \to P_\varkappa(\lambda)$ by $f(P) = \{\alpha \in P |\ P \in t(\alpha)\}$. Let now $A \subseteq \lambda$ be such that $S = \{P \in P_\varkappa(\lambda) |\ A \cap P = f(P)\}$ is stationary. Define $t'\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ as follows: $t'(\alpha) = t(\alpha)$ if $\alpha \in A$ and $t'(\alpha) = P_\varkappa(\lambda) - t(\alpha)$ if $\alpha \notin A$.

Suppose $P \in S$ (i.e. $P \cap A = f(P)$) and let $\alpha \in P$. If $\alpha \in A$ then $\alpha \in f(P)$ and $P \in t(\alpha) = t'(\alpha)$. If $\alpha \notin A$ then $\alpha \notin f(P)$ and $P \notin t(\alpha)$, hence $P \in P_\varkappa(\lambda) - t(\alpha) = t'(\alpha)$. In any case, $P \in t'(\alpha)$. So we have shown that $S \subseteq \underset{\alpha<\lambda}{\triangle} t'(\alpha)$, which must therefore be stationary.

Conversely, let $f\colon P_\varkappa(\lambda) \to P_\varkappa(\lambda)$ be such that for all P $f(P) \subseteq P$. Define $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ by $t(\alpha) = \{P \in P_\varkappa(\lambda) |\ \alpha \in f(P)\}$. Let t' be such that $t' \sim t$ and $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary.

Put $A = \bigcup \{f(P) |\ P \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)\}$. Let's check that if $P \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$ then $P \cap A = f(P)$. Obviously $f(P) \subseteq P \cap A$, so we have only to check that $A \cap P \subseteq f(P)$. By the definition of A, this is true if for any $Q \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$ we have $P \cap f(Q) \subseteq f(P)$. Indeed, if $Q \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$, let $\alpha \in P \cap f(Q)$ then $Q \in t(\alpha)$. On the other hand as $Q \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$, $Q \in t'(\alpha)$ so $t'(\alpha) = t(\alpha)$. But $\alpha \in P$ and $P \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$, so $P \in t'(\alpha) = t(\alpha)$ and hence $\alpha \in f(P)$. ■

LEMMA 2. *If $\varkappa$ is λ-supercompact then for each $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ there is a $t' \sim t$ such that $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary.*

Proof. Let μ be a normal measure on $P_\varkappa(\lambda)$, and let $t\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$. Define $t'\colon \lambda \to \mathscr{P}(P_\varkappa(\lambda))$ by

$$t'(\alpha) = \begin{cases} t(\alpha) & \text{if } \mu(t(\alpha)) = 1\,, \\ P_\varkappa(\lambda) - t(\alpha) & \text{otherwise}\,. \end{cases}$$

Then $\mu(\underset{\alpha<\lambda}{\triangle} t'(\alpha)) = 1$ and therefore $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary. ■

LEMMA 3. *Property* (iii) *of the theorem implies that $\varkappa$ is supercompact.*

Proof. Let $\gamma = 2^{(\lambda^{\underset{\sim}{\varkappa}})}$, and well order the subsets of $P_\varkappa(\lambda)$ by ordinals less than γ. So for each $\alpha < \gamma$, A_α is the αth subset of $P_\varkappa(\lambda)$.

For every $A \subseteq P_\varkappa(\lambda)$ let $F(A) = \{Q \in P_\varkappa(\gamma) |\ Q \cap \lambda \in A\}$, and define $t\colon \gamma \to \mathscr{P}(P_\varkappa(\gamma))$ by $t(\alpha) = F(A_\alpha)$ for all $\alpha < \gamma$. Let t' be such that $t' \sim t$ and $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary.

We define a filter U on $P_\varkappa(\lambda)$ as follows: if $A \subseteq P_\varkappa(\lambda)$, then $A = A_\alpha$ for some $\alpha < \gamma$. Let $A \in U$ if and only if $t'(\alpha) = t(\alpha)$.

It is easy to check that U is in fact a filter on $P_\varkappa(\lambda)$. We will now prove that U defines a fine, $\varkappa$-additive, normal measure on $P_\varkappa(\lambda)$.

a) To show that U is fine, let $P \in P_\varkappa(\lambda)$, then $\hat{P} = A_\alpha$ for some $\alpha < \gamma$. If $\hat{P} \notin U$ then $t'(\alpha) = P_\varkappa(\gamma) - F(A_\alpha)$. And if $Q \in (\underset{\beta<\gamma}{\triangle} t'(\beta)) \cap \{\hat{\alpha}\}$ then $Q \in P_\varkappa(\gamma) - F(A_\alpha)$, i.e. $P \nsubseteq Q$. We have then shown $(\underset{\beta<\gamma}{\triangle} t'(\beta)) \cap \{\hat{\alpha}\} \cap \{Q |\ Q \supseteq P\} = \emptyset$ which contradicts the fact that $\underset{\beta<\gamma}{\triangle} t'(\beta)$ is stationary.

b) To see that U is an ultrafilter, let $A_\alpha \subseteq P_\varkappa(\lambda)$. If $A_\alpha \notin U$ then $t'(\alpha) = P_\varkappa(\lambda) - F(A_\alpha)$. Let $A_\beta = P_\varkappa(\lambda) - A_\alpha$. Suppose $A_\beta \notin U$ then $t'(\beta) = P_\varkappa(\gamma) - F(A_\beta) = F(P_\varkappa(\lambda) - A_\beta) = F(A_\alpha)$. So $(\underset{\xi<\gamma}{\triangle} t'(\xi)) \cap \widehat{\{\alpha, \beta\}} = \emptyset$, a contradiction. Hence $A_\beta = P_\varkappa(\lambda) - A_\alpha \in U$. Similarly if $A_\alpha \in U$ then $P_\varkappa(\lambda) - A_\alpha \notin U$.

c) U is $\varkappa$-complete. Let $P \in P_\varkappa(\gamma)$ be such that for any $\alpha \in P$, $A_\alpha \in U$. Suppose $\bigcap \{A_\alpha |\ \alpha \in P\} \notin U$. First, $\bigcap \{A_\alpha |\ \alpha \in P\} = A_\beta$ for some $\beta < \gamma$, so $t'(\beta) = P_\varkappa(\gamma) - F(A_\beta) = F(P_\varkappa(\lambda) - A_\beta)$. Let $Q \in P_\varkappa(\gamma)$ s.t. $Q \supseteq P \cup \{\beta\}$. If $Q \in \underset{\xi<\gamma}{\triangle} t'(\xi)$ then $Q \in F(A_\alpha)$ for all $\alpha \in P$ and $Q \in F(P_\varkappa(\lambda) - A_\beta)$. That is to say $Q \cap \lambda \in A_\alpha$ for all $\alpha \in P$ and

$Q\cap\lambda\notin A_\beta = \bigcap\{A_\alpha|\ \alpha\in P\}$, a contradiction. So we have shown that if $\bigcap\{A_\alpha|\ \alpha\in P\}\notin U$ then $(\underset{\xi<\gamma}{\triangle} t'(\xi))\cap\hat{P}\cap\widehat{\{\beta\}} = \emptyset$ which contradicts the fact that $\underset{\xi<\gamma}{\Delta} t'(\xi)$ is stationary.

d) U is normal. Let $\{B_\alpha\}_{\alpha<\lambda}$ be a collection of subsets of $P_\varkappa(\lambda)$ such that $B_\alpha\in U$ for all $\alpha<\lambda$. We will show that $\underset{\alpha<\lambda}{\triangle} B_\alpha\in U$. Let $f\colon \lambda\to\gamma$ be such that for each $\alpha<\lambda$, $B_\alpha = A_{f(\alpha)}$.

CLAIM. *$C = \{Q\in P_\varkappa(\gamma)|\ f(\alpha)\in Q$ for all $\alpha\in Q\cap\lambda\}$ is closed and unbounded.*

C is closed. Suppose $Q_0\subseteq Q_1\subseteq\ldots\subseteq Q_\xi\subseteq\ldots$ is an increasing sequence of elements of C of length $\delta<\varkappa$. Then $Q=\bigcup\{Q_\xi|\ \xi<\delta\}\in C$ because if $\alpha\in Q\cap\lambda$ then $\alpha\in Q_\xi\cap\lambda$ for some $\xi<\delta$ and so $f(\alpha)\in Q_\xi\subseteq Q$.

C is unbounded. Let $P\in P_\varkappa(\lambda)$, and construct an increasing sequence as follows: $Q_0=P$, $Q_{n+1}=Q_n\cup\{f(\alpha)|\ \alpha\in Q_n\cap\lambda\}$. Let $Q=\bigcup\{Q_n|\ n<\omega\}$. Then $P\subseteq Q$ and $Q\in C$. This concludes the proof of the claim.

If $\underset{\alpha<\lambda}{\triangle} B_\alpha\notin U$ then by b) $P_\varkappa(\lambda)-(\underset{\alpha<\lambda}{\triangle} B_\alpha)\in U$. Let $A_\beta = P_\varkappa(\lambda)-\underset{\alpha<\lambda}{\triangle} B_\alpha$. We have

$$t'(\beta)=F(A_\beta)=\{Q\in P_\varkappa(\gamma)|\ \exists\,\alpha\in Q\cap\lambda \text{ such that } Q\cap\lambda\notin B_\alpha\}.$$

Let $P\in(\underset{\xi<\gamma}{\triangle} t'(\xi))\cap\widehat{\{\beta\}}$ then $\exists\,\alpha\in P\cap\lambda$ such that $P\cap\lambda\notin B_\alpha$ hence $P\notin C$ (otherwise $f(\alpha)\in P$ and so $P\in t'(f(\alpha))=F(A_{f(\alpha)})$, thus $P\cap\lambda\in A_{f(\alpha)}=B_\alpha$) so if $\Delta B_\alpha\notin U$, we contradict the fact that $\underset{\xi<\gamma}{\triangle} t'(\xi)$ is stationary (because $(\underset{\xi<\gamma}{\triangle} t'(\xi))\cap\widehat{\{\beta\}}\cap C=\emptyset$). This completes the proof of the lemma and of Theorem 1.

A similar result holds for strongly compact cardinals. First we will give a definition.

A cardinal $\varkappa$ is *mildly λ-ineffable* if for any $f\colon P_\varkappa(\lambda)\to P_\varkappa(\lambda)$ such that $f(P)\subseteq P$ for all $P\in P_\varkappa(\lambda)$ there is a set $A\subseteq\lambda$ such that for all $P\in P_\varkappa(\lambda)$,

$$\{Q\in P_\varkappa(\lambda)|\ Q\cap P\cap A=f(Q)\cap P\}$$

is unbounded.

THEOREM 2. *The following are equivalent*

(i) *$\varkappa$ is strongly compact.*

(ii) *$\varkappa$ is mildly λ-ineffable for all $\lambda\geqslant\varkappa$.*

(iii) *$\forall\,\lambda\geqslant\varkappa$ if $t\colon\lambda\to\mathcal{P}(P_\varkappa(\lambda))$ there is a $t'\sim t$ such that $\forall\,P\in P_\varkappa(\lambda)$ $\bigcap_{\alpha\in P} t'(\alpha)$ is unbounded.*

Proof. The proof is quite similar to that of Theorem 1. Once again (ii) $\Leftrightarrow$ (iii) can be done level-by-level (i.e., (ii) and (iii) are equivalent for any fixed $\lambda\geqslant\varkappa$). Note that in Lemma 3 all the properties of the measure obtained, except normality, rely only on the fact that $\bigcap_{\alpha\in P} t'(\alpha)$ is unbounded for any $P\in P_\varkappa(\lambda)$.

We now turn our attention to a different flipping property equivalent to supercompactness. In the first theorem we flipped a given sequence to obtain one with stationary diagonal intersection. As in the case of ineffable cardinals (see [1]) the notion of stationary can be sidetracked by requiring instead that the diagonal intersection of the flipped sequence be unbounded when taken in any order.

THEOREM 3. *$\varkappa$ is supercompact if and only if $\forall\,\lambda\geqslant\varkappa$ if $t\colon\lambda\to\mathcal{P}(P_\varkappa(\lambda))$ there is a $t'\sim t$ such that for all permutations $\pi\colon\lambda\to\lambda$, $\underset{\alpha<\lambda}{\triangle} t'\circ\pi(\alpha)$ is unbounded.*

Proof. Just as in Lemma 2 if $\varkappa$ is supercompact and $t\colon\lambda\to\mathcal{P}(P_\varkappa(\lambda))$ we can flip t to obtain $t'\sim t$ such that $\underset{\alpha}{\triangle} t'(\alpha)$ is stationary. Now it is easy to see that if $\pi\colon\lambda\to\lambda$ is any permutation $\underset{\alpha}{\triangle} t'\circ\pi(\alpha)$ is also stationary and therefore unbounded. In fact, if $\pi\colon\lambda\to\lambda$ is any permutation, the set $C=\{P\in P_\varkappa(\lambda)|\ \alpha\in P$ if and only if $\pi(\alpha)\in P\}$ is closed and unbounded. Hence $(\underset{\alpha}{\triangle} t'(\alpha))\cap C$ is stationary. As $(\underset{\alpha}{\triangle} t'(\alpha))\cap C\subseteq\underset{\alpha}{\triangle} t'\circ\pi(\alpha)$ we have the desired result.

The most obvious way to attack the converse would be to show that if $\underset{\alpha<\lambda}{\triangle} t(\alpha)$ is not stationary then there is a $\pi\colon\lambda\to\lambda$ such that $\underset{\alpha<\lambda}{\triangle} t\circ\pi(\alpha)$ is not unbounded. This would establish the equivalence of the two flipping properties. It appears, however, that one would need the additional assumption $|P_\varkappa(\lambda)|=|\lambda|$ to make this argument go.

Instead we proceed as in Lemma 3. Again let $\{A_\alpha|\ \alpha<\gamma\}$ be an enumeration of all the subsets of $P_\varkappa(\lambda)$ (where $\gamma\geqslant 2^{(\lambda^{\varkappa})}$). For each $\alpha<\gamma$ let $t(\alpha)=F(A_\alpha)=\{P\in P_\varkappa(\gamma)|\ P\cap\lambda\in A_\alpha\}$, and let $t'\sim t$ be such that $\underset{\alpha}{\triangle} t'\circ\pi(\alpha)$ is unbounded for each permutation π of γ. Define μ by $\mu(A_\alpha)=1$ if and only if $F(A_\alpha)=t'(\alpha)$. The other properties of μ being routine to check, we shall just present the normality argument.

Suppose that for each $\alpha<\lambda$ we have a subset B_α of $P_\varkappa(\lambda)$ such that $\mu(B_\alpha)=1$ and that $\mu(\underset{\alpha<\lambda}{\triangle} B_\alpha)=0$. We would like to tinker with the B_α sequence so as to eliminate any possible repetitions. The new sequence, B_α, should have the original properties assumed for the B_α, and should in addition be nonrepeating. One such tinkering procedure is:

$$B'(\alpha)=B(\alpha)\cup\{\alpha\}-\{\{\delta\}|\ \delta<\alpha\}.$$

It is easy to see the B'_α have the desired property. Let β be such that $P_\varkappa(\lambda)-\underset{\alpha<\lambda}{\triangle} B'_\alpha=A_\beta$, and $f\colon\lambda\to\gamma$ be such that for each $\alpha<\lambda$, $B'_\alpha=A_{f(\alpha)}$ (f is one to one).

Now consider $\pi\colon\gamma\to\gamma$ be the permutation given by

$$\pi(\alpha)=\begin{cases} f(\alpha) & \text{for all } \alpha<\lambda,\\ \delta & \text{if } \alpha=f(\delta),\\ \alpha & \text{otherwise}.\end{cases}$$

As $\underset{\alpha<\lambda}{\triangle} t' \circ \pi(\alpha)$ is unbounded, take $P \in \underset{\alpha<\lambda}{\triangle} t' \circ \pi(\alpha)$ such that $\pi^{-1}(\beta) \in P$, then $P \in t'(\beta) = F(A_\beta)$ (since $\mu(A_\beta) = 1$) and so $\exists\, \alpha \in P \cap \lambda$ such that $P \cap \lambda \notin B'_\alpha$. On the other hand $\forall\, \alpha \in P, P \in t' \circ \pi(\alpha)$, in particular for every $\alpha \in P \cap \lambda$, $P \in t' \circ \pi(\alpha) = t'(f(\alpha)) = F(A_{f(\alpha)})$ so $P \cap \lambda \in A_{f(\alpha)} = B'_\alpha$, a contradiction.

Then $\mu(\underset{\alpha<\lambda}{\triangle} B'_\alpha) = 1$, and μ is normal.

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On saturated sets of ideals and Ulam's problem

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Abstract. A set $\mathscr{I}$ of countably complete ideals on ω_1 is called λ-saturated iff for every collection $\{X_\alpha\colon \alpha<\lambda\} \subseteq \mathscr{P}(\omega_1) - \bigcup \mathscr{I}$ there exists $\{\alpha, \beta\} \in [\lambda]^2$ such that $X_\alpha \cap X_\beta \notin \bigcap \mathscr{I}$. An old problem of Ulam asks if there can exist a 2-saturated set $\mathscr{I}$ of size ω_1. We show that a weak version of Kurepa's hypothesis implies that if $|\mathscr{I}| \leqslant \omega_1$ then $\mathscr{I}$ is not even ω_2-saturated. This answers a question of Prikry. Some related results are obtained and several questions are stated.

§ 0. Introduction. Over thirty years ago S. Ulam raised the following question (see [6]). Let $\varkappa$ be an uncountable cardinal less than the first weakly inaccessible cardinal. What is the smallest cardinal λ having the property that there exists a family of λ two valued countable additive measures defined for the subsets of $\varkappa$ (singletons having measure 0 and $\varkappa$ having measure 1 for each of them) such that every subset of $\varkappa$ is measurable with respect to at least one of these measures? The following version of this question was stated as Problem 81 of [7] and will be referred to here as *Ulam's problem*.

Problem (S. Ulam). Can one define $\aleph_1$ σ-additive 0-1 measures on ω_1 so that each subset is measurable with respect to one of them?

In this paper we will consider several generalizations of Ulam's problem. Several new results are obtained and many older results from the literature are collected together. Some eighteen open problems are also stated.

We begin by establishing some notation. ν will denote an arbitrary cardinal, while λ and μ will be reserved for infinite cardinals and $\varkappa$ for an uncountable cardinal. We will use the phrase "ideal on $\varkappa$" to mean "proper uniform ideal on $\varkappa$". (An ideal I on $\varkappa$ is called *uniform* iff $[\varkappa]^{<\varkappa} \subseteq I$.) The (normal) ideal of non-stationary subsets of the regular cardinal $\varkappa$ is denoted by $NS_\varkappa$.

If I is an ideal on $\varkappa$ then I^+ denotes $\mathscr{P}(\varkappa) - I$ (the sets of "positive I-measure") and I^* denotes $\{X \subseteq \varkappa\colon \varkappa - X \in I\}$ (the sets of "I-measure one"). If $A \in I^+$ then the restriction of I to A is the ideal

$$I \upharpoonright A = \{X \subseteq \varkappa\colon X \cap A \in I\}.$$

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