strategy for Player I in the game $G(\mathrm{Dim}_n, X)$. Finally, by Proposition 3, we have $\dim X \leqslant n$.

COROLLARY 3. *If a normal space X has a σ-closure-preserving cover $\mathscr{F}$ such that each $F \in \mathscr{F}$ is countably compact, closed and $\dim F \leqslant n$, then $\dim X \leqslant n$.*

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On some Marcus problem concerning functions possessing the derivative at points of discontinuity only

by

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Abstract. In this paper we obtain (Theorem 1) affirmative answer on the following Marcus' problem [1]:

Does there exist a function with a dense and, at the same time, boundary set of points of continuity, which possesses the derivative at each point of discontinuity and which possesses no unilateral derivative at any point of continuity?

Theorem 3 generalizes the obtained solution so that any dense, denumerable set given earlier is a set of points of discontinuity of some function satisfying the conditions formulated in the Marcus problem. As an auxiliary theorem we use Theorem 2 on the existence of homeomorphisms of the class C_1, mapping the given, arbitrary, dense, denumerable set onto another such set.

1. Introduction. It is known that the existence of the derivative at a point for a function of a real variable does not depend on the continuity of the function at this point. All the same, the derivative exists only in an at most denumerable subset of the set of points of discontinuity. Therefore the condition that the derivative exists at each point of discontinuity can be satisfied only for the functions whose set of points of discontinuity is at most denumerable. In connection with the above, it is interesting to know whether there exist functions singular in the sense that they have the derivative in the set of points of discontinuity, which is denumerable and infinite, and they have no derivative at the remaining points. With this question the following problem of S. Marcus is connected (see [1], p. 13, Problem 5):

Does there exist a function with a dense and, at the same time, boundary set of points of continuity, which possesses the derivative at each point of discontinuity and which possesses no unilateral derivative (neither the left-hand nor the right-hand one) at any point of continuity?

Theorem 1 of this paper gives an affirmative answer to the above question, and Theorem 3 generalizes the obtained solution so that any dense denumerable set given earlier is a set of points of discontinuity of some function satisfying the conditions formulated in the Marcus problem. As an auxiliary theorem we use Theorem 2 on the existence of homeomorphisms of the class C_1, mapping the given arbitrary dense denumerable set onto another such set.

In the paper we consider real functions defined in the set R of real numbers. The symbols $\underline{f}^-(x)$, $\bar{f}^-(x)$, $\underline{f}^+(x)$ and $\bar{f}^+(x)$ denote the Dini derivatives of the function f at the point x, respectively, the lower left-hand, the upper left-hand, the lower right-hand, and the upper right-hand ones.

2. Solution of the Marcus problem. Let $(x_1, x_2, \ldots)_2 = \sum_{k=1}^{\infty} 2^{-k}\cdot x_k$ be a dyadic expansion of the number $x \in \langle 0, 1)$, containing infinitely many terms equal to zero. Such a representation is determined uniquely by the number x. Denote by $\bar{x}_n$ the non-negative remainder of dividing the number

$$\sum_{k=1}^{n} |x_k - x_{k-1}| \quad (x_0 = 0)$$

by 6, and put

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots),$$

$$x'_n = \begin{cases} 0, & \text{when } 3 | \bar{x}_n, \\ 1, & \text{when } 3 \nmid \bar{x}_n \end{cases}$$

and

$$g(x) = (x'_1, x'_2, \ldots)_2 = \sum_{n=1}^{\infty} 2^{-n} x'_n .$$

It is easily seen that the sum $\sum_{k=1}^{n} |x_k - x_{k-1}|$ is a number of changes of the values in the sequence $(x_0, x_1, \ldots, x_n)$ and that:

$$x_n = 0 \Leftrightarrow \bar{x}_n \text{ is an even number } (0, 2 \text{ or } 4),$$

$$x_n = 1 \Leftrightarrow \bar{x}_n \text{ is an odd number } (1, 3 \text{ or } 5).$$

LEMMA 1. *The function g is continuous from the right at each point $x \in \langle 0, 1)$.*

Proof. Let $x = (x_1, x_2, \ldots)_2$, and let ε be any positive number. We choose an m such that $2^{-m} < \varepsilon$. In the sequence $(x_1, x_2, \ldots)$ there are infinitely many terms equal to zero, therefore we have $x_n = 0$ for some $n > m$. For any $y = (y_1, y_2, \ldots)_2 \in \langle 0, 1)$ satisfying the inequality

(1) $$0 < y - x < 2^{-n}$$

we have

(2) $$x_k = y_k \quad \text{for} \quad k < n .$$

Indeed, if $x_l < y_l$ for some $l < n$, then, considering the smallest such l, we would obtain the inequality

$$y - x = \sum_{k=1}^{\infty} 2^{-k}(y_k - x_k) \geqslant 2^{-l} - \sum_{k=l+1}^{n-1} 2^{-k} - \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}$$

contradicting (1). Similarly, the supposition that $x_l > y_l$ for some $l < n$ leads now to the inequality

$$x - y \geqslant 2^{-l} - \sum_{k=l+1}^{\infty} 2^{-k} \geqslant 0$$

which contradicts (1). From the relations (2) already proved there follows an estimate

$$|g(y) - g(x)| \leqslant \sum_{k=n}^{\infty} 2^{-k} |y'_k - x'_k| \leqslant 2^{-n+1} \leqslant 2^{-m} < \varepsilon$$

implying the equality $\lim_{y \to x+} g(y) = g(x)$. ■

LEMMA 2. *The function g is continuous from the left at each point $x \in (0, 1)$ whose dyadic expansion $(x_1, x_2, \ldots)_2$ contains infinitely many terms equal to* 1.

Proof. Let ε be any positive number, and m a natural number such that $2^{-m} < \varepsilon$. Since in the sequence $(x_1, x_2, \ldots)$ infinitely many terms $x_k = 1$, there exists an $n > m$ such that $x_n = 1$. If the number $y = (y_1, y_2, \ldots)_2 \in \langle 0, 1)$ satisfies the inequality

$$0 < x - y < 2^{-n},$$

then $x_k = y_k$ for $k < n$. Indeed, if $y_l < x_l$ and $y_k = x_k$ for $k < l$, where $l < n$, then

$$x - y = \sum_{k=1}^{\infty} 2^{-k}(x_k - y_k) \geqslant 2^{-l} - \sum_{l+1}^{n-1} 2^{-k} - \sum_{n+1}^{\infty} 2^{-k} = 2^{-n},$$

which contradicts the inequality $x - y < 2^{-n}$. Similarly, if $y_l > x_l$ for some $l < n$, and $x_k = y_k$ for $k < l$, then we would obtain the inequality $y \geqslant x$ contradicting the assumption. From the equality $x_k = y_k$ we now obtain the estimate

$$|g(x) - g(y)| \leqslant \sum_{n}^{\infty} 2^{-k} |x'_k - y'_k| \leqslant 2^{-n+1} \leqslant 2^{-m} < \varepsilon$$

which proves the left-hand continuity of the function g at the point x. ■

LEMMA 3. *If, for the number $x = (x_1, x_2, \ldots)_2 \in (0, 1)$, there exists an n such that $x_n = 1$, and $x_k = 0$ for $k > n$, then*

(3) $$\lim_{y \to x-} g(y) = \begin{cases} g(x) - 2^{-n} & \textit{when } \bar{x}_n = 1, \\ g(x) & \textit{when } \bar{x}_n = 3 \textit{ and } x_{n-1} = 0 \\ & \textit{or } \bar{x}_n = 5 \textit{ and } x_{n-1} = 1, \\ g(x) + 2^{-n} & \textit{when } \bar{x}_n = 3 \textit{ and } x_{n-1} = 1 \\ & \textit{or } \bar{x}_n = 5 \textit{ and } x_{n-1} = 0 . \end{cases}$$

Proof. Reasoning similarly as in the proof of Lemma 2, we get the following statement:

(4) If $0 < x - y < 2^{-n}$, then $y_k = x_k$ for $k < n$, where $y = (y_1, y_2, \ldots)_2$.

Let $m > n$ be any natural number. We shall now prove the proposition:

(5) If $0 < x - y < 2^{-m}$, then $x_k = y_k$ for $k < n$, $y_n = 0$ and $y_k = 1$ for $n < k \leqslant m$.

Suppose that the proposition (5) is false. Then, in virtue of (4), there exists an index l such that $n < l \leqslant m$ and $y_l = 0$. Hence we obtain the inequality

$$x - y \geqslant 2^{-n} - \sum_{n+1}^{l-1} 2^{-k} - \sum_{l+1}^{\infty} 2^{-k} = 2^{-l} \geqslant 2^{-m}$$

contradicting the assumption of the proposition (5).

Let y be any point satisfying the inequality

$$0<x-y<2^{-m}.$$

From the assumption of the lemma and from the proposition (5) it follows that

$$x=(x_1,\ldots,x_{n-1},1,0,0,\ldots)_2$$

and

$$y=(x_1,\ldots,x_{n-1},0,1,\ldots,1,y_{m+1},y_{m+2},\ldots)_2.$$

If $\bar{x}_n=1$ and $x_{n-1}=0$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},0,1,2,2,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},0,0,1,\ldots,1,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},0,1,1,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},0,0,1,\ldots,1,y'_{m+1},y'_{m+2},\ldots)_2,\end{aligned}$$

$$g(y)-g(x)=-2^{-n}+\sum_{m+1}^{\infty}2^{-k}(y'_k-1)$$

and

$$\lim_{y\to x-}g(y)=g(x)-2^{-n},$$

for with $y\to x-$, we have $m\to\infty$ and

$$-2^{-m}\leqslant\sum_{m+1}^{\infty}2^{-k}(y'_k-1)\leqslant 0.$$

If $\bar{x}_n=1$ and $x_{n-1}=1$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},1,1,2,2,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},1,2,3,\ldots,3,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},1,1,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},1,1,0,\ldots,0,y'_{m+1},y'_{m+2},\ldots)_2,\end{aligned}$$

$$g(y)-g(x)=-2^{-n}+2^{-m}+\sum_{m+1}^{\infty}2^{-k}(y'_k-1)$$

and

$$\lim_{y\to x-}g(y)=g(x)-2^{-n}.$$

If $\bar{x}_n=3$ and $x_{n-1}=0$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},2,3,4,4,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},2,2,3,\ldots,3,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},1,0,1,1,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},1,1,0,\ldots,0,y'_{m+1},y'_{m+2},\ldots)_2,\end{aligned}$$

$$g(y)-g(x)=2^{-m}+\sum_{m+1}^{\infty}2^{-k}(y'_k-1)$$

and

$$\lim_{y\to x-}g(y)=g(x).$$

If $\bar{x}_n=3$ and $x_{n-1}=1$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},3,3,4,4,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},3,4,5,\ldots,5,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},0,0,1,1,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},0,1,1,\ldots,1,y'_{m+1},y'_{m+2},\ldots)_2,\end{aligned}$$

$$g(y)-g(x)=2^{-n}+\sum_{m+1}^{\infty}2^{-k}(y'_k-1)$$

and

$$\lim_{y\to x-}g(y)=g(x)+2^{-n}.$$

If $\bar{x}_n=5$ and $x_{n-1}=0$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},4,5,0,0,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},4,4,5,\ldots,5,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},1,1,0,0,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},1,1,1,\ldots,1,y'_{m+1},y'_{m+2},\ldots)_2,\end{aligned}$$

$$g(y)-g(x)=2^{-n}-2^{-m}+\sum_{m+1}^{\infty}2^{-k}y'_k$$

and

$$\lim_{y\to x-}g(y)=g(x)+2^{-n}.$$

Finally, if $\bar{x}_n=5$, and $x_{n-1}=1$, then

$$\begin{aligned}\bar{x}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},5,5,0,0,\ldots),\\ \bar{y}&=(\bar{x}_1,\ldots,\bar{x}_{n-2},5,0,1,\ldots,1,\bar{y}_{m+1},\bar{y}_{m+2},\ldots),\\ g(x)&=(x'_1,\ldots,x'_{n-2},1,1,0,0,\ldots)_2,\\ g(y)&=(x'_1,\ldots,x'_{n-2},1,0,1,\ldots,1,y'_{m+1},y'_{m+2},\ldots),\end{aligned}$$

$$g(y)-g(x)=-2^{-m}+\sum_{m+1}^{\infty}2^{-k}y'_k$$

and

$$\lim_{y\to x-}g(y)=g(x).\ \blacksquare$$

LEMMA 4. *If, for the number $x=(x_1,x_2,\ldots)_2\in(0,1)$, there exists an n such that $x_n=1$, and $x_k=0$ for $k>n$, then*

(6) $\underline{g}^-(x)\leqslant-\frac{7}{5}$, $\bar{g}^-(x)\geqslant-1$, $\underline{g}^+(x)\leqslant-1$ *and* $\bar{g}^+(x)\geqslant-\frac{1}{3}$ *when* $\bar{x}_n=3$ *and* $x_{n-1}=0$;

(7) $\underline{g}^-(x)\leqslant 0$, $\bar{g}^-(x)\geqslant\frac{1}{3}$, $\underline{g}^+(x)\leqslant\frac{4}{3}$ *and* $\bar{g}^+(x)\geqslant 2$ *when* $\bar{x}_n=5$ *and* $x_{n-1}=1$.

Proof. We shall define the sequences $(u(m))$, $(v(m))$, $(y(m))$ and $(z(m))$ $(m=n+1,n+2,\ldots)$, convergent to x, and such that $u(m),v(m)<x<y(m),z(m)$. Let $u(m)=(u_1,u_2,\ldots)_2$, $v(m)=(v_1,v_2,\ldots)_2$, $y(m)=(y_1,y_2,\ldots)_2$ and $z(m)$

$= (z_1, z_2, \ldots)_2$, and, in the case of $\bar{x}_n = 3$ and $x_{n-1} = 0$, let there be:

$$u_k = v_k = y_k = z_k = x_k \quad \text{for} \quad k<n\,;$$
$$u_n = u_{m+1} = u_{m+4} = u_{m+5} = \ldots = 0$$

and

$$u_{n+1} = u_{n+2} = \ldots = u_m = u_{m+2} = u_{m+3} = 1\,;$$
$$v_n = v_{m+1} = v_{m+2} = \ldots = 0 \quad \text{and} \quad v_{n+1} = v_{n+2} = \ldots = v_m = 1\,;$$
$$y_{n+1} = y_{n+2} = \ldots = y_{m-1} = y_{m+1} = y_{m+2} = \ldots = 0 \quad \text{and} \quad y_n = y_m = 1\,;$$
$$z_{n+1} = z_{n+2} = \ldots = z_{m-1} = z_{m+2} = z_{m+3} = \ldots = 0 \quad \text{and} \quad z_n = z_m = z_{m+1} = 1.$$

Then

$$g(x) = (x'_1, x'_2, \ldots)_2, \quad g\big(u(m)\big) = (u'_1, u'_2, \ldots)_2\,,$$
$$g\big(v(m)\big) = (v'_1, v'_2, \ldots)_2, \quad g\big(y(m)\big) = (y'_1, y'_2, \ldots)_2$$

and

$$g\big(z(m)\big) = (z'_1, z'_2, \ldots)_2\,,$$

where

$$x'_k = u'_k = v'_k = y'_k = z'_k \quad \text{for} \quad k<n\,;$$
$$x'_n = 0 \quad \text{and} \quad x'_{n+1} = x'_{n+2} = \ldots = 1\,;$$
$$u'_{n+1} = u'_{n+2} = \ldots = u'_m = u'_{m+4} = u'_{m+5} = \ldots = 0$$

and

$$u'_n = u'_{m+1} = u'_{m+2} = u'_{m+3} = 1\,;$$
$$v'_{n+1} = v'_{n+2} = \ldots = v'_m = 0 \quad \text{and} \quad v'_n = v'_{m+1} = v'_{m+2} = \ldots = 1\,;$$
$$y'_n = y'_{m+1} = y'_{m+2} = \ldots = 0 \quad \text{and} \quad y'_{n+1} = y'_{n+2} = \ldots = y'_m = 1\,;$$
$$z'_n = z'_{m+2} = z'_{m+3} = \ldots = 0 \quad \text{and} \quad z'_{n+1} = z'_{n+2} = \ldots = z'_{m+1} = 1\,.$$

Hence we obtain the equalities

$$x-u(m) = 5\cdot 2^{-m-3}, \quad g(x)-g\big(u(m)\big) = -7\cdot 2^{-m-3} \quad \text{and} \quad \frac{g(x)-g\big(u(m)\big)}{x-u(m)} = -\frac{7}{5}\,;$$

$$x-v(m) = 2^{-m}, \quad g(x)-g\big(v(m)\big) = -2^{-m} \quad \text{and} \quad \frac{g(x)-g\big(v(m)\big)}{x-v(m)} = -1\,;$$

$$y(m)-x = 2^{-m}, \quad g\big(y(m)\big)-g(x) = -2^{-m} \quad \text{and} \quad \frac{g\big(y(m)\big)-g(x)}{y(m)-x} = -1\,;$$

$$z(m)-x = 3\cdot 2^{-m-1}, \quad g\big(z(m)\big)-g(x) = -2^{-m-1} \quad \text{and} \quad \frac{g\big(z(m)\big)-g(x)}{z(m)-x} = -\frac{1}{3}\,,$$

which imply the relations (6).

In the case of $\bar{x}_n = 5$ and $x_{n-1} = 1$, we assume

$$u_k = v_k = y_k = z_k = x_k \quad \text{for} \quad k<n\,;$$
$$u_n = u_{m+1} = u_{m+2} = \ldots = 0 \quad \text{and} \quad u_{n+1} = u_{n+2} = \ldots = u_m = 1\,;$$
$$v_n = v_{m+1} = v_{m+3} = v_{m+4} = \ldots = 0 \quad \text{and} \quad v_{n+1} = v_{n+2} = \ldots = v_m = v_{m+2} = 1\,;$$
$$y_{n+1} = y_{n+2} = \ldots = y_{m-1} = y_{m+2} = y_{m+3} = \ldots = 0 \quad \text{and} \quad y_n = y_m = y_{m+1} = 1\,;$$
$$z_{n+1} = z_{n+2} = \ldots = z_{m-1} = z_{m+1} = z_{m+2} = \ldots = 0 \quad \text{and} \quad z_n = z_m = 1\,.$$

Then we have

$$x'_k = u'_k = v'_k = y'_k = z'_k \quad \text{for} \quad k<n\,;$$
$$x'_n = 1 \quad \text{and} \quad x'_{n+1} = x'_{n+2} = \ldots = 0\,;$$
$$u'_n = 0 \quad \text{and} \quad u'_{n+1} = u'_{n+2} = \ldots = 1\,;$$
$$v'_n = v'_{m+2} = 0 \quad \text{and} \quad v'_{n+1} = v'_{n+2} = \ldots = v'_{m+1} = v'_{m+3} = v'_{m+4} = \ldots = 1\,;$$
$$y'_{n+1} = y'_{n+2} = \ldots = y'_{m-1} = 0 \quad \text{and} \quad y'_n = y'_m = y'_{m+1} = \ldots = 1\,;$$
$$z'_{n+1} = z'_{n+2} = \ldots = z'_{m-1} = 0 \quad \text{and} \quad z'_n = z'_m = z'_{m+1} = \ldots = 1\,;$$

$$x-u(m) = 2^{-m}, \quad g(x)-g[u(m)] = 0, \quad \frac{g(x)-g[u(m)]}{x-u(m)} = 0\,;$$

$$x-v(m) = 3\cdot 2^{-m-2}, \quad g(x)-g[v(m)] = 2^{-m-2}, \quad \frac{g(x)-g[v(m)]}{x-v(m)} = \frac{1}{3}\,;$$

$$y(m)-x = 3\cdot 2^{-m-1}, \quad g[y(m)]-g(x) = 4\cdot 2^{-m-1}, \quad \frac{g[y(m)]-g(x)}{y(m)-x} = \frac{4}{3}\,;$$

$$z(m)-x = 2^{-m}, \quad g[z(m)]-g(x) = 2\cdot 2^{-m}, \quad \frac{g[z(m)]-g(x)}{z(m)-x} = 2\,.$$

From the above equalities there follow the relations (7).

LEMMA 5. *If the dyadic expansion $(x_1, x_2, \ldots)_2$ of the number $x \in \langle 0, 1)$ contains infinitely many terms equal to* 1, *then*

$$\underline{g}^-(x) \leqslant -\tfrac{1}{2}, \quad \bar{g}^-(x) \geqslant \tfrac{1}{2}, \quad \underline{g}^+(x) \leqslant -\tfrac{1}{2} \quad \textit{and} \quad \bar{g}^+(x) \geqslant \tfrac{1}{4}\,. \tag{8}$$

Proof. Consider any natural number m. There exists for it a number $m_0>m$ such that $\bar{x}_{m_0} = 0$ and $\bar{x}_{m_0-1} = 5$.

Denote by m_s, where $s = 1, 2, \ldots, 6$, the smallest of the natural numbers $n>m_0$ satisfying the condition $\bar{x}_{m_s} = s$ ($\bar{x}_{m_6} = 0$). We have

$$m<m_0<m_1<\ldots<m_6\,;$$
$$x_n = 0 \text{ and } x'_n = 0, \quad \text{when} \quad m_0\leqslant n<m_1\,;$$
$$x_n = 1 \text{ and } x'_n = 1, \quad \text{when} \quad m_1\leqslant n<m_2 \text{ or when } m_5\leqslant n<m_6\,;$$
$$x_n = 0 \text{ and } x'_n = 1, \quad \text{when} \quad m_2\leqslant n<m_3 \text{ or when } m_4\leqslant n<m_5\,;$$
$$x_n = 1 \text{ and } x'_n = 0, \quad \text{when} \quad m_3\leqslant n<m_4\,.$$

We assume

$$u(m) = (u_1, u_2, \dots)_2, \quad v(m) = (v_1, v_2, \dots)_2,$$
$$y(m) = (y_1, y_2, \dots)_2 \quad \text{and} \quad z(m) = (z_1, z_2, \dots)_2,$$

where

$$u_k = x_k \text{ for } k<m_1, \quad u_k = 0 \text{ for } k \geqslant m_1;$$
$$v_k = x_k \text{ for } k<m_3, \quad v_k = 0 \text{ for } k \geqslant m_3;$$
$$y_k = x_k \text{ for } k<m_4, \quad y_k = 1 \text{ for } m_4 \leqslant k < m_6, \quad y_k = 0 \text{ for } k \geqslant m_6;$$
$$z_k = x_k \text{ for } k<m_6, \quad z_{m_6} = z_{m_6+2} = 1$$

and

$$z_{m_6+1} = z_k = 0 \quad \text{for} \quad k \geqslant m_6+3.$$

By calculating we now obtain

$$u'_k = x'_k \text{ for } k<m_1, \quad u'_k = 0 \text{ for } k \geqslant m_1,$$
$$0 < x-u(m) = \sum_{k=m_1}^{\infty} 2^{-k} x_k \leqslant 2^{-m_1+1},$$
$$g(x) - g[u(m)] = \sum_{k=m_1}^{\infty} 2^{-k} x'_k \geqslant 2^{-m_1}$$

and

(9) $$\frac{g(x)-g[u(m)]}{x-u(m)} \geqslant \frac{1}{2};$$

$$v'_k = x'_k \text{ for } k<m_3, \quad v'_k = 1 \text{ for } k \geqslant m_3,$$
$$0 < x-v(m) = \sum_{k=m_3}^{\infty} 2^{-k} x_k \leqslant 2^{-m_3+1},$$
$$g(x) - g[v(m)] = \sum_{k=m_3}^{\infty} 2^{-k} (x'_k - 1) \leqslant -2^{-m_3}$$

and

(10) $$\frac{g(x)-g[v(m)]}{x-v(m)} \leqslant -\frac{1}{2};$$

$$y'_k = x'_k \text{ for } k<m_4, \quad y'_k = 0 \text{ for } m_4 \leqslant k < m_6 \quad \text{and} \quad y'_k = 1 \text{ for } k \geqslant m_6,$$
$$0 < y(m)-x = \sum_{k=m_4}^{m_6-1} 2^{-k}(1-x_k) - \sum_{k=m_6}^{\infty} 2^{-k} x_k \leqslant 2^{-m_4+1},$$
$$g[y(m)] - g(x) = -\sum_{k=m_4}^{m_6-1} 2^{-k} x'_k + \sum_{k=m_6}^{\infty} 2^{-k}(1-x'_k) \leqslant -2^{-m_4},$$

(11) $$\frac{g[y(m)]-g(x)}{y(m)-x} \leqslant -\frac{1}{2};$$

$$z'_k = x'_k \text{ for } k<m_6, \quad z'_{m_6} = 1, \quad z'_{m_6+1} = 0, \quad z'_k = 1 \text{ for } k \geqslant m_6+2,$$

$$0 < z(m)-x = 2^{-m_6} - 2^{-m_6-1} x_{m_6+1} + 2^{-m_6-2}(1-x_{m_6+2}) - \sum_{k=m_6+3}^{\infty} 2^{-k} x_k$$
$$\leqslant 2^{-m_6} + 2^{-m_6-2} \leqslant 2^{-m_6+1},$$
$$g[z(m)] - g(x) = 2^{-m_6} - 2^{-m_6-1} x'_{m_6+1} + \sum_{k=m_6+2}^{\infty} 2^{-k}(1-x'_k) \geqslant 2^{-m_6-1},$$

(12) $$\frac{g[z(m)]-g(x)}{z(m)-x} \geqslant \frac{1}{4}.$$

Since $u(m), v(m) < x < y(m), z(m)$, and $u(m), v(m), y(m)$ and $z(m)$ tend to x as $m \to \infty$, therefore from (9)-(12) there follow the inequalities (8). ■

DEFINITION. We assume

$$f(x+n) = f(x) = \tfrac{1}{2}[g(x-)+g(x+)] \quad \text{for} \quad x \in \langle 0, 1) \text{ and } n = 0, \pm 1, \pm 2, \dots,$$

where

$$g(x-) = \lim_{y \to x-} g(y) \quad \text{and} \quad g(x+) = \lim_{y \to x+} g(y) \quad (g(0-) = g(1-)).$$

THEOREM 1. *There exists a function $f\colon R \to R$ with a dense and, at the same time, boundary set of points of continuity, which possesses the derivative at each point of discontinuity and which possesses neither the left-hand derivative nor the right-hand one at any point of continuity.*

The function f defined above is such a function. This easily follows from Lemmas 1-5.

3. On a homeomorphism mapping a denumerable set onto another denumerable set.

THEOREM 2. *For any denumerable sets A and B, dense in $(0, 1)$ and for any number $a \in (0, 1)$, there exists a function $f\colon \langle 0, 1\rangle \xrightarrow{\text{onto}} \langle 0, 1\rangle$ such that $f(A) = B$, f' is continuous in $\langle 0, 1\rangle$ ($f \in C_1$), and $1-a \leqslant f'(x) \leqslant 1+a$ for $x \in \langle 0, 1\rangle$.*

Proof. Let $A = \{a_2, a_3, \dots\}$ and $B = \{b_2, b_3, \dots\}$. Put $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$. For the sequences $(m_0, m_1, \dots)$ and $(n_0, n_1, \dots)$, defined below, we assume:

(1) $$f(a_{m_k}) = b_{n_k} \quad \text{for} \quad k = 0, 1, 2, \dots;$$

$$c_k^- = \max\{a_{m_j}\colon j = 0, 1, \dots, k-1;\ a_{m_j} < a_{m_k}\},$$
$$c_k^+ = \min\{a_{m_j}\colon j = 0, 1, \dots, k-1;\ a_{m_j} > a_{m_k}\},$$
$$d_k^- = \max\{b_{n_j}\colon j = 0, 1, \dots, k-1;\ b_{n_j} < b_{n_k}\}$$

and

$$d_k^+ = \min\{b_{n_j}\colon j = 0, 1, \dots, k-1;\ b_{n_j} > b_{n_k}\} \quad \text{for} \quad k = 2, 3, \dots$$

We define the sequences (m_k) and (n_k) of natural numbers in such a way that they should be one-to-one and satisfy the conditions:

(2) $$\{m_0, m_1, \ldots, m_{2k}\} \supset \{0, 1, \ldots, k+1\} \quad \text{for} \quad k = 1, 2, \ldots;$$

(3) $$\{n_0, n_1, \ldots, n_{2k}\} \supset \{0, 1, \ldots, k\} \quad \text{for} \quad k = 0, 1, \ldots;$$

(4) $$f_k(x) = f_{k-1}(x) \quad \text{for} \quad x \in \langle 0, 1\rangle \setminus (c_k^-, c_k^+),\ k = 1, 2, \ldots;$$

(5) $$0 < f_{k-1}(x) - f_k(x) < 2^{-k}a \quad \text{for} \quad x \in (c_k^-, c_k^+),\ k = 1, 2, \ldots$$

Moreover, we define the functions f_k in such a way to satisfy additionally the conditions:

(6) $$f_k(a_{m_j}) = b_{n_j} \quad \text{for} \quad j = 0, 1, \ldots, k;$$

(7) $$|f_k'(x) - f_{k-1}'(x)| < 2^{-k}a;$$

(8) $$1 - a + 2^{-k}a < f_k(x) < 1 + a - 2^{-k}a.$$

We assume $m_0 = n_0 = 0$ and $m_1 = n_1 = 1$ and $f_1(x) = x$. Suppose that for $j < 2k$ we have defined the function f_j and the terms m_j and n_j. We put

$$m_{2k} = \min\{n \in N\colon n \neq m_j \text{ for } j = 0, 1, \ldots, 2k-1\}.$$

We define term n_{2k} and function f_{2k} in such a way that, for $b_{n_{2k}} = f_{2k}(a_{m_{2k}})$, the conditions (4), (5) and (7) are satisfied.

We assume in turn

$$n_{2k+1} = \min\{n \in N\colon n \neq n_j \text{ for } j = 0, 1, \ldots, 2k\}$$

and select f_{2k} and m_{2k+1} such that, for $a_{m_{2k+1}} = f_{2k+1}^{-1}(b_{n_{2k+1}})$, the conditions (4), (5) and (7) are satisfied. The existence of the terms n_{2k} and m_{2k+1} satisfying the above conditions follows from the density of the sets A and B in $(0, 1)$. The existence of the functions f_{2k} and f_{2k+1} satisfying the conditions (4), (5) and (7) is obvious.

Now it is easily seen that the sequences (m_k) and (n_k) of natural numbers are one-to-one and satisfy the conditions (2)-(5).

The function f defined so far in the set $\{a_0, a_1, a_2, \ldots\}$ will be continued onto the entire interval $\langle 0, 1\rangle$.

The function f_k thus defined satisfies, of course, the conditions (4)–(7), and, on account of the inequality

$$1 - a + 2^{-k+1}a < f_{k-1}'(x) < 1 + a - 2^{-k+1}a,$$

the condition (8), too.

From the condition (7) it follows that the sequence (f_k') is uniformly convergent in $\langle 0, 1\rangle$. Since $f_k(0) = 0$, therefore, in virtue of the elementary theorem of mathematical analysis, the sequence (f_k) is in the interval $\langle 0, 1\rangle$ uniformly convergent to the function $\lim f_k$ which is of the class C_1. With that

$$(\lim f_k)' = \lim f_k'.$$

From the condition (6) we deduce that

$$\lim_{k\to\infty} f_k(a_{m_j}) = b_{n_j} \quad \text{for} \quad j = 0, 1, \ldots$$

This means that the function $\lim f_k$ is a continuation of the function f defined by the condition (1). In this context, we assume

$$f(x) = \lim_{k\to\infty} f_k(x) \quad \text{for} \quad x \in \langle 0, 1\rangle.$$

So we have $f(a_{m_j}) = b_{n_j}$, and in particular, $f(0) = 0$, $f(1) = 1$; in view of the inclusions (2) and (3), $f(A) = B$. The function f is in $\langle 0, 1\rangle$ of the class C_1, and its derivative $f' = \lim f_k'$. Hence, and from the inequality (8), we obtain the estimate

$$1 - a \leqslant f'(x) \leqslant 1 + a \quad \text{for} \quad x \in \langle 0, 1\rangle$$

which, in virtue of the assumption $a \in (0, 1)$ and the equalities $f(0) = 0$ and $f(1) = 1$, means that the function f is increasing and it maps the interval $\langle 0, 1\rangle$ onto itself.

COROLLARY. *For any denumerable sets A and B dense in R, there exists a homeomorphism of the class C_1, $f\colon R \xrightarrow{\text{onto}} R$, such that*

$$f(A) = B \quad \text{and} \quad \pi^{-2} < f'(x) < \pi^2 \quad \text{for} \quad x \in R.$$

Proof. Let

$$\varphi(x) = -\operatorname{ctg} \pi x \quad \text{for} \quad x \in (0, 1),$$

and let $g\colon \langle 0, 1\rangle \xrightarrow{\text{onto}} \langle 0, 1\rangle$ be a homeomorphism satisfying the conditions:

$$\varphi^{-1}(B) = g[\varphi^{-1}(A)] \quad \text{and} \quad \tfrac{1}{2} < g'(x) < 2 \quad \text{for} \quad x \in \langle 0, 1\rangle.$$

Such a homeomorphism exists on the basis of Theorem 2.

Put

$$f(x) = \varphi\{g[\varphi^{-1}(x)]\} \quad \text{for} \quad x \in R.$$

Then, f is a homeomorphism mapping the set A onto the set B, and its derivative f' is a continuous and bounded function:

$$\pi^{-2} < f'(x) < \pi^2 \quad \text{for} \quad x \in R.$$

4. Remarks and generalization of the solution.

Remark 1. There exists a convex function satisfying the conditions of Theorem 2.

Remark 2. There exists a homeomorphism f satisfying the conditions of the above corollary and the inequality $1 - a < f'(x) < 1 + a$ for $x \in R$, where a is an arbitrarily fixed number from the interval $(0, 1)$.

THEOREM 3. *For each denumerable set A dense in R, there exists a function $h\colon R \to R$ whose set of points of discontinuity is the set A whose derivative $h'(x)$ exists for $x \in A$, and for which none of the unilateral derivatives $h_-'(x)$ and $h_+'(x)$ exists for $x \in R \setminus A$.*

Proof. Denote by B a set of points of discontinuity of the function f from Theorem 1. The set B is dense in R. Thus, in virtue of the corollary, there exists a homeomorphism of the class C_1, $g\colon R \xrightarrow{\text{onto}} R$, such that

$$g(A) = B \quad \text{and} \quad \pi^{-2} < g'(x) < \pi^2 \quad \text{for} \quad x \in R.$$

Put

$$h(x) = f[g(x)] \quad \text{for} \quad x \in R.$$

Then, the function h satisfies the conditions of Theorem 3. ■

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Some existence and non-existence theorems for k-regular maps

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Abstract. A continuous map $f\colon X \to \mathbf{R}^n$ is said to be *k-regular* if whenever $x_1, \ldots, x_j$ are distinct points of X with $j \leqslant k$, then $f(x_1), \ldots, f(x_j)$ are linearly independent. Using some algebraic-topological machinery, a non-existence theorem for $2k$-regular maps on a disjoint union of k closed manifolds is obtained. In the other direction, we show explicitly that if X is a metric space and A is a closed neighborhood retract in X, then existence of k-regular maps $X-A \to \mathbf{R}^m$ and $A \to \mathbf{R}^n$ implies existence of a k-regular map $X \to \mathbf{R}^{m+n}$. Some applications of these existence and non-existence results are given.

1. Introduction. The study of k-regular maps is motivated by the theory of Čebyšev approximation. In that theory, a set of n real-valued continuous functions on X is called a *k-Čebyšev set of length n* if these functions are the components of a k-regular map of X into $\mathbf{R}^n$. The reader is referred to [8, pp. 237–242] for the significance of this concept.

Closely related to k-regularity is the concept of *affine k-regularity*: $f\colon X \to \mathbf{R}^n$ is affinely k-regular if and only if $g\colon X \to \mathbf{R}^{n+1}$ given by $g(x) = (1, f(x))$ is $(k+1)$-regular. Affine k-regularity has been considered in [2], [1], and [7]. Some previous work on k-regularity, using algebraic-topological tools, appears in [5], [3], and [4].

The problem we consider is the following: Given X, k, and n, does there exist a k-regular map $X \to \mathbf{R}^n$? In Section 2 we prove the following:

THEOREM 2.4. *Let $M_1, \ldots, M_k$ be closed, connected manifolds of dimensions $n_1, \ldots, n_k$, respectively. Suppose, for $1 \leqslant i \leqslant k$, the q_i-th dual Stiefel–Whitney class of M_i is non-zero. If a $2k$-regular map of the disjoint union $\coprod_{i=1}^{k} M_i$ into $\mathbf{R}^N$ exists, then $N \geqslant 2k + \sum_{i=1}^{k} (n_i + q_i)$.*

In [1], an affine analogue of the following is proved by geometric methods:

THEOREM 1.1 (Boltjanskiĭ–Ryškov–Šaškin). *Let $n \geqslant 1$. If a $2k$-regular map of $\mathbf{R}^n$ into $\mathbf{R}^N$ exists, then $N \geqslant (n+1)k$.*