[28] W. E. Ritter, *Representability of partial recursive functions in formal theories*, Proc. Amer. Math. Soc. 18 (1967), pp. 647–651.
[29] J. B. Rosser, *Extensions of some theorems of Gödel and Church*, J. Symbolic Logic 1 (1936), pp. 87–91.
[30] — *An informal exposition of proofs of Gödel's theorem and Church's theorem*, J. Symbolic Logic 4 (1939), pp. 53–60.
[31] D. Scott, *Algebras of sets binumerable in complete extensions of arithmetic*, in: Recursive Function Theory (AMS, Providence), 1962.
[32] J. C. Shepherdson, *Representability of recursively enumerable sets in formal theories*, Arch. Math. Logic 5 (1960), pp. 119–127.
[33] C. Smoryński, *Calculating self-referential formulae, I; explicit calculations*, to appear in the Proceedings of the 1977 Logic Colloquium at Wrocław.
[34] — *Avoiding self-referential statements*, to appear.

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Accepté par la Rédaction le 17. 4. 1978

On order locally finite and closure-preserving covers

by

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Abstract. The present paper deals with structural properties of the covers and contains the order locally finite sum theorem and the closure-preserving sum theorem for the covering dimension.

The purpose of this paper is to study structural properties of the covers and, besides, by mean of that properties, to derive two general sum theorems for the covering dimension dim. Section 1 contains a characterization of order star-finite open covers. There are many results (cf. [2], [7], [8] and [9]) dealing with spaces endowed with two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ such that E_ξ is closed and has a topological property $\mathscr{P}$, while U_ξ is an open neighborhood of E_ξ for each $\xi<\alpha$. In Section 2 a structure of such spaces is described and, in particular, the order locally finite sum theorem for the covering dimension is established. Finally, Section 3 is concerned with closure-preserving closed covers consisting of countably compact sets, where the closure-preserving sum theorem for the covering dimension is proved. The last result turns out to be a special case of a statement established by a topological game.

The set of natural numbers $1, 2, 3, \ldots$ is denoted by N, while natural numbers by k, m and n. Ordinal numbers are denoted by α, ξ, η and ζ.

Let $\{A_i\colon i\in I\}$ be an indexed family of subsets of a space X. We shall denote by $\{A_i\colon i\in I\}^{\#}$ the set of all points $x\in X$ such that the set $\{i\in I\colon U\cap A_i\neq 0\}$ is infinite for each neighborhood U of x.

Let us note several properties of the operation $\#$:

(a) $\{A_i\colon i\in I\}^{\#}=0$ iff $\{A_i\colon i\in I\}$ is locally finite.

(b) $\{A_i\colon i\in I\}^{\#}$ is closed in X.

(c) If $U\supset\{A_i\colon i\in I\}^{\#}$, where U is open, then $\{A_i-U\colon i\in I\}^{\#}=0$.

(d) If $B_i\subset A_i$ for each $i\in I$, then $\{B_i\colon i\in I\}^{\#}\subset\{A_i\colon i\in I\}^{\#}$.

1. Order star-finite covers. A family $\{A_i\colon i\in I\}$ of subsets of a space X is said to be *order star-finite* [9], if one can introduce a well ordering $<$ in the index set I so that for each $i\in I$ the set A_i meets at most finitely many A_j with $j<i$. Since every well ordered set is order isomorphic to an initial segment of ordinal numbers, we may use the notation $\{A_\xi\colon \xi<\alpha\}$ instead of $\{A_i\colon i\in I\}$.

LEMMA 1. *Let* $\{E_\xi\colon \xi<\alpha\}$ *and* $\{U_\xi\colon \xi<\alpha\}$ *be order star-finite covers of a space* X, *where* E_ξ *is closed and* U_ξ *is an open neighborhood of* E_ξ *for each* $\xi<\alpha$. *Then there is a partition* $\{I_n\colon n\in N\}$ *of* $\{\xi\colon \xi<\alpha\}$ *so that for each* $n\in N$:

1.1. *if* $\xi,\eta\in I_n$ *and* $\xi\neq\eta$, *then* $U_\xi\cap U_\eta=0$;

1.2. *if* $\xi\in I_n$ *and* $m<n$, *then* $\{\eta\in I_m\colon U_\xi\cap U_\eta\neq 0\}$ *is finite*;

1.3. $\{U_\xi\colon \xi\in I_n\}^{\#}\subset\bigcup\{E_\eta\colon \eta\in I_m$ *and* $m<n\}$;

1.4. $F_n=\bigcup\{E_\xi\colon \xi\in I_m$ *and* $m\leqslant n\}$ *is closed in* X.

Proof. For each $\xi<\alpha$, we define a sequence $\langle T_\xi^1, T_\xi^2, \ldots\rangle$ of subsets of $\{\xi\colon \xi<\alpha\}$ as follows. We set $T_\xi^1=\{\eta<\xi\colon U_\eta\cap U_\xi\neq 0\}$. Assume that T_ξ^n is already defined. Then we set $T_\xi^{n+1}=\bigcup\{T_\xi^1\colon \eta\in T_\xi^n\}$. Now we set $T_\xi=\{\xi\}\cup\bigcup\{T_\xi^n\colon n\in N\}$. It is easy to show (by induction with respect to ξ) that the sets T_ξ are finite. Let us observe that

1.5. if $\eta<\xi<\alpha$ and $U_\eta\cap U_\xi\neq 0$, then $T_\eta\subset T_\xi\setminus\{\xi\}$, and therefore $\operatorname{card}T_\eta<\operatorname{card}T_\xi$.

For each $n\in N$ we set $I_n=\{\xi<\alpha\colon \operatorname{card}T_\xi=n\}$. Clearly, $\{I_n\colon n\in N\}$ is a partition of $\{\xi\colon \xi<\alpha\}$. The condition 1.1 immediately follows from 1.5. Let $\xi\in I_n$ and $m<n$. We set $T=\{\eta\in I_m\colon U_\eta\cap U_\xi\neq 0\}$. If $\eta>\xi$ and $U_\eta\cap U_\xi\neq 0$, then $\operatorname{card}T_\eta>\operatorname{card}T_\xi$ by 1.5, and thus $\eta\notin T$. Hence $T\subset\{\eta\colon \eta<\xi\}$, and therefore $T\subset T_\xi^1$, i.e., the condition 1.2 is fulfiled. Let $x\in\{U_\eta\colon \eta\in I_n\}^{\#}$. There is $m\in N$ and $\xi\in I_m$ such that $x\in E_\xi$. The set $S=\{\eta\in I_n\colon U_\eta\cap U_\xi\neq 0\}$ is infinite, because $x\in U_\xi$. Since the set $\{\eta<\xi\colon U_\eta\cap U_\xi\neq 0\}$ is finite, there is $\eta\in S$ with $\eta>\xi$. Hence $\operatorname{card}T_\eta>\operatorname{card}T_\xi$, i.e., $n>m$. Therefore the condition 1.3 is satisfied. Finally, we claim that the set F_n is closed. The set F_1 is closed, because $\{E_\xi\colon \xi\in I_1\}^{\#}=0$ by 1.3. Let $n>1$ and $x\in\overline{F}_n$. If $x\in F_{n-1}$, then $x\in F_n$ because $F_{n-1}\subset F_n$. If $x\notin F_{n-1}$, then $x\notin\{E_\xi\colon \xi\in I_n\}^{\#}$, and hence $x\in E_\xi$ for some $\xi\in I_n$. Therefore F_n is closed for each $n\in N$.

PROPOSITION 1. *An open cover* $\{U_i\colon i\in I\}$ *of a space* X *is order star-finite iff the index set* I *admits a partition* $\{I_n\colon n\in N\}$ *such that for each* $n\in N$ *the family* $\{U_i\colon i\in I_n\}$ *consists of pairwise disjoint sets and the set* $\{j\in I_m\colon U_j\cap U_i\neq 0$ *and* $m<n\}$ *is finite for each* $i\in I_n$.

Proof. The "only if" part of the assertion follows from the proof of Lemma 1. The "if" part is almost immediate, since one can introduce a well ordering $<$ in I so that $i<j$ for each $i\in I_m$, $j\in I_n$ and $m<n$.

COROLLARY 1. *Each order star-finite family of open sets is* σ-*disjoint.*

2. Order locally finite covers. A family $\{A_i\colon i\in I\}$ of subsets of a space X is said to be *order locally finite* [2], if one can introduce a well ordering $<$ in the index set I so that for each $i\in I$ the family $\{A_j\colon j<i\}$ is locally finite at each point of A_i. Since every well ordered set is order isomorphic to an initial segment of ordinal numbers, we may use the notation $\{A_\xi\colon \xi<\alpha\}$ instead of $\{A_i\colon i\in I\}$.

LEMMA 2. *Let* X *be a space with order locally finite covers* $\{E_\xi\colon \xi<\alpha\}$ *and* $\{U_\xi\colon \xi<\alpha\}$, *where* E_ξ *is closed and* U_ξ *is an open neighborhood of* E_ξ *for each* $\xi<\alpha$.

Then X *has a closed cover* $\{E_{\xi,n}\colon \xi<\alpha$ *and* $n\in N\}$ *such that for each* $n\in N$:

2.1. $E_{\xi,n}\subset E_{\xi,n+1}$ *and* $\bigcup\{E_{\xi,n}\colon n\in N\}=E_\xi$ *for each* $\xi<\alpha$;

2.2. $\{E_{\eta,n}\colon \eta<\alpha\}^{\#}\subset\bigcup\{E_{\xi,k}\colon \xi<\alpha$ *and* $k<n\}$;

2.3. $F_n=\bigcup\{E_{\xi,n}\colon \xi<\alpha\}$ *is closed in* X.

Proof. For each $\xi<\alpha$, $x\in X$ and $n\in N$ we set $T_{\xi,x}=\{\eta\leqslant\xi\colon x\in U_\eta\}$ and $E_{\xi,n}=\{x\in E_\xi\colon \operatorname{card}T_{\xi,x}\leqslant n\}$. Since the set $T_{\xi,x}$ is finite for each $x\in U_\xi$ and $\xi<\alpha$, we have $\bigcup\{E_{\xi,n}\colon n\in N\}=E_\xi$. Therefore the condition 2.1 is fulfiled. Now we shall show that $E_{\xi,n}$ is closed in E_ξ. Let $x\in E_\xi-E_{\xi,n}$. Then $\operatorname{card}T_{\xi,x}\geqslant n+1$ and $U=\bigcap\{U_\eta\colon \eta\in T_{\xi,x}\}$ is an open neighborhood of x in X. If $y\in U\cap E_\xi$, then $\operatorname{card}T_{\xi,y}\geqslant n+1$. Hence $U\cap E_{\xi,n}=0$ and therefore $E_{\xi,n}$ is closed. In order we shall show the inclusion in 2.2. Let $x\in\{E_{\eta,n}\colon \eta<\alpha\}^{\#}$. There are $\xi<\alpha$ and $k\in N$ such that $x\in E_{\xi,k}$. Let U be an open neighborhood of x contained in $\bigcap\{U_\eta\colon \eta\in T_{\xi,x}\}$ and such that the set $\{\eta\leqslant\xi\colon U\cap E_\eta\neq 0\}$ is finite. Since $x\in\{E_{\eta,n}\colon \eta<\alpha\}^{\#}$, the set $\{\eta<\alpha\colon U\cap E_{\eta,n}\neq 0\}$ is infinite. Hence there is $\eta>\xi$ such that $U\cap E_{\eta,n}\neq 0$. Let us pick a point $y\in U\cap E_{\eta,n}$. Then $y\in U_\eta\cap\bigcap\{U_\mu\colon \mu\in T_{\xi,x}\}$ and thus $\operatorname{card}T_{\eta,y}\geqslant 1+\operatorname{card}T_{\xi,x}$, i.e., $n\geqslant 1+k$. Consequently, $x\in E_{\xi,k}$ where $k<n$. In particular, we have $\{E_{\eta,1}\colon \eta<\alpha\}^{\#}=0$, and hence the set F_1 is closed. Assume that F_n is closed for some $n\in N$. Let $x\in\overline{F}_{n+1}-F_n$. Then, by 2.2, $x\notin\{E_{\xi,n+1}\colon \xi<\alpha\}^{\#}$. Thus the point x has an open neighborhood which meets at most finitely many $E_{\xi,n+1}$ with $\xi<\alpha$. Hence $x\in E_{\xi,n+1}$ for some $\xi<\alpha$ and therefore $x\in F_{n+1}$. By the same the condition 2.3 is verified.

Remark 1. Under the same assumptions as in Lemma 2 one can point out that X has an open cover $\{U_{\xi,n}\colon \xi<\alpha$ and $n\in N\}$ such that for each $n\in N$: $U_{\xi,n}\subset U_{\xi,n+1}$, $\bigcup\{U_{\xi,n}\colon n\in N\}=U_\xi$, where $\xi<\alpha$, and $\{U_{\eta,n}\colon \eta<\alpha\}^{\#}\subset\bigcup\{U_{\xi,k}\colon \xi<\alpha$ and $k<n\}$. The set $U_{\xi,n}$ is defined by setting $U_{\xi,n}=\{x\in U_\xi\colon \operatorname{card}S_{\xi,x}\leqslant n\}$, where $S_{\xi,x}=\{\eta\leqslant\xi\colon x\in\overline{U}_\eta\}$. The inclusion $F_{\xi,n}\subset U_{\xi,n}$, however, does not hold in general. Let us notice, besides, that an analogous statement to Lemma 2 holds for spaces with two order point finite covers.

Let X be a space and let $\underline{K}$ be a family of closed subsets of X such that (i) if $E\in\underline{K}$ and F is a closed subset of E, then $F\in\underline{K}$, and (ii) if $E\in\underline{K}$ and $F\in\underline{K}$, then $E\cup F\in\underline{K}$. A space X is said to be *locally* $\underline{K}$ *at a point* x, if x has an open neighborhood U for which $\overline{U}\in\underline{K}$. Furthermore, X is said to be $\underline{K}$-*scattered* [10], if each closed nonvoid subset of X is locally $\underline{K}$ at a point.

The next proposition is an immediate consequence of Lemma 2.

PROPOSITION 2. *Let* X *be a space with order locally finite covers* $\{E_\xi\colon \xi<\alpha\}$ *and* $\{U_\xi\colon \xi<\alpha\}$ *such that* E_ξ *is closed,* $E_\xi\in\underline{K}$ *and* U_ξ *is an open neighborhood of* E_ξ *for each* $\xi<\alpha$. *Then* X *is the union of a sequence* $F_1\subset F_2\subset\ldots$ *of its* $\underline{K}$-*scattered closed subsets, where, moreover,* F_1 *is locally* $\underline{K}$ *and* $F_{n+1}-F_n$ *is locally* $\underline{K}$ *for each* $n\in N$.

Proposition 2 is a generalization of Theorem 3 in [9], where X was assumed to be paracompact and $\underline{K}$ was the class of compact spaces.

LEMMA 3 (C. H. Dowker [1]). *Let F be a closed subset of a normal space X with* $\dim F \leqslant n$. *If* $\dim E \leqslant n$ *for each closed set E with* $E \cap F = 0$, *then* $\dim X \leqslant n$.

THEOREM 1. *If a normal space X has order locally finite covers* $\{E_\xi\colon \xi<\alpha\}$ *and* $\{U_\xi\colon \xi<\alpha\}$ *such that E_ξ is closed,* $\dim E_\xi \leqslant n$ *and U_ξ is an open neighborhood of E_ξ for each* $\xi<\alpha$, *then* $\dim X \leqslant n$.

Proof. By Lemma 2 we have $X = \bigcup\{F_k\colon k\in N\}$, where $F_k = \bigcup\{E_{\xi,k}\colon \xi<\alpha\}$. By the locally finite sum theorem we infer that $\dim F_1 \leqslant n$. Assume that $\dim F_k \leqslant n$ for some $k\in N$. If E is a closed subset of X contained in $F_{k+1}-F_k$, then, by 2.2, the family $\{E_{\xi,k+1}\cap E\colon \xi<\alpha\}$ is locally finite in X. Thus, by the locally finite sum theorem, $\dim E\leqslant n$. Hence, by Lemma 3, $\dim F_{k+1}\leqslant n$. Finally, by the countable sum theorem, $\dim X\leqslant n$.

From Theorem 1 we get

COROLLARY 2 (K. Morita [4]). *If a normal space X has covers* $\{E_\xi\colon \xi<\alpha\}$ *and* $\{U_\xi\colon \xi<\alpha\}$ *such that E_ξ is closed,* $\dim E_\xi\leqslant n$, *U_ξ is an open neighborhood of E_ξ and* $\{U_\eta\colon \eta<\xi\}$ *is locally finite for each* $\xi<\alpha$, *then* $\dim X\leqslant n$.

Remark 2. In one of earlier versions of this paper we proved Theorem 1 under the assumption of the paracompactness of X. However, Professor K. Nagami has informed us kindly that the normality of X is sufficient to get the result. The proof, he has indicated, consists in the following: a given continuous map from a closed subset of X into the n-sphere S^n is extendable to X by transfinite induction with respect to $\xi<\alpha$. Moreover, the referee of this paper has also presented an argument which eliminates the paracompactness of X. His suggestion has essentially contributed to the present form of Lemma 2.

3. Closure-preserving covers. The following lemma is well known (e.g., it follows from Proposition 1.3.8 and Lemma 3.2.4 of [5]).

LEMMA 4. *Let X be a normal space and let E be a closed subset of X with* $\dim E\leqslant n$, *where* $n\geqslant 0$. *Then each continuous map f from a closed subset F of X into the n-sphere S^n admits a continuous extension* $g\colon \overline{V}\to S^n$, *where V is an open set containing* $E\cup F$.

Now we shall make use of the topological game $G(\underline{K}, X)$ introduced and studied in [10]. Let $\mathrm{Dim}_n = \{Y\colon Y$ is normal and $\dim Y\leqslant n\}$.

PROPOSITION 3. *Let X be a normal space. If Player I has a winning strategy in* $G(\mathrm{Dim}_n, X)$, *then* $\dim X\leqslant n$. *(The converse implication is obvious.)*

Proof. Let s be a winning strategy of Player I in the game $G(\mathrm{Dim}_n, X)$, where X is a normal space. To prove the inequality $\dim X\leqslant n$, it suffices to show that each continuous map from a closed subset of X into S^n admits a continuous extension to the whole space ([5], Theorem 3.2.5). Let $f\colon F\to S^n$ be a continuous map, where F is a closed subset of X. Put $E_0 = X$ and $E_1 = s(E_0)$. Then $\dim E_1\leqslant n$. Thus, by Lemma 4, f admits a continuous extension $f_1\colon \overline{V}_1\to S^n$, where V_1 is an open set containing $F\cup E_1$. Put $E_2 = X\setminus V_1$ and $E_3 = s(E_0, E_1, E_2)$. Then $\dim E_3\leqslant n$. Thus, by Lemma 4, f_1 admits a continuous extension $f_2\colon \overline{V}_2\to S^n$, where V_2 is an

open set containing $\overline{V}_1\cup E_3$. Put $E_4 = X\setminus V_2$ and $E_5 = s(E_0, E_1, E_2, E_3, E_4)$. Continuing in that manner, we get a play $\langle E_0, E_1, E_2, \ldots\rangle$ of $G(\mathrm{Dim}_n, X)$, an increasing sequence $\langle V_1, V_2, \ldots\rangle$ of open sets, and a sequence $\langle f, f_1, f_2, \ldots\rangle$ of continuous maps such that $E_{2k} = X\setminus V_k$, $F\cup E_1\subset V_1$, $\overline{V}_{k-1}\cup E_{2k-1}\subset V_k$, $f_k\colon \overline{V}_k\to S^n$ and $f_k|\overline{V}_{k-1} = f_{k-1}$ for each $k>2$. Since s is a winning strategy of Player I, we have $\bigcap\{E_{2k}\colon k\in N\} = 0$. Hence $\bigcup\{V_k\colon k\in N\} = X$. Let g be the map defined by setting $g(x) = f_k(x)$ whenever $x\in V_k$. Then $g\colon X\to S^n$ is a continuous extension of $f\colon F\to S^n$. Therefore $\dim X\leqslant n$.

A family $\mathscr{F}$ of subsets of a space X is said to be *closure-preserving* [3] if $\overline{\bigcup\mathscr{E}} = \bigcup\{\overline{E}\colon E\in\mathscr{E}\}$ for each $\mathscr{E}\subset\mathscr{F}$. Furthermore, a family $\mathscr{F}$ is said to be σ-closure-preserving if $\mathscr{F} = \bigcup\{\mathscr{F}_n\colon n\in N\}$, where each $\mathscr{F}_n$ is closure-preserving.

The following lemma is an improvement of Lemma 5 in [6]; nevertheless their proofs are different.

LEMMA 5. *Let X be a space with a closure-preserving cover $\mathscr{F}$ consisting of countably compact closed sets. Then to each closed subset E of X one can assign a discrete family $\mathscr{S}_E$ of countably compact closed subsets of E so that $\mathscr{S}_E$ refines $\mathscr{F}$ and the following condition holds:*

5.1. *If* $\langle E_1, E_2, \ldots\rangle$ *is a decreasing sequence of closed subsets of X for which* $E_1\cap\bigcup\mathscr{S}_X = 0$ *and* $E_{n+1}\cap\bigcup\mathscr{S}_{E_n} = 0$ *for each* $n\in N$, *then* $\bigcap\{E_n\colon n\in N\} = 0$.

Proof. Let E be a closed subset of X and let $\mathscr{S}_E$ be a maximal disjoint subfamily of $\{F\cap E\colon F\in\mathscr{F}\}$. Then the family $\mathscr{S}_E$ is closure-preserving and thus it is discrete. It remains to show that the condition 5.1 is fulfiled. For, let $\langle E_1, E_2, \ldots\rangle$ be a decreasing sequence of closed subsets of X such that $E_1\cap\bigcup\mathscr{S}_X = 0$ and $E_{n+1}\cap\bigcup\mathscr{S}_{E_n} = 0$ for each $n\in N$. Suppose that $\bigcap\{E_n\colon n\in N\}\neq 0$. Then there is an $F_0\in\mathscr{F}$ such that $F_0\cap E_n\neq 0$ for each $n\in N$. Now we claim that $F_0\cap E_n\notin\mathscr{S}_{E_n}$ for each $n\in N$. For, if $F_0\cap E_n\in\mathscr{S}_{E_n}$ for some $n\in N$, then $F_0\cap E_n\cap E_{n+1}\subset\bigcup\mathscr{S}_{E_n}\cap E_{n+1} = 0$, and hence $F_0\cap E_{n+1} = 0$, because $E_{n+1}\subset E_n$. The contradiction shows that our claim is true. Let $n\in N$. By the maximality of $\mathscr{S}_{E_n}$ we infer that there is $F_n\in\mathscr{F}$ such that $F_n\cap E_n\in\mathscr{S}_{E_n}$ and $F_n\cap F_0\cap E_n\neq 0$. Let $x_n\in F_n\cap F_0\cap E_n$. Then $x_n\notin E_{n+1}$. Hence we have $x_m\neq x_n$ for $m\neq n$. Since $\{x_n\colon n\in N\}$ is an infinite subset of F_0, it has in F_0 a cluster point x_0. We claim that $x_0\in\bigcap\{E_n\colon n\in N\}$. Indeed, if $x_0\notin E_n$ for some $n\in N$, then $x_m\notin E_n$ for some $m>n$, and that gives the contradiction with $x_m\in E_m\subset E_n$. Now we claim that $x_0\notin\bigcup\{F_n\colon n\in N\}$. For, if $x_0\in F_n$ for some $n\in N$, then $x_0\in F_n\cap E_n\in\mathscr{S}_{E_n}$ and thus $x_0\notin E_{n+1}$. Having the contradiction with the preceding claim we infer that the last claim is true. On the other hand, we have $\{x_n\colon n\in N\}\subset\bigcup\{F_n\colon n\in N\}$. Since $\mathscr{F}$ is closure-preserving, it follows that the set $\bigcup\{F_n\colon n\in N\}$ is closed. Hence we have $x_0\in\bigcup\{F_n\colon n\in N\}$. The contradiction shows that $\bigcap\{E_n\colon n\in N\} = 0$.

THEOREM 2. *If a normal space X has a closure-preserving cover $\mathscr{F}$ such that each* $F\in\mathscr{F}$ *is countably compact, closed and* $\dim F\leqslant n$, *then* $\dim X\leqslant n$.

Proof. By Lemma 5, we infer that $s(E_0, \ldots, E_{2k}) = \bigcup\mathscr{S}_{E_{2k}}$ is a winning strategy for Player I in the game $G(\mathrm{Dim}_n, X)$. Finally, by Proposition 3, we have $\dim X \leqslant n$.

COROLLARY 3. *If a normal space X has a σ-closure-preserving cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is countably compact, closed and $\dim F \leqslant n$, then $\dim X \leqslant n$.*

References

[1] C. H. Dowker, *Local dimension of normal spaces*, Quart. Journ. Math. Oxford 6 (1955), pp. 101–120.

[2] Y. Katuta, *A theorem on paracompactness of product spaces*, Proc. Japan Acad. 43 (1967), pp. 615–618.

[3] E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. 8 (1957), pp. 822–828.

[4] K. Morita, *On the dimension of normal spaces II*, J. Math. Soc. Japan 2 (1950), pp. 16–33.

[5] A. R. Pears, *Dimension Theory of General Spaces*, Cambridge University Press, 1975.

[6] H. B. Potoczny, *Closure-preserving families of compact sets*, General Top. and Appl. 3 (1973), pp. 243–248.

[7] K. Tamano and Y. Yajima, *On total paracompactness and total metacompactness*, Fund. Math. 105 (1979), pp. 73–78.

[8] R. Telgársky, *C-scattered and paracompact spaces*, ibid. 73 (1971), pp. 59–74.

[9] — *Closure-preserving covers*, ibid. 85 (1974), pp. 165–175.

[10] — *Spaces defined by topological games*, ibid. 88 (1975), pp. 193–223.

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Accepté par la Rédaction le 17. 4. 1978

On some Marcus problem concerning functions possessing the derivative at points of discontinuity only

by

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Abstract. In this paper we obtain (Theorem 1) affirmative answer on the following Marcus' problem [1]:

Does there exist a function with a dense and, at the same time, boundary set of points of continuity, which possesses the derivative at each point of discontinuity and which possesses no unilateral derivative at any point of continuity?

Theorem 3 generalizes the obtained solution so that any dense, denumerable set given earlier is a set of points of discontinuity of some function satisfying the conditions formulated in the Marcus problem. As an auxiliary theorem we use Theorem 2 on the existence of homeomorphisms of the class C_1, mapping the given, arbitrary, dense, denumerable set onto another such set.

1. Introduction. It is known that the existence of the derivative at a point for a function of a real variable does not depend on the continuity of the function at this point. All the same, the derivative exists only in an at most denumerable subset of the set of points of discontinuity. Therefore the condition that the derivative exists at each point of discontinuity can be satisfied only for the functions whose set of points of discontinuity is at most denumerable. In connection with the above, it is interesting to know whether there exist functions singular in the sense that they have the derivative in the set of points of discontinuity, which is denumerable and infinite, and they have no derivative at the remaining points. With this question the following problem of S. Marcus is connected (see [1], p. 13, Problem 5):

Does there exist a function with a dense and, at the same time, boundary set of points of continuity, which possesses the derivative at each point of discontinuity and which possesses no unilateral derivative (neither the left-hand nor the right-hand one) at any point of continuity?

Theorem 1 of this paper gives an affirmative answer to the above question, and Theorem 3 generalizes the obtained solution so that any dense denumerable set given earlier is a set of points of discontinuity of some function satisfying the conditions formulated in the Marcus problem. As an auxiliary theorem we use Theorem 2 on the existence of homeomorphisms of the class C_1, mapping the given arbitrary dense denumerable set onto another such set.