On the number of models of the Kelley-Morse theory of classes

by

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Abstract. A result on the number of β-models for various "second-order type" theories is proved. The technique of a definable quantifier introduced by Keisler and Mostowski and thoroughly investigated by Krivine and McAloon [2] has numerous applications. Here, we employ it to calculate the number of extensions of a given model (with a fixed definable subset). We prove a lemma on a definable quantifier in the case of the Kelley–Morse theory of classes — which is our principal interest — and then generalize it to a class of theories called "set theory like". This allows us to handle cases of higher order arithmetics, higher order set theories etc.

We generalize the results of Mostowski and Srebrny [4], and Keisler's and our own [3].

Section 1. Models of the Kelley–Morse theory of classes. Models of the Kelley–Morse (abbr. KM) theory of classes are of the form $M = \langle C^M, E\rangle$, where the universe C^M is a set and E is a binary (membership) relation on C^M. For general model theoretic reasons we may assume that the universal class V^M of the model M is a set and every proper class $A \in C^M$ is a subset of V^M. Thus, the membership E between sets and proper classes of M is standard, although we do not assume that M is standard.

Note, that every finite subset of V^M is in a natural way codable as an element of V^M and hence we may assume that

$$P_\omega(V^M) = \{x \subseteq V^M\colon x \text{ — finite}\} \subseteq V^M .$$

Let M be a fixed countable model of KM. By a definable quantifier Q over M we mean a definable, monotone, additive and σ-additive quantifier, i.e. Q is supposed to satisfy the following conditions:

1) $M \models QxF \to (\mathrm{E}x, y)[x \neq y \,\&\, F(x) \,\&\, F(y)]$,

2) if $M \models QxF$ and $M \models (x)F \to G$ then $M \models QxG$,

3) $M \models Qx[F \vee G]$ iff $M \models QxF \vee QxG$,

4) $M \models Qx(\mathrm{E}y)_V F \to (\mathrm{E}y)_V QxF$.

Every model of KM plus the scheme of choice has a definable quantifier satisfying 1)-4).

We mention two examples:

a) "There are uncodably many x such that F"

$$\neg(\mathrm{E}y)(x)[F(x) \equiv x\,\eta\, y]$$

where $x\,\eta\,y$ is $(\mathrm{E}z)[x = \{u: \langle z, u\rangle \in y\}]$.

b) "There are arbitrarily large wellorderings x such that F"

$$(x)\{\mathrm{WO}(x) \to (\mathrm{E}y)[\mathrm{WO}(y) \,\&\, x \prec y \,\&\, F(y)]\}$$

where $x \prec y$ means that x is embeddable into y,

We treat the set $P(V^M)$ as a topological space with the usual product topology.

LEMMA 1.1. *Let M be a countable model of* KM *with a definable quantifier Q and let W be an F_σ subset of $P(V^M)$ disjoint from M. Then, there is a proper countable elementary extension M_1 of M such that $V^{M_1} = V^M$ and $M_1 \cap W = \emptyset$.*

Proof. The set W is of the form $W = \bigcup_{n\in\omega} W_n$, with all W_n's closed, i.e. $W_n = \bigcap_{s\in b_n} (P(V^M)\setminus U_s)$, where $s = \langle s_1, s_2\rangle$ is a pair of finite subsets of $P(V^M)$ and $x \in U_s$ iff $s_1 \subseteq x \,\&\, s_2 \cap x = \emptyset$.

Let $\psi(s, x)$ be a formula of KM such that

$$x \notin U_s \quad \text{iff} \quad \psi(s, x)\,.$$

Let L be the language of M, i.e. L has constants c_a for every $a \in C^M$. $L(d)$ denotes the language obtained from L by adding a new constant d. Let p_n be an enumeration of all sentences of L true in M, q_n — an enumeration of all sentences of $L(d)$ and, finally, r_n — an enumeration of all sentences of $L(d)$ of the form $(\mathrm{E}x)(x \notin V \,\&\, F)$.

We may assume that all the above-mentioned formulas are restricted, that is all quantifiers are followed by $x \in V$ or $x \notin V$. As in Mostowski's original proof, we define inductively the sequence Z_n of finite sets of sentences of $L(d)$, such that the set $Z = \bigcup_{n\in\omega} Z_n$ has a model satisfying the conclusion of the lemma.

We start with $Z_0 = \{d \notin V\}$. Assume that Z_j, $j<n$, are defined and satisfy the following conditions (c_n is an enumeration of constants denoting elements of V^M):

(i) $p_{j-1} \in Z_j$ and $d \neq c_{j-1} \in Z_j$,

(ii) $q_{j-1} \in Z_j$ or $\neg q_{j-1} \in Z_j$,

(iii) if a sentence of the form $(\mathrm{E}x)F$ is in Z_j, then $F(c_k)$ is in Z_j, for some c_k,

(iv) if $r_{K(j)} = (\mathrm{E}x)[x \notin V \,\&\, F]$ is in Z_j, then for some $s \in b_{L(j)}$ the sentence $(\mathrm{E}x)[x \notin V \,\&\, F \,\&\, \neg\psi(s, x)]$ is in Z_j (here K and L are the converses of a pairing function: $\omega\times\omega \to \omega$),

(v) $M \models Qx \bigwedge\bigwedge Z_j(x/d)$ where Q is a definable quantifier over M, $\bigwedge\bigwedge Z_j$ is the conjunction of Z_j and x/d is the substitution of x for d.

To define Z_n we proceed as follows: we put $Z_n' = Z_{n-1} \cup \{p_{n-1}, d \neq c_{n-1}\}$. Obviously, $M \models Qx \bigwedge\bigwedge Z_n'(x/d)$, since Q is nontrivial.

Then, since $M \models Qx[\bigwedge\bigwedge Z_n' \,\&\, q_{n-1} \vee \bigwedge\bigwedge Z_n' \,\&\, \neg q_{n-1}](x/d)$ holds, we adjoin to Z_n' the sentence q_{n-1} or $\neg q_{n-1}$ depending on whether the first or the second part is true. In this way we have

$$Z_n'' = Z_n' \cup \{\pm q_{n-1}\} \quad \text{and} \quad M \models Qx \bigwedge\bigwedge Z_n''(x/d) \quad \text{holds}.$$

In order to satisfy (iii) we use the σ-additivity of Q in an obvious way. Finally, let $r_{k(n)} = (\mathrm{E}x)[x \notin V \,\&\, F]$ be in Z_n'' and let $b = b_{L(n)}$.

We have to find an $s \in b$ in such a way that the sentence $(\mathrm{E}x)[x \in V \,\&\, F \,\&\, \neg\psi(s, x)]$ can be adjoined to Z_n'' without destroying (v).

Replacing variables if necessary, we may write the sentence $\bigwedge\bigwedge Z_n''$ in the form:

$$(\mathrm{E}x)[x \notin V \,\&\, F \,\&\, F_1]\,.$$

Consider the class

$$S = \{s\colon \neg Qz[(\mathrm{E}x)(x \notin V \,\&\, F \,\&\, F_1 \,\&\, \neg\psi(s, x))](z/d)\}\,.$$

Since Q is a definable quantifier, $S \in M$. By σ-additivity of Q we have

$$M \models \neg Qz(\mathrm{E}s, x)[s \in S \,\&\, x \notin V \,\&\, F \,\&\, F_1 \,\&\, \neg\psi(s, x)](z/d)\,,$$

or, equivalently,

$$M \models \neg Qz[\mathrm{E}x)\{x \notin V \,\&\, F \,\&\, F_1 \,\&\, (\mathrm{E}s)[s \in S \,\&\, \neg\psi(s, x)]\}(z/d)\,.$$

Suppose that $b \subseteq S$. Then

$$M \models (x)(\mathrm{E}s)[s \in S \,\&\, \neg\psi(s, x)]$$

since $W_{L(n)} \cap M = \emptyset$ and hence such an s can be found in $b = b_{L(n)}$.

It follows that

$$M \models \neg Qz(\mathrm{E}x)[x \notin V \,\&\, F \,\&\, F_1](z/d)\,,$$

which implies $M \models \neg Qz \bigwedge\bigwedge Z_n''(z/d)$, a contradiction. Thus $b - S \neq \emptyset$ and for $s \in b - S$ we have

$$M \models Qz(\mathrm{E}x)[x \notin V \,\&\, F \,\&\, F_1 \,\&\, \neg\psi(s, x)](z/d)\,.$$

Now, we adjoin the sentence $(\mathrm{E}x)[x \notin V \,\&\, F \,\&\, \neg\psi(s, x)]$ to obtain Z_n and $M \models Qz \bigwedge\bigwedge Z_n(z/d)$ holds.

Conditions (i)-(v) imply that the set $Z = \bigcup_{n\in\omega} Z_n$ is consistent, complete, and V^M-closed and that the types $\{x \neq c_n\colon n \in \omega\}$ and $\{\psi(s, x)\colon s \in b_n\}$ are non-principal with respect to Z. By the omitting types theorem Z has a countable model M_1 which omits all these types. Obviously M_1 is an elementary extension of M and $V^M = V^{M_1}$. ■

We now prove the main theorem of this section:

THEOREM 1.2. *Every countable model M of* KM *having a definable quantifier has 2^{ω_0} countable elementary extensions and 2^{ω_1} elementary extensions of cardinality ω_1. In addition, all the above-mentioned extensions have the same universal class, namely that of M.*

Proof. Let $T = \bigcup_{\xi<\omega_1} 2^\xi$ be the full binary tree of height ω_1. With each $\varphi \in T$ we associate a countable $M(\varphi)$ in such a way that $\varphi \subseteq \psi$ implies $M(\varphi) \prec M(\psi)$.

Note that if a model has a definable quantifier, then its elementary extension also has such. We put $M(\emptyset) = M$. Let $\varphi \in T$ and $\mathrm{Dm}(\varphi) \notin \mathrm{Lim}$.

Assuming inductively that all $M(\psi)$ with $\mathrm{Dm}(\psi) \leqslant \mathrm{Dm}(\varphi)$ are defined and the countable set

$$X = \bigcup_{\xi < \mathrm{Dm}(\varphi)} M(\varphi|\xi+1^\frown\langle 1-\varphi(\xi+1)\rangle) \setminus M(\varphi|\xi+1)$$

is disjoint from $M(\varphi)$, we apply the lemma to find a countable $M(\varphi^\frown\langle 0\rangle) \succ M(\varphi)$ which omits X and whose universal class is $V^{M(\varphi)}$. Again by the lemma we find $M(\varphi^\frown\langle 1\rangle)$ omitting

$$X \cup (M(\varphi^\frown\langle 0\rangle) \setminus M(\varphi)).$$

If $\mathrm{Dm}(\varphi) \in \mathrm{Lim}$ and all $M(\varphi|\xi)$, for $\xi < \mathrm{Dm}(\varphi)$ are given, then we put $M(\varphi)$ to be the union of the elementary chain $M(\varphi|\xi)$.

By construction all $M(\varphi)$'s are distinct, all are elementary extensions of $M(\emptyset) = M$ and $V^{M(\varphi)} = V^M$. ■

Section 2. Generalization. Given a countable 1-st order language containing a binary predicate ε, we say that a consistent theory Y in this language is "set-theory like w.r.t. $U(x)$" (where $U(x)$ is a formula with a single variable x) iff the following holds:

a) Y contains extensionality axioms for ε and the comprehension schema in the form

$$(Ey)(x)[x \,\varepsilon\, y \equiv U(x) \,\&\, F]$$

where F is an arbitrary formula not containing y.

b) For each $n \in \omega$ there is an operation definable in X, which to any sequence $\langle x_1, \ldots, x_n\rangle$ of elements of U assigns an element of U.

It is easy to check that the proof of Lemma 1.1 works in the case of "set-theory like" theories. Hence, we have

THEOREM 2.1. *Let Y be "set-theory like" and let M be a countable model of Y having a definable quantifier Q satisfying $M \models \neg(Qx)U(x)$. If W is an F_σ subset of $\gamma(U^M)$ disjoint from M, then M has 2^{ω_0} countable elementary extensions omitting W and 2^{ω_1} elementary extensions of cardinality ω_1. In addition, the class U in these extensions is equal to U^M.*

We shall now list some applications of Theorem 2.1.

2.2 (Keisler). If M is a countable ω-model of A_2 (2-nd order arithmetic with choice scheme), then M has 2^{ω_0} countable elementary extensions and 2^{ω_1} elementary extensions of cardinality ω_1, which are ω-models.

2.3. Generalization of 2.2 to arbitrary (not ω-) models of A_2.

2.3 — Theorem 1.2 of this paper.

2.4. Let M be a countable model of A_n, $n \geqslant 2$, and let $k \leqslant n-1$. There are 2^{ω_0} countable elementary extensions and 2^{ω_1} elementary extensions of cardinality ω_1 of M with the same objects of order $\leqslant k$.

2.5. An analogue of 2.4 for higher order set theories.

Consider now the case of standard models. Let Y be the following theory:

$$Y = \mathrm{ZFC}(-) \text{ plus "}P(\omega)\text{ exists" plus "}\omega_2\text{ exists" plus "}|P(\omega)| \geqslant \omega_2\text{".}$$

We say that a model M of A_2 has property (A) iff M is the continuum of a countable model of Y whose ω_1 is standard.

Obviously, if M has property (A), then M is a β-model. The theory Y is "set-theory like w.r.t. $U(x)$", where $U(x)$ is "$x \in \omega_1$". There is also a suitable definable quantifier Q — "there is more than ω_1 ...".

THEOREM 2.6. *Every β-model M of A_2 having property (A) has 2^{ω_0} countable elementary extensions and 2^{ω_1} elementary extensions of cardinality ω_1, which are also β-models. In addition, all these extensions have the same height, namely that of M.*

Proof. By property (A), there is a countable M_1 such that $M_1 \models Y$, $M = P(\omega)^{M_1}$ and $\omega_1^{M_1}$ is standard. We extend M_1 using Theorem 2.1 with the above-mentioned quantifier Q. Since $|P(\omega)| \geqslant \omega_2$ in M_1, the continuum has been enlarged. On the other hand, ω_1 in the extension is standard as it is equal to $\omega_1^{M_1}$. Thus, the continuum of the extension is a β-model. ■

In a similar manner we treat the case of models of KM with the scheme of choice. Let Z be the following theory:

$$Z = \mathrm{ZFC}(-) \text{ plus "an inaccessible cardinal } \alpha \text{ exists" plus "}\alpha^{++}\text{ exists" plus "}P(\alpha)\text{ exists" plus "}|P(\alpha)| \geqslant \alpha^{++}\text{".}$$

A model M of KMC has property (B) iff M is $R_{\alpha+1}$ of a model N of Z with standard $(\alpha^+)^N$ and also $V^M = R_\alpha^N$ holds.

An analogous reasoning gives

THEOREM 2.7. *Every countable β-model of* KMC *with property (B) has 2^{ω_0} countable elementary extensions and 2^{ω_1} elementary extensions of cardinality ω_1, which are also β-models.*

Obviously an analogous property (C_n) can be formulated for A_n and an analogous theorem can be proved.

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