Proof. Denote by B a set of points of discontinuity of the function f from Theorem 1. The set B is dense in R. Thus, in virtue of the corollary, there exists a homeomorphism of the class C_1, $g\colon R \xrightarrow{\text{onto}} R$, such that

$$g(A) = B \quad \text{and} \quad \pi^{-2} < g'(x) < \pi^2 \quad \text{for} \quad x \in R.$$

Put

$$h(x) = f[g(x)] \quad \text{for} \quad x \in R.$$

Then, the function h satisfies the conditions of Theorem 3. ■

Reference

[1] S. Marcus, *Sur les proprietés differentielles des fonctions dont les points de continuité forment un ensemble frontière partout dense*, Annales Scientifiques de l'École Normale Supérieure (3), 79, 1 (1958), pp. 1–21.

INSTITUTE OF MATHEMATICS
ŁÓDŹ UNIVERSITY

Accepté par la Rédaction le 28. 4. 1978

Some existence and non-existence theorems for k-regular maps

by

David Handel (Detroit, Mich.)

Abstract. A continuous map $f\colon X \to \boldsymbol{R}^n$ is said to be *k-regular* if whenever $x_1, \ldots, x_j$ are distinct points of X with $j \leqslant k$, then $f(x_1), \ldots, f(x_j)$ are linearly independent. Using some algebraic-topological machinery, a non-existence theorem for $2k$-regular maps on a disjoint union of k closed manifolds is obtained. In the other direction, we show explicitly that if X is a metric space and A is a closed neighborhood retract in X, then existence of k-regular maps $X-A \to \boldsymbol{R}^m$ and $A \to \boldsymbol{R}^n$ implies existence of a k-regular map $X \to \boldsymbol{R}^{m+n}$. Some applications of these existence and non-existence results are given.

1. Introduction. The study of k-regular maps is motivated by the theory of Čebyšev approximation. In that theory, a set of n real-valued continuous functions on X is called a *k-Čebyšev set of length n* if these functions are the components of a k-regular map of X into $\boldsymbol{R}^n$. The reader is referred to [8, pp. 237–242] for the significance of this concept.

Closely related to k-regularity is the concept of *affine k-regularity*: $f\colon X \to \boldsymbol{R}^n$ is affinely k-regular if and only if $g\colon X \to \boldsymbol{R}^{n+1}$ given by $g(x) = (1, f(x))$ is $(k+1)$-regular. Affine k-regularity has been considered in [2], [1], and [7]. Some previous work on k-regularity, using algebraic-topological tools, appears in [5], [3], and [4].

The problem we consider is the following: Given X, k, and n, does there exist a k-regular map $X \to \boldsymbol{R}^n$? In Section 2 we prove the following:

THEOREM 2.4. *Let $M_1, \ldots, M_k$ be closed, connected manifolds of dimensions $n_1, \ldots, n_k$, respectively. Suppose, for $1 \leqslant i \leqslant k$, the q_i-th dual Stiefel–Whitney class of M_i is non-zero. If a $2k$-regular map of the disjoint union $\coprod_{i=1}^{k} M_i$ into $\boldsymbol{R}^N$ exists, then $N \geqslant 2k + \sum_{i=1}^{k} (n_i + q_i)$.*

In [1], an affine analogue of the following is proved by geometric methods:

THEOREM 1.1 (Boltjanskiĭ–Ryškov–Šaškin). *Let $n \geqslant 1$. If a $2k$-regular map of $\boldsymbol{R}^n$ into $\boldsymbol{R}^N$ exists, then $N \geqslant (n+1)k$.*

In [3], 1.1 is improved in the case $n = 2$, using algebraic topology. An immediate corollary of 2.4 is the following improvement of 1.1 in another direction:

COROLLARY 2.5. *Let $n \geqslant 1$. If a $2k$-regular map of a disjoint union of k copies of S^{n-1} into $\boldsymbol{R}^N$ exists, then $N \geqslant (n+1)k$.*

In Section 3, the following existence theorem is proved:

THEOREM 3.1. *Let X be a metric space, and A a closed subspace of X which is a retract of some neighborhood in X. Suppose there exist k-regular maps of $X-A$ and A into $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, respectively. Then there exists a k-regular map of X into $\boldsymbol{R}^{m+n}$.*

Results of Sections 2 and 3 are used in Section 4 to obtain some best-possible results for k-regular maps on certain 1-dimensional complexes. Results of 2, 3, and 4 are used in Section 5 to obtain some existence and non-existence results for k-regular maps on certain 2-dimensional manifolds, but there is a dimensional gap between these existence and non-existence results.

2. A non-existence theorem. Let $F(X, k)$ denote the kth configuration space of X, i.e., the subspace of X^k consisting of all ordered k-tuples of distinct points of X. The symmetric group Σ_k acts freely on $F(X, k)$, and orthogonally on $\boldsymbol{R}^k$, by permuting factors. Thus, for each subgroup G of Σ_k and each Hausdorff space X, we obtain a real k-plane bundle $F(X, k) \times_G \boldsymbol{R}^k \to F(X, k)/G$ which we denote by $F(X, k, G)$. The following proposition is a slight extension of Corollary 2.2 of [3]:

PROPOSITION 2.1. *If a k-regular map $f\colon X \to \boldsymbol{R}^N$ exists, then for each subgroup G of Σ_k, $F(X, k, G)$ admits an N-k-plane inverse.*

Proof. Write $(x; t)_G$ for the point in $F(X, k) \times_G \boldsymbol{R}^k$ determined by $(x, t) \in F(X, k) \times \boldsymbol{R}^k$. Let $g\colon F(X, k) \times_G \boldsymbol{R}^k \to \boldsymbol{R}^N$ be given by $g(x_1, \ldots, x_k; t_1, \ldots, t_k)_G = \sum_i t_i f(x_i)$. g is well-defined, and the k-regularity of f implies that the restriction of g to each fibre is an $\boldsymbol{R}$-monomorphism.

We recall the following result of Wu [9, p. 380]:

THEOREM 2.2 (Wu). *Let M be a closed, connected n-dimensional manifold. Suppose q is the largest integer such that $\overline{w}_q(M)$, the q-th dual Stiefel–Whitney class of M, is non-zero. Let v denote the first Stiefel–Whitney class of the double covering $F(M, 2) \to F(M, 2)/\Sigma_2$. Then $v^{n+q} \neq 0$, $v^{n+q+1} = 0$.*

LEMMA 2.3. *Let X be Hausdorff, and let L denote the real line bundle associated with the double covering $F(X, 2) \to F(X, 2)/\Sigma_2$. Then the real 2-plane bundle $F(X, 2, \Sigma_2)$ is isomorphic to the Whitney sum of L and a trivial line bundle.*

Proof. The total space $E(L)$ of L is $F(X, 2) \times_{\Sigma_2} \boldsymbol{R}$ where the generator of Σ_2 acts on $\boldsymbol{R}$ by multiplication by -1. Write $[x, y; t]$ for the point in $E(L)$ determined by $(x, y; t) \in F(X, 2) \times \boldsymbol{R}$. Define $f\colon E(L) \times \boldsymbol{R} \to F(X, 2) \times_{\Sigma_2} \boldsymbol{R}^2$ by $f([x, y; t], s) = (x, y; s+t, s-t)_{\Sigma_2}$. Then f is well-defined and is the desired bundle isomorphism.

THEOREM 2.4. *Let $M_1, \ldots, M_k$ be closed, connected manifolds of dimensions $n_1, \ldots, n_k$, respectively. Suppose, for $1 \leqslant i \leqslant k$, q_i is the largest integer for which $\overline{w}_{q_i}(M_i) \neq 0$. If a $2k$-regular map of the disjoint union $\coprod_{i=1}^{k} M_i$ into $\boldsymbol{R}^N$ exists, then $N \geqslant 2k + \sum_{i=1}^{k} (n_i + q_i)$.*

Proof. Write $X = \coprod_{i=1}^{k} M_i$ and let $G = \underbrace{\Sigma_2 \times \ldots \times \Sigma_2}_{k} \subset \Sigma_{2k}$, where the generator of the ith factor is the transposition which interchanges $2i-1$ and $2i$, $1 \leqslant i \leqslant k$. By 2.1, it suffices to show that $\overline{w}_r(F(X, 2k, G)) \neq 0$ for $r = \sum_{i=1}^{k} (n_i + q_i)$. We have a map of $2k$-plane bundles

$$F(M_1, 2, \Sigma_2) \times \ldots \times F(M_k, 2, \Sigma_2) \to F(X, 2k, G)$$

given by

$$((x_1, y_1; s_1, t_1)_{\Sigma_2}, \ldots, (x_k, y_k; s_k, t_k)_{\Sigma_2}) \mapsto (x_1, y_1, \ldots, x_k, y_k; s_1, t_1, \ldots, s_k, t_k)_G .$$

Thus, if suffices to show that the r-dimensional component of $\overline{w}(F(M_1, 2, \Sigma_2)) \times \times \ldots \times \overline{w}(F(M_k, 2, \Sigma_2))$ is non-zero, where r is as above. By 2.3,

$$\overline{w}(F(M_i, 2, \Sigma_2)) = (1+v_i)^{-1} = \sum_{j \geqslant 0} v_i^j$$

where v_i is the first Stiefel–Whitney class of the double covering $F(M_i, 2) \to F(M_i, 2)/\Sigma_2$. Thus by 2.2, the dimension of the highest non-zero component of $\overline{w}(F(M_i, 2, \Sigma_2))$ is $n_i + q_i$, and the theorem now follows.

COROLLARY 2.5. *Let $n \geqslant 1$. If a $2k$-regular map of a disjoint union of k copies of S^{n-1} into $\boldsymbol{R}^N$ exists, then $N \geqslant (n+1)k$.*

Since the G used in the proof of 2.4 is such a small subgroup of Σ_{2k}, it is expected that 2.4 can be improved. However, we shall see in Section 4 that 2.5 is a best possible result when $n = 2$. (It is trivially best possible when $n = 1$).

3. An existence theorem.

THEOREM 3.1. *Let X be a metric space, and A a closed subspace of X which is a retract of some neighborhood in X. Suppose there exist k-regular maps of $X-A$ and A into $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, respectively. Then there exists a k-regular map of X into $\boldsymbol{R}^{m+n}$.*

Proof. Let $r\colon U \to A$ be a retraction of an open set U onto A. Choose continuous maps $\alpha, \beta\colon X \to I$ such that $\alpha(A) = 0$, $\alpha(X-A) \subset (0, 1]$, $\beta(A) = 1$, $\beta(X-U) = 0$. Suppose $f\colon A \to \boldsymbol{R}^m$, $g\colon X-A \to \boldsymbol{R}^n$ are k-regular. We can suppose f and g are bounded (e.g. replace $f(x)$ by $f(x)/\|f(x)\|$). Define $h\colon X \to \boldsymbol{R}^{m+n} = \boldsymbol{R}^m \times \boldsymbol{R}^n$ by

$$h(x) = \begin{cases} (f(x), 0) & \text{if } x \in A, \\ (\beta(x) f(r(x)), \alpha(x) g(x)) & \text{if } x \in U-A, \\ (0, \alpha(x) g(x)) & \text{if } x \in X-U. \end{cases}$$

Then h is k-regular.

DEFINITION 3.2. Let X be a topological space. A closed subspace A of X is an *n-complement in X* if A is a retract of a neighborhood in X, and $X-A$ is topologically embeddable in $\boldsymbol{R}^n$.

We conclude this section by recalling some known facts which will be used in Sections 4 and 5.

PROPOSITION 3.3. *The maps* $R \to R^k$ *and* $R^2 = C \to R^{2k-1} = R \times C^{k-1}$ *given by* $x \to (1, x, x^2, \ldots, x^{k-1})$ *are k-regular.*

This is immediate from the non-vanishing of the Vandermonde determinant.

PROPOSITION 3.4. *Suppose a k-regular map* $f\colon X \to R^n$ *exists. Then for each* $x_0 \in X$, *there exists a* $(k-1)$*-regular map* $X - \{x_0\} \to R^{n-1}$.

In fact f, followed by projection on the orthogonal complement of $f(x_0)$, yields the desired $(k-1)$-regular map on $X - \{x_0\}$. (See [2, p. 355], [5, Theorem 2.2]).

4. 1-dimensional complexes.

THEOREM 4.1. *Let* X_i *be a one-point union of a finite or countably infinite number of circles,* $1 \leqslant i \leqslant m$, *and let X be the disjoint union of the* X_i. *Then for each* $k \geqslant 1$, *there exists a k-regular map of X into* $R^{\min\{2k, k+m\}}$. *Moreover, this result is best possible if either* a) $k = 2m$, *or* b) $k > 2m$ *and at least one of the* X_i *is a one-point union of at least* 2 *circles.*

Proof. X has a 1-complement A consisting of m points (1 point from each X_i). Thus there exists a k-regular map of A into R^m. On the other hand, since A is embeddable in R, it follows from 3.3 that there exists a k-regular map of A into R^k. Thus there exists a k-regular map of A into $R^{\min\{k,m\}}$. Since $X - A$ is embeddable in R, it follows from 3.3 that there exists a k-regular map of $X - A$ into R^k. The existence part now follows from 3.1.

Since X contains a disjoint union of m circles, the non-existence part under hypothesis a) follows from 2.5.

Suppose hypothesis b) holds and that there exists a k-regular map of X into R^{k+m-1}. There exists a disjoint union of m circles Y contained in X such that $X - Y$ is an infinite set. In particular, $X - Y$ contains $k - 2m$ distinct points. Then, by repeated applications of 3.4, there would exist a $2m$-regular map of Y into $R^{k+m-1-(k-2m)} = R^{3m-1}$, contradicting 2.5.

4.1 will be used in Section 5. Results on various other 1-dimensional complexes, some of them best possible, can be obtained by the above methods. For example:

THEOREM 4.2. *Let X be any countable* 1*-dimensional complex. Then for any* $k \geqslant 1$, *there exists a k-regular map of X into* R^{2k}.

Proof. X has a 1-complement A which is a finite or countably infinite discrete space. Thus A and $X - A$ are both embeddable in R, and so by 3.3, both admit k-regular maps into R^k. The result now follows by 3.1.

5. 2-dimensional manifolds.

THEOREM 5.1. *Let* $M_1, \ldots, M_k$ *be closed, connected* 2*-dimensional manifolds, and let M be the disjoint union of the* M_i. *Suppose exactly r of the* M_i *are non-orientable. Then there does not exist a* $2k$*-regular map of M into* R^{4k+r-1}. *If the* M_i *are all* 2*-spheres, there exists a* $2k$*-regular map of M into* R^{5k-1}. *If exactly t of the* M_i *are not spheres and* $t > 0$, *there exists a* $2k$*-regular map of M into* R^{6k+t-1}.

Proof. If M_i is non-orientable then $\overline{w}_1(M_i)$ is the top non-zero dual Stiefel–Whitney class of M_i, while if M_i is orientable then $\overline{w}_0(M_i)$ is the top non-zero dual Stiefel–Whitney class of M_i ([6, p. 120, Cor. 11.4 and p. 148, Prob. 12-A]). Thus the $n_i + q_i$ of 2.4 is 3 if M_i is non-orientable, 2 if M_i is orientable. Hence $2k + \sum_{i=1}^{k} (n_i + q_i) = 2k + 3r + 2(k-r) = 4k + r$, and the non-existence part now follows from 2.4.

If the M_i are all spheres, then M has a 2-complement A consisting of k points (one from each sphere). Thus there exists a $2k$-regular map of A into R^k. By 3.3, there exists a $2k$-regular map of $M - A$ into R^{4k-1}. Hence by 3.1, there exists a $2k$-regular map of M into R^{5k-1}.

If M_i is not a sphere, M_i has a 2-complement consisting of a one-point union of a finite number of circles. Thus if t of the M_i are not spheres, M has a 2-complement A consisting of the disjoint union of a discrete space having $k - t$ points with a disjoint union of t one-point unions of circles. By embedding the discrete part of A in a circle, A is embeddable in a disjoint union of t one-point unions of circles. Thus by 4.1, there exists a $2k$-regular map of A into R^{2k+t}. By 3.3, there exists a $2k$-regular map of $M - A$ into R^{4k-1}, and so it follows from 3.1 that M admits a $2k$-regular map into R^{6k+t-1}.

Note that any improvement of 3.3 on the existence of k-regular maps on R^2 will yield a corresponding improvement of the existence part of 5.1.

References

[1] V. G. Boltjanskiĭ, S. S. Ryškov, and Ju. A. Šaškin, *On k-regular imbeddings and their application to the theory of approximation of functions*, Uspehi Mat. Nauk 15 (1960), 6 (96), pp. 125–132 (Russian); Amer. Math. Soc. Transl. (2) 28 (1963), pp. 211–219.

[2] K. Borsuk, *On the k-independent subsets of the Euclidean space and of the Hilbert space*, Bull. Acad. Polon. Sci. 5 (1957), pp. 351–356.

[3] F. R. Cohen and D. Handel, *k-regular embeddings of the plane*, Proc. Amer. Math. Soc. 72 (1978), pp. 201–204.

[4] D. Handel, *Obstructions to 3-regular embeddings* Houston J. Math. 5 (1979), pp. 339–343.

[5] — and J. Segal, *On k-regular embeddings of spaces in Euclidean space*, Fund. Math. 106 (1980), pp. 231–237.

[6] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. of Math. Studies No. 76, Princeton University Press, Princeton, New Jersey (1974).

[7] Ju. A. Šaškin, *Topological properties of sets connected with approximation theory*, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), pp. 1085–1094 (Russian).

[8] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, 1970.

[9] W.-t. Wu, *On the realization of complexes in Euclidean space*, II, Scientia Sinica 7 (1958), pp. 365–387.

WAYNE STATE UNIVERSITY
Detroit, Michigan

Accepté par la Rédaction le 15. 5. 1978