Whitney-reversible properties

by

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Abstract. The setting of the paper is the hyperspace of all nonempty subcontinua of a metric continuum. A converse to the notion of a Whitney property is introduced. This converse notion, called a Whitney-reversible property, is investigated in connection with the following: Dimension, contractibility, shape, tree-like, acyclicity, hereditary indecomposability, and certain circle-like continua. Some unanswered questions are raised.

1. Introduction. Since this is the first paper on this topic, and since we introduce some new concepts related to other relatively new ideas, we will give in this section a somewhat lengthy discussion which hopefully clarifies and motivates the rest of the paper.

Let X be a continuum, i.e., a compact connected metric space which is *non-degenerate* (= consists of more than one point). By a *subcontinuum of* X we mean a nonempty (but possibly degenerate) compact connected subset of X. By the hyper-space of X we mean

$$C(X) = \{A \subset X: A \text{ is a subcontinuum of } X\}$$

with the Hausdorff metric (see [17] or [26]). The term *mapping* is synonymous with continuous function. In 1932, Whitney [35] showed that there is a mapping μ: $C(X) \to [0, +\infty)$ such that (a) if $A, B \in C(X)$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$; (b) $\mu(\{x\}) = 0$ for each $x \in X$. Throughout this paper the symbol μ will always denote *any* mapping from $C(X)$ into $[0, +\infty)$ satisfying (a) and (b) above; μ will be called a *Whitney map*. In [9, p. 1032] it was proved that Whitney maps are *monotone*, i.e., $\mu^{-1}(t)$ is connected for each t. Hence, $\mu^{-1}(t)$ is a continuum whenever $0 \leqslant t < \mu(X)$. Thus, the property of being a continuum passes from X to $\mu^{-1}(t)$. It has recently been of interest to see what other properties of X must also be properties of $\mu^{-1}(t)$. Precisely: A topological property P is said to be a *Whitney property* [23] provided that whenever a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and each t, $0 < t < \mu(X)$. Much work, by many authors, has been done on Whitney properties. A comprehensive treatment of what is known is in [26,

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Chapter XIV] — some results, appropriate to this paper, will be mentioned later. Now, however, we discuss the notion of Whitney-reversible properties.

Before defining the notion of a Whitney-reversible property, we give a geometric interpretation of it. Let X be a continuum, let μ be a Whitney map for $C(X)$, and let $t_n \in [0, \mu(X)]$ such that $\{t_n\}_{n=1}^{\infty} \to 0$ as $n \to \infty$. For each $n = 1, 2, \ldots$, $\mu^{-1}(t_n)$ is a continuum which we visualize as being a horizontal level in $C(X)$ [see Fig. 1].

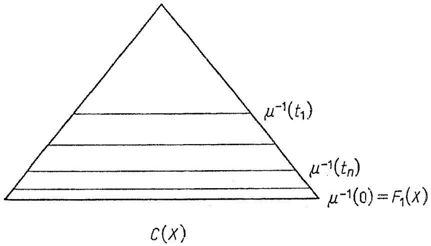


Fig. 1.

Since μ is an open mapping [9, p. 1032], the levels $\mu^{-1}(t_n)$ converge to $\mu^{-1}(0)$ as $n \to \infty$. We use the symbol $F_1(X)$ to denote the space $\mu^{-1}(0)$. It follows easily from the definition of μ above that

$$F_1(X) = \{\{x\} \in C(X) \colon x \in X\}$$

and, hence, $F_1(X)$ is called the space of singletons. Note that X and $F_1(X)$ are homeomorphic in a natural way by associating x with $\{x\}$ for each $x \in X$. Thus, as $n \to \infty$, the levels $\mu^{-1}(t_n)$ "approximate" X closer and closer. *Roughly speaking, Whitney-reversible properties are those properties which are preserved by this approximation.*

More precisely, we have the following definitions [26, Chapter XIV, E]: A topological property P is said to be a *strong Whitney-reversible property* (respectively, a *Whitney-reversible property*) provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property P for some Whitney map (respectively, all Whitney maps) μ and each t, $0<t<\mu(X)$, then X has property P. Thus: Whitney properties are those topological properties which pass from X to each $\mu^{-1}(t)$, whereas Whitney-reversible properties are those topological properties which pass from $\mu^{-1}(t)$, $0<t<\mu(X)$, to X. Hence, these notions are "converses" to each other. However, we have made a distinction between *strong* Whitney-reversible properties and Whitney-reversible properties. In this connection, let us note the following. There are continua X, which have a certain topological property P, and Whitney maps $\mu_1, \mu_2 \colon C(X) \to [0, +\infty)$ such that $\mu_1^{-1}(t)$ has property P whenever $0<t<\mu_1(X)$ but $\mu_2^{-1}(s)$ does not have property P for some s, $0<s<\mu_2(X)$. This happens, for example, when P is the property of being contractible [29]. When this happens, of course P is not a Whitney property. It is not known if the "reverse" can happen,

i.e., it is not known if there is a Whitney-reversible property which is not a strong Whitney-reversible property [26, (14.56)].

As regards relationships between Whitney properties and reversible properties, we make the following comments. There are Whitney properties which are not Whitney-reversible properties; such is the case for the property of being arcwise connected [26, (14.48)]. There are strong Whitney-reversible properties which are not Whitney properties; for example, the property of being indecomposable [26, p. 454]. Clearly, the negation of a Whitney property is a strong Whitney-reversible property [26, (14.46)]. Finally, we mention that there are properties which are both Whitney properties and strong Whitney-reversible properties; for example, the property of being locally connected [26, (14.48)].

The purpose of this paper is to determine some strong Whitney-reversible properties. The paper is a continuation of the work done in [26, Chapter XIV, E]. In particular, some of the results here answer questions raised in [26]. We mention that some of our results show that certain Whitney properties are strong Whitney-reversible properties [see (3.3) and (5.5)]. Thus, these results are "converses" to the corresponding (known) results about Whitney properties.

For some properties P, we will show the following: If X is a continuum such that there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty} \to 0$ as $n \to \infty$ such that $\mu^{-1}(t_n)$ has property P for each $n = 1, 2, \ldots$, then X has property P. We will call such a property P a *sequential strong Whitney-reversible property*. It is not known if every strong Whitney-reversible property is sequential [26, (14.55.1)].

Throughout this paper we adopt the following notation. The symbol $A \setminus B$ denotes the complement of B in A. For groups G_1 and G_2, the symbol $G_1 \approx G_2$ means G_1 is isomorphic to G_2. For each $n = 1, 2, \ldots$, R^n denotes Euclidean n-space and S^n denotes the n-sphere in R^{n+1},

$$S^n = \{v \in R^{n+1} \colon \|v\| = 1\}.$$

By the symbol ANR we mean a (possibly non-compact) absolute neighborhood retract for the class of all metrizable spaces. Recall from above that the symbol X always denotes a continuum. Since X and $F_1(X)$ are homeomorphic in a natural way (see above), we will use the symbols X and $F_1(X)$ interchangeably.

A mapping from a space Z_1 into a space Z_2 is said to be *inessential* provided that it is homotopic to a constant mapping; otherwise, it is said to be *essential*. The space Z_1 is said to be *contractible with respect to* Z_2 provided that every mapping from Z_1 into Z_2 is inessential [24, p. 370]. A mapping f from Z_1 to Z_2 is called an *r-map* provided that there is a mapping g from Z_2 to Z_1 such that $f \circ g \colon Z_2 \to Z_2$ is the identity map [4, p. 7]. If $f \colon Z_1 \to Z_2$ is an r-map, then Z_2 is said to be an *r-image* of Z_1 [4, p. 8].

Terminology not defined above may be found later or in appropriate references at the end of the paper.

The following result will be used many times in this paper.

(1.1) *If* $\mathcal{S} \subset C(X)$ *is a subcontinuum of* $C(X)$ *such that* $[\bigcup \Sigma] \in \mathcal{S}$ *for each* *subcontinuum Σ of $\mathscr{S}$, then $\mathscr{S}$ is an r-image of the hyperspace $C(\mathscr{S})$* [22, Section 1]. *Hence, $\mathscr{S}$ has all those properties which hyperspaces have and which are preserved by r-maps.*

Proof. Define $f\colon C(\mathscr{S})\to\mathscr{S}$ by $f(\Sigma)=\bigcup\Sigma$ for each $\Sigma\in C(\mathscr{S})$ and define $g\colon\mathscr{S}\to C(\mathscr{S})$ by $g(A)=\{A\}$ for each $A\in\mathscr{S}$. Then f is continuous [17, p. 23], g is continuous, and $f\circ g$ is the identity map on $\mathscr{S}$. Therefore, $\mathscr{S}$ is the r-image of $C(\mathscr{S})$ by f.

The most frequent application of (1.1) will be to the set $\mu^{-1}([t,\mu(X)])$. By monotoneness of μ [9, p. 1032], $\mu^{-1}([t,\mu(X)])$ is a subcontinuum of $C(X)$. Thus, by using [17, 1.2], we see that $\mu^{-1}([t,\mu(X)])$ satisfies all the hypotheses on $\mathscr{S}$ in (1.1). Hence:

(1.2) *For any $t\in[0,\mu(X)]$, $\mu^{-1}([t,\mu(X)])$ has all those properties which hyperspaces have and which are preserved by r-maps.*

2. Dimension. In [26, (14.57)] it was asked if the property of being one-dimensional is a strong Whitney-reversible property. In [1] the answer was shown to be "yes" for the special case of finite-dimensional continua X. We will show in (2.8) that the answer is "yes" with no restrictions. We show even more in (2.6) and (2.7), which are the main results of this section. We begin with the following two general theorems which will be used in the proof of (2.6).

(2.1) Theorem. *For any t such that $0\leqslant t<\mu(X)$, $\dim[\mu^{-1}(t)]\leqslant\dim[\mu^{-1}((t,\mu(X)])]$.*

Proof. Let $\Gamma=\mu^{-1}((t,\mu(X)])$. Since the theorem is obvious when $\dim[\Gamma]=\infty$, assume for the purpose of proof that $\dim[\Gamma]=n<\infty$. Suppose that

$$(*)\qquad \dim[\mu^{-1}(t)\mu(t)]\geqslant n+1\,.$$

Then, by [12, VI4, p. 83], there is a closed subset Λ of $\mu^{-1}(t)$ and a mapping $f\colon\Lambda\to S^n$ such that f can not be extended to all of $\mu^{-1}(t)$. Hence, [12, p. 87], f is essential. Let

$$\mathscr{S}=\{B\in\mu^{-1}([t,\mu(X)])\colon\ B\supset A\ \text{for some}\ A\in\Lambda\}\,.$$

By using the definition of μ (in Section 1), it follows easily that $[\mathscr{S}\setminus\Lambda]\subset\Gamma$. Thus, $\dim[\mathscr{S}\setminus\Lambda]\leqslant n$. Hence, since Λ is a closed subset of $\mathscr{S}$, we have by [12, Corollary, p. 84] that f can be extended to a mapping $g\colon\mathscr{S}\to S^n$. Now observe that $\mathscr{S}$ is a subcontinuum of $C(X)$ since it is compact (by an easy sequence argument using compactness of Λ and of $C(X)$) and arcwise connected (by using [17, 2.3 and 2.6]). Thus, by using [17, 1.2], it follows easily that $\mathscr{S}$ satisfies all the hypotheses of (1.1). Hence, since hyperspaces are contractible with respect to ANR's [18, 1.6], we have by (1.1) and [24, p. 371] that $\mathscr{S}$ is contractible with respect to S^n. Therefore, g is inessential which implies that f is inessential, a contradiction. Thus, $(*)$ is false and, therefore, we have proved (2.1).

As we will see in the proof of (2.6), the value of (2.1) for our purposes is to show that $\dim[\mu^{-1}(t)]$ is finite if $\dim[\mu^{-1}((t,\mu(X)])]$ is finite.

(2.2) Theorem. *For any t such that $0\leqslant t<\mu(X)$, $\dim[\mu^{-1}([t,\mu(X)])]=\dim[\mu^{-1}((t,\mu(X)])]$.*

Proof. Let $\Gamma=\mu^{-1}((t,\mu(X)])$ and let $\Omega=\mu^{-1}([t,\mu(X)])$. Since the theorem is obvious when $\dim[\Gamma]=\infty$, assume for the purpose of proof that $\dim[\Gamma]=n<\infty$. Then, by (2.1), $\dim[\mu^{-1}(t)]\leqslant n$. Thus, since

$$\Omega=\mu^{-1}(t)\cup\Big[\bigcup_{i=1}^{\infty}\mu^{-1}([t+2^{-i},\mu(X)])\Big]\,,$$

it follows using the Sum Theorem [12, III2, p. 30] that $\dim[\Omega]\leqslant n$. Therefore, since $\Gamma\subset\Omega$ and $\dim[\Gamma]=n$, we have that $\dim[\Omega]=\dim[\Gamma]$ which proves (2.2).

Let us note the following corollary.

(2.3) Corollary. *For any continuum X, $\dim[X]\leqslant\dim[C(X)\setminus F_1(X)]=\dim[C(X)]$.*

Proof. Recalling from Section 1 that $F_1(X)=\mu^{-1}(0)$ and that $F_1(X)$ is homeomorphic to X, we see that (2.3) follows immediately from (2.1) and (2.2) by setting $t=0$.

Our next result is a sharpening of (2.1) for the case when $\dim[\mu^{-1}(t)]$ is finite. It will be used in the proof of (2.6), and it is a generalization of [18, 2.3] — see (2.5).

(2.4) Theorem. *If $0\leqslant t<\mu(X)$ and if $\dim[\mu^{-1}(t)]<\infty$, then $\dim[\mu^{-1}(t)]<\dim[\mu^{-1}([t,\mu(X)])]$.*

Proof. Assume that $0\leqslant t<\mu(X)$ and that $\dim[\mu^{-1}(t)]=n<\infty$. Let $\mathscr{S}=\mu^{-1}((t,\mu(X)])$. By (2.2), it suffices to prove that

$$(*)\qquad n<\dim[\mathscr{S}]\,.$$

To prove $(*)$ first observe that, since hyperspaces are acyclic [34, 1.2] and since r-maps preserve acyclicity, we have by (1.2) that

(1) $\mathscr{S}$ is acyclic.

Assume $n=1$ (the proof in this case will be similar to the proof in [9, p. 1029]). Since $t<\mu(X)$, there exist $A,B\in\mu^{-1}(t)$ such that $A\neq B$. By [17, 2.3 and 2.6] there are arcs α and β in $\mathscr{S}$ such that $\alpha\cap\mu^{-1}(t)=\{A\}$, $\beta\cap\mu^{-1}(t)=\{B\}$, and $X\in[\alpha\cap\beta]$. Since $\mu^{-1}(t)$ is a continuum, it follows easily that $\mu^{-1}(t)\cup\alpha\cup\beta$ is a non-acyclic (in fact, non-unicoherent) subcontinuum of $\mathscr{S}$. Thus, by (1), $\dim[\mathscr{S}]>1$. Next, assume $n>1$ (the proof in this case is a modification of the first part of the proof in [18, p. 709]). Let $Y_1=\mu^{-1}(t)$. Since $\dim[Y_1]=n$, there is a compact subset Λ of Y_1 and an $(n-1)$-dimensional cycle γ^{n-1} in Λ such that γ^{n-1} is essential in Λ and bounds in Y_1 [2, p. 21]. Let $Y_2=\{B\in\mathscr{S}\colon B\supset A$ for some $A\in\Lambda\}$. By using the definition of μ (in section 1), it follows easily that $Y_1\cap Y_2=\Lambda$. Note that Y_2, being analogous to the set $\mathscr{S}$ in the proof of (2.1), is acyclic by (1.1). Thus, γ^{n-1} bounds in Y_2. Recall from (1) that $\mathscr{S}$ is acyclic. Hence, by letting $Y=\mathscr{S}$, we see that Y, Y_1, and Y_2 satisfy the hypotheses of [18, 2.4]. Hence, since $n>1$, we have by [18, 2.4] that $\dim[Y]>n$. Therefore, we have proved $(*)$.

(2.5) Remark. The result in [18, 2.3] says that if $\dim[X]<\infty$, then $\dim[X]<\dim[C(X)]$. It is also known that if $\dim[X]\geqslant 3$, then $\dim[C(X)]=\infty$ [32, Theorem 5]. With respect to the two results just mentioned and (2.4), let us note the following: For each $n = 2, 3, \ldots$, there is a continuum X_n with Whitney map μ_n for $C(X_n)$ such that, for a suitably chosen t_n, $\dim[\mu_n^{-1}(t_n)] = n-1$ and $\dim[\mu_n^{-1}([t_n, \mu_n(X_n)])] = n$. Such is seen to be the case when X_n consists of n perpendicular segments $A_1, \ldots, A_n$ emanating from the origin in R^n and

$$t_n = \max\{\mu_n(A_1), \ldots, \mu_n(A_n)\}$$

where μ_n is any Whitney map for $C(X_n)$; in fact, $\mu_n^{-1}(t_n)$ is an $(n-1)$-cell and $\mu_n^{-1}([t_n, \mu(X_n)])$ is an n-cell.

Now we come to one of the main results of this section.

(2.6) Theorem. *Let t be fixed, $0 \leqslant t < \mu(X)$. If $\dim[\mu^{-1}(s)] \leqslant n$ for each $s > t$ and for some fixed $n < \infty$, then $\dim[\mu^{-1}(t)] \leqslant n$ and $\dim[\mu^{-1}([t, \mu(X)])] \leqslant n+1$.*

Proof. Let μ_* denote the restriction of μ to $\mu^{-1}((t, \mu(X)])$. Then, using the compactness of $C(X)$, it follows easily that μ_* is a closed mapping of $\mu^{-1}((t, \mu(X)])$ to $(t, \mu(X)]$. Also, by hypothesis, $\dim[\mu_*^{-1}(s)] \leqslant n$ for each $s \in (t, \mu(X)]$. Hence, by [12, p. 92],

$$(\#) \qquad \dim[\mu_*^{-1}((t, \mu(X)])] \leqslant n+1 .$$

By (2.1) and (#) we see that $\dim[\mu^{-1}(t)] < \infty$. Thus, by (2.4) and (#), $\dim[\mu^{-1}(t)] \leqslant n$. By (2.2) and (#), $\dim[\mu^{-1}([t, \mu(X)])] \leqslant n+1$. We have proved (2.6).

Our next main result is an easy consequence of (2.6). Another application of (2.6) is in the proof of (3.4).

(2.7) Theorem. *For any given $n < \infty$, the property of being of dimension $\leqslant n$ is a strong Whitney-reversible property.*

Proof. Use (2.6) with $t = 0$.

We state the following corollary since it answers the question in [26, (14.57)] mentioned at the beginning of this section.

(2.8) Corollary. *The property of being one-dimensional is a strong Whitney-reversible property.*

Except for the case when $n \leqslant 2$, the following result gives more information about $\dim[X]$ than is contained in (2.6).

(2.9) Theorem. *If $\dim[\mu^{-1}(s)] \leqslant n$ for each $s > 0$ and for some fixed $n < \infty$, then $\dim[X] \leqslant 2$.*

Proof. Setting $t = 0$ in (2.6), we see that $\dim[C(X)] \leqslant n+1 < \infty$. Hence, by the result in [32] mentioned in (2.5), we have $\dim[X] \leqslant 2$.

The result in (2.9) and the full generality of (2.6) should be compared with the discussion in (2.5).

We have completed the results in this section. Answers to the following questions are not known.

(2.10) Question. Are the properties in (2.7) and (2.8) sequential strong Whitney-reversible properties? In fact, can the result in (2.6) be generalized so as to be "sequential"? We recall that it is not known if there is a strong Whitney-reversible property which is not sequential [26, (14.55.1)].

(2.11) Question. Is the property of being finite-dimensional a strong (or, sequential strong) Whitney-reversible property? The result in (2.7) provides a partial answer.

(2.12) Question. Can (2.9) be strengthened, under the same hypotheses, to conclude that $\dim[X] = 1$? By using (2.7) we see easily that this question is equivalent to the following question raised in [28, 2.2]: If $\dim[C(X)] < \infty$, then must $\dim[X] = 1$?

3. Contractibility with respect to ANR's. In (3.1) we prove that *contractibility with respect to an* ANR is a sequential strong Whitney-reversible property. We will then deduce a number of consequences of this result, the most interesting one being perhaps (3.4). We mention that in (3.7) we will show that *contractibility* is not a Whitney-reversible property.

(3.1) Theorem. *Let t be fixed, $0 \leqslant t \leqslant \mu(X)$. Let Y be an* ANR. *If there is a sequence $\{t_n\}_{n=1}^{\infty} \to t$ as $n \to \infty$ such that $t_n \geqslant t$ and $\mu^{-1}(t_n)$ is contractible with respect to Y for each $n = 1, 2, \ldots$, then $\mu^{-1}(t)$ is contractible with respect to Y. Hence, the property of being contractible with respect to Y is a sequential strong Whitney-reversible property.*

Proof. Let $f\colon \mu^{-1}(t) \to Y$ be a mapping. We will show that f is inessential. Let $\mathscr{S} = \mu^{-1}([t, \mu(X)])$. Since Y is an ANR and $\mu^{-1}(t)$ is a closed subset of $\mathscr{S}$, we have by [4, p. 88] that there is an open subset U of $\mathscr{S}$, with $\mu^{-1}(t) \subset U$, and a continuous extension $g\colon U \to Y$ of f. Since $\{t_n\}_{n=1}^{\infty} \to t$ as $n \to \infty$ and since μ is an open mapping [9, p. 1032], we infer that there exists $s = t_n$ for some fixed n such that $\mu^{-1}([t, s]) \subset U$. Let g_s denote the restriction of g to $\mu^{-1}(s)$. Then, since $\mu^{-1}(s) = \mu^{-1}(t_n)$ is contractible with respect to Y, g_s is inessential. Hence, since Y is an ANR, we have by the homotopy extension theorem [4, 8.1, p. 94] that g_s can be extended to a mapping

$$h\colon \mu^{-1}([s, \mu(X)]) \to Y .$$

Since $h(A) = g_s(A) = g(A)$ for each $A \in \mu^{-1}(s)$, the formula

$$k(A) = \begin{cases} g(A), & \text{if } A \in \mu^{-1}([t, s]), \\ h(A), & \text{if } A \in \mu^{-1}([s, \mu(X)]) \end{cases}$$

defines a mapping k from all of $\mathscr{S}$ to Y. Noting that hyperspaces are contractible with respect to ANR's [18, 1.6], we have by (1.2) and [24, p. 371] that $\mathscr{S}$ is contractible with respect to Y. Hence, k is inessential. Therefore, since k is an extension of f, f is inessential. This completes the proof of (3.1) (since the second part of (3.1) follows from the first part by setting $t = 0$).

A number of seemingly different properties are known to be equivalent to contractibility with respect to every ANR. We list some of them in the following corollary to (3.1):

(3.2) COROLLARY. *The result in* (3.1) *holds when the property "contractible with respect to* Y*" is replaced by any one of the following properties: Having trivial shape, being a fundamental absolute retract, being a weak proximate absolute retract, and being absolutely neighborhood contractible.*

Proof. For continua it is known that each of the properties listed in (3.2) is equivalent to contractibility with respect to every ANR (see for example [5, p. 95], [13], and [19, 2.1]). Therefore, (3.2) follows from (3.1).

Some properties which are equivalent to contractibility with respect to every ANR, but which are not listed in (3.2), may be found in the references in the proof of (3.2). These properties are in particular, therefore, sequential strong Whitney-reversible properties.

Some properties are equivalent to contractibility with respect to each member of a certain class of ANR's. Such is the case for the properties in (3.3) and (3.4) below.

(3.3) COROLLARY. *The result in* (3.1) *holds when the property "contractible with respect to* Y*" is replaced by the property of being acyclic in dimension one* (i.e., $H^1(\cdot) \approx 0$).

Proof. Any given continuum is acyclic in dimension one if and only if it is contractible with respect to the circle S^1 [8, 8.1]. Hence, (3.3) is a consequence of (3.1).

The result in (3.3) is a special case of results in section 4. Let us note that (3.3) can be interpreted as being a converse to [25, Theorem 4] and [31, Corollary 6], where it is shown that acyclicity in dimension one is a Whitney property.

The next result is our main application of (3.1). It gives a partial answer to the following question asked in [26, (14.57)]: Is the property of being chainable a strong Whitney-reversible property? For more discussion, see (5.6).

(3.4) THEOREM. *Let* t *be fixed,* $0 \leqslant t < \mu(X)$. *If* $\mu^{-1}(s)$ *is tree-like for each* s, $t < s < \mu(X)$, *then* $\mu^{-1}(t)$ *is tree-like. Hence, the property of being tree-like is a strong Whitney-reversible property.*

Proof. Since tree-like continua are one-dimensional, $\dim[\mu^{-1}(s)] = 1$ for each s such that $t < s < \mu(X)$. Hence, by (2.6), $\dim[\mu^{-1}(t)] = 1$. Case and Chamberlin [7, Theorem 1] have proved the following result: (*) *A one-dimensional continuum is tree-like if and only if it is contractible with respect to every linear graph.* By (*), $\mu^{-1}(s)$ is contractible with respect to every linear graph for each $s > t$. Hence, since linear graphs are ANR's, we have by (3.1) that $\mu^{-1}(t)$ is contractible with respect to every linear graph. Therefore, since $\dim[\mu^{-1}(t)] = 1$, we have by (*) that $\mu^{-1}(t)$ is tree-like.

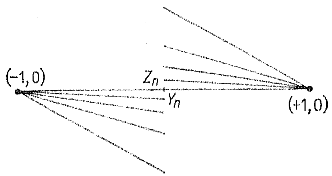
(3.5) Remark. Recently there has been an increased interest in a-triodic tree-like continua (see for example [11], [14], and [15]). For this reason we mention that, by (3.4) and [23, 3.3], it follows that if $\mu^{-1}(t)$ is tree-like for each t, $0 < t < \mu(X)$, then X is an a-triodic tree-like continuum. More generally, by [23, 3.3]: If, for a continuum X and a Whitney map μ for $C(X)$, we have $\dim[\mu^{-1}(t)] \leqslant n$ for each $t > 0$, then X does not contain an $(n+2)$-odd.

Another result about tree-like continua is in (5.5). We mention the following question whose answer is not known.

(3.6) QUESTION. Is the property of being tree-like a sequential strong Whitney-reversible property? From the proof of (3.4) we see that if (2.6) were a "sequential theorem" [see (2.10)], then the question just asked would have an affirmative answer.

We have seen in (3.1) that *contractibility with respect to an* ANR is a Whitney-reversible property. The following example shows that *contractibility* is not a Whitney-reversible property. This answers a question in [26, (14.57)].

(3.7) EXAMPLE. Let $X = X_0 \cup [\bigcup_{n=1}^{\infty} Y_n] \cup [\bigcup_{n=1}^{\infty} Z_n]$ where X_0 is the convex arc in R^2 from $(-1, 0)$ to $(+1, 0)$ and, for each $n = 1, 2, \ldots$, Y_n (respectively, Z_n) is the convex arc in R^2 from $(-1, 0)$ (respectively, $(+1, 0)$) to $(0, -2^{-n})$ (respectively, $(0, +2^{-n})$) — see Fig. 2. Then: X is a non-contractible dendroid such that $\mu^{-1}(t)$ is contractible for any Whitney map μ for $C(X)$ and any t such that $0 < t < \mu(X)$



X

Fig. 2

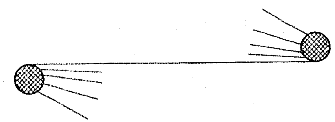

$\mu^{-1}(t)$

Fig. 3

The fact that $\mu^{-1}(t)$ is contractible for $t > 0$ can be seen by observing that, for fixed $t > 0$, the members of $C(Z_i) \cap \mu^{-1}(t)$ are uniformly, for all i and j, far away from the members of $C(Y_j) \cap \mu^{-1}(t)$ — see Fig. 3. The details are omitted. We mention that this example shows, in contrast to (3.1), that contractibility with respect to a given non-ANR need not be a Whitney-reversible property (since $\mu^{-1}(t)$ is contractible with respect to X, whenever $0 < t < \mu(X)$, but X is not contractible).

4. Acyclicity. It is known that being acyclic in dimension one is a Whitney property ([25, Theorem 4] or [31, Corollary 61]), but that being acyclic in dimension two is not a Whitney property [29]. In [26, (14.57)] it was asked if being acyclic is a strong Whitney-reversible property. We answer this question affirmatively in (4.3) below. First, we prove the following general result:

(4.1) THEOREM. *Let* X *be a continuum and let* μ *be a Whitney map for* $C(X)$. *Then, for any* n *and any* s *and* t *such that* $0 \leqslant s \leqslant t \leqslant \mu(X)$, *we have*

$$H^n(\mu^{-1}(t)) \approx H^n(\mu^{-1}([s, t]))\,.$$

Proof. Let $Y_1 = \mu^{-1}([s, t])$, let $Y_2 = \mu^{-1}([t, \mu(X)])$, let $A = Y_1 \cap Y_2 = \mu^{-1}(t)$, and let $Y = Y_1 \cup Y_2 = \mu^{-1}([s, \mu(X)])$. Since $(Y; Y_1, Y_2)$ is a compact triad, it is a proper triad [10, p. 257]. Hence, the Mayer–Vietoris cohomology sequence

$$(1) \qquad \ldots \to H^n(Y) \to H^n(Y_1) \oplus H^n(Y_2) \to H^n(A) \to H^{n+1}(Y) \to \ldots$$

is exact [10, 15.3c, p. 43]. Since hyperspaces are acyclic [34, 1.2] and r-maps preserve acyclicity, we have by (1.2) that Y and Y_2 are each acyclic. Therefore, by exactness of (1), $H^n(Y_1) \approx H^n(A)$.

The result in (4.1) has a geometric interpretation, namely: The cohomology of the band $\mu^{-1}([s, t])$ is completely determined by the cohomology of the top level $\mu^{-1}(t)$.

We will now use (4.1) to obtain the following stability theorem for acyclicity of levels:

(4.2) Theorem. *Let X be a continuum and let μ be a Whitney map for $C(X)$. Let $s \in [0, \mu(X)]$. If there is a sequence $\{t_i\}_{i=1}^{\infty} \to s$ as $i \to \infty$ such that, for each $i = 1, 2, \ldots$, $t_i \geqslant s$ and $H^n(\mu^{-1}(t_i)) \approx 0$ for some fixed n, then $H^n(\mu^{-1}(s)) \approx 0$.*

Proof. By (4.1), $H^n(\mu^{-1}([s, t_i])) \approx H^n(\mu^{-1}(t_i)) \approx 0$ for each $i = 1, 2, \ldots$ Therefore, since

$$\mu^{-1}(s) = \bigcap_{i=1}^{\infty} \mu^{-1}([s, t_i]),$$

we have by the continuity of cohomology [10, p. 260] that $H^n(\mu^{-1}(s)) \approx 0$.

The following corollary answers the question in [26, (14.57)] mentioned at the beginning of this section.

(4.3) Corollary. *The property of being acyclic in dimension n is a sequential strong Whitney-reversible property.*

Proof. Use (4.2) with $s = 0$.

5. Hereditarily indecomposable continua. A continuum X is *hereditarily indecomposable* if and only if whenever K and L are subcontinua of X such that $K \cap L \neq \emptyset$, $K \subset L$ or $L \subset K$. The pseudo-arc is the unique hereditarily indecomposable chainable continuum [3, Theorem 1] and a *pseudo-solenoid* is any hereditarily indecomposable circle-like continuum which is not chainable (i.e., not the pseudo-arc) — see [33, p. 580].

In this section we will be concerned primarily with the types of continua mentioned above. Our main results are in (5.3) and (5.5). The one in (5.5) generalizes and augments (14.54) of [26]. The reasons for our interest in these results will be discussed in (5.6).

Let us note the following definitions. A mapping f from a continuum Y to a continuum Z is said to be an *ε-map* provided that the diameter of $f^{-1}[f(y)]$ is less than ε for each $y \in Y$. Let $\mathscr{C}$ be a given nonempty class of continua. Then: A continuum Y is said to be *$\mathscr{C}$-like* provided that for each $\varepsilon > 0$ there exists an ε-map from Y *onto*

some member of $\mathscr{C}$. Let us note the following well-known and easy-to-prove fact which will be used in the proof of (5.3):

(5.1) *If $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$ are collections of continua such that each member of $\mathscr{C}_1$ is $\mathscr{C}_2$-like and each member of $\mathscr{C}_2$ is $\mathscr{C}_3$-like, then each member of $\mathscr{C}_1$ is $\mathscr{C}_3$-like.*

The following fact will be used several times in this section:

(5.2) *Let t be fixed, $0 < t < \mu(X)$. If $B_1, B_2 \in \mu^{-1}(t)$ such that $B_1 \cap B_2 \neq \emptyset$ and $B_1 \neq B_2$, then there is an arc $\beta \subset \mu^{-1}(t)$ such that B_1 and B_2 are the end points of β* ([27, Lemma 1] *or* [33, 3.6]).

(5.3) Theorem. *Let t be fixed, $0 \leqslant t < \mu(X)$, and let $\mathscr{C}$ be a given class of continua. If there is a sequence $\{t_n\}_{n=1}^{\infty} \to t$ as $n \to \infty$ such that $t_n \geqslant t$ and $\mu^{-1}(t_n)$ is hereditarily indecomposable $\mathscr{C}$-like for each $n = 1, 2, \ldots$, then $\mu^{-1}(t)$ is hereditarily indecomposable $\mathscr{C}$-like.*

Proof. Assume the hypotheses of (5.3). We define functions $f_n\colon \mu^{-1}(t) \to \mu^{-1}(t_n)$, $n = 1, 2, \ldots$, as follows. Let n be fixed. Let $A \in \mu^{-1}(t)$. Since $\mu^{-1}(t_n)$ contains no arc, it follows easily from (5.2) that there is at most one $B \in \mu^{-1}(t_n)$ such that $B \supset A$; furthermore, by using [17, 2.3 and 2.6], such a B exists. Let $f_n(A) = B$. An easy sequence argument using the compactness of $\mu^{-1}(t_n)$ shows that f_n is continuous for each $n = 1, 2, \ldots$ To see that $\mu^{-1}(t)$ is hereditarily indecomposable, let A_1 and A_2 be subcontinua of $\mu^{-1}(t)$ such that $A_1 \cap A_2 \neq \emptyset$. Then, for each $n = 1, 2, \ldots$, $f_n[A_1] \cap f_n[A_2] \neq \emptyset$ and, since f_n is continuous, $f_n[A_1]$ and $f_n[A_2]$ are subcontinua of $\mu^{-1}(t_n)$. Hence, since each $\mu^{-1}(t_n)$ is hereditarily indecomposable, we have that for any given $n = 1, 2, \ldots$, $f_n[A_1] \subset f_n[A_2]$ or $f_n[A_2] \subset f_n[A_1]$. By going to a subsequence (if necessary), we assume without loss of generality that $f_n[A_1] \subset f_n[A_2]$ for each $n = 1, 2, \ldots$ By using the definition of μ in section 1, it follows easily that the sequences $\{f_n[A_1]\}_{n=1}^{\infty}$ and $\{f_n[A_2]\}_{n=1}^{\infty}$ converge to A_1 and A_2 respectively. Thus, $A_1 \subset A_2$. Therefore, from what we have shown, we infer that $\mu^{-1}(t)$ is hereditarily indecomposable. To see that $\mu^{-1}(t)$ is $\mathscr{C}$-like, first observe by using [17, 2.3 and 2.6] that f_n maps $\mu^{-1}(t)$ onto $\mu^{-1}(t_n)$ for each $n = 1, 2, \ldots$ Easy computations show that for any given $\varepsilon > 0$, there exists $n(\varepsilon)$ such that $f_{n(\varepsilon)}$ is an ε-map onto $\mu^{-1}(t_{n(\varepsilon)})$. Hence, it follows using (5.1) that $\mu^{-1}(t)$ is $\mathscr{C}$-like. This completes the proof of (5.3).

The following proposition will be used in the proof of (5.5). Though some results in the literature could be combined to replace its use in proving (5.5), the proposition would seem to be of some independent interest (comp., [26, (14.73.13)]).

(5.4) Proposition. *Let t be fixed, $0 \leqslant t < \mu(X)$. If $\mu^{-1}(t)$ contains no arc, then*

$$\sigma_t\colon C(\mu^{-1}(t)) \to \mu^{-1}([t, \mu(X)]),$$

defined by $\sigma_t(\mathscr{A}) = \bigcup \mathscr{A}$ for each $\mathscr{A} \in C(\mu^{-1}(t))$, is one-to-one.

Proof. When $t = 0$, σ_t is one-to-one with no assumptions about $\mu^{-1}(t)$. If $t > 0$, then (5.4) follows easily using (5.2).

(5.5) THEOREM. *Let t be fixed, $0 \leqslant t < \mu(X)$. If there is a sequence $\{t_n\}_{n=1}^{\infty} \to t$ as $n \to \infty$ such that, for each $n = 1, 2, \ldots$, $t_n \geqslant t$ and $\mu^{-1}(t_n)$ is* (1) *hereditarily indecomposable,* (2) *the pseudo-arc,* (3) *any particular (fixed) pseudo-solenoid, or* (4) *hereditarily indecomposable tree-like, then $\mu^{-1}(t)$ is as in* (1), (2), (3), *or* (4) *respectively. Hence, the properties in* (1) *through* (4) *are sequential strong Whitney-reversible properties.*

Proof. To prove (1), simply let $\mathscr{C} = \{\mu^{-1}(t_n): n = 1, 2, \ldots\}$ and apply (5.3). To prove (2), let $\mathscr{C} = \{[0, 1]\}$ and use (5.3) and [26, Theorem 1]. Now we prove (3). By taking $\mathscr{C} = \{S^1\}$ and applying (5.3), we see that $\mu^{-1}(t)$ is an hereditarily indecomposable circle-like continuum. Hence, (i) $\mu^{-1}(t)$ is the pseudo-arc or a particular pseudo-solenoid. Let us note the following known facts: (ii) The property of being the pseudo-arc is a Whitney property [9, p. 1032], as is the property of being a particular pseudo-solenoid [33, 4.11]; (iii) If σ_t, defined in (5.4), is one-to-one and if $\mu^{-1}(t)$ has a certain Whitney property P, then $\mu^{-1}(s)$ has property P for all $s \geqslant t$ [30, Corollary 26]. Now, by using (5.4) and (i) through (iii) above, it follows easily that $\mu^{-1}(t)$ is the particular pseudo-solenoid that $\mu^{-1}(t_n)$ is for each $n = 1, 2, \ldots$ This completes the proof of (3). To prove (4), let $\mathscr{C}$ denote the class of trees (i.e., simply connected linear graphs) and apply (5.3).

(5.6) Remark. It is known that being any of the types of continua in (1) through (4) of (5.5) is a Whitney property (see [9, p. 1032], [20, 4.2], and [33, 4.11]). Thus, (5.5) provides a converse to these known results. Our interest in (5.5) is also due to the following considerations. In [26, (14.57)], the following question was asked: Is being (a) *chainable* or (b) *circle-like* a strong Whitney-reversible property? In (5.5) we showed that the answer to this question is "yes" in the hereditarily indecomposable case. Moreover, in (5.5) we saw that topological type was preserved (in general this does not happen). The answer to the question above is not known. Since chainable continua are tree-like, we see from (3.5) that a continuum X, which would show that the answer to (a) is "no", would need to be an a-triodic nonchainable tree-like continuum. On the other hand, if an affirmative answer were obtained to the question above, then we would have converses to the results in [21, 6.2]. Besides (3.4) and (5.5), we mention the following partial answers to the question above. In [1] it was shown that being (c) *hereditarily decomposable* and (d) *hereditarily decomposable chainable* are each sequential strong Whitney-reversible properties. Let us note the following analogue of (d) for circle-like continua:

(5.7) THEOREM. *The property of being hereditarily decomposable circle-like is a sequential strong Whitney-reversible property.*

Proof. Assume there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty} \to 0$ $n \to \infty$ such that $\mu^{-1}(t_n)$ is hereditarily decomposable circle-like for each $n = 1, 2, \ldots$ By (c) of (5.6), X is hereditarily decomposable. It remains to show that X is circle-like. Note the following two facts: (1) each nondegenerate proper subcontinuum of each $\mu^{-1}(t_n)$ is a hereditarily decomposable chainable continuum; (2) if Y is a nondegenerate subcontinuum of X, then the restriction of μ to $C(Y)$ is a Whitney map for $C(Y)$. Using these two facts, together with (d) of (5.6), it follows that each nondegenerate proper subcontinuum of X is chainable. Hence, since X is decomposable, X is chainable or circle-like by [16, Theorem 4]. Choose and fix n. Suppose that X is chainable. Then, since chainability is a Whitney property [21, 6.2a], $\mu^{-1}(t_n)$ is chainable. Thus, since $\mu^{-1}(t_n)$ is also circle-like, $\mu^{-1}(t_n)$ contains a nondegenerate indecomposable continuum by [6, Theorem 3]. This contradicts the hereditary decomposability assumption on $\mu^{-1}(t_n)$. Therefore, we have proved that X is circle-like.

In (5.5) through (5.7) we have seen that certain subclasses of the chainable and circle-like continua are reversible. It would be interesting to obtain some other such classes. In this connection we mention that the property of being an arc (respectively, a circle) is a sequential strong Whitney-reversible property (26, (14.50) and (14.51)). These two results give converses to the results in [21, 6.4].

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