On families of σ-complete ideals

by

Adam Krawczyk and Andrzej Pelc (Warszawa)

Abstract. Our main results are the following: Assume Martin's Axiom. Then 1. For every \( \lambda < 2^n \) and every family \( \{ I_\alpha : \alpha < \lambda \} \) of two-valued uniform measures on \( 2^n \) there exists an \( X \subseteq 2^n \) non-measurable with respect to any of them.

2. For every cardinal \( \kappa \) such that \( 2^n < \kappa < \text{cardinal} \) carrying a \( 2^n \)-complete \( 2^n \)-saturated ideal the following holds: if \( \lambda < 2^n \) and \( \{ I_\alpha : \alpha < \lambda \} \) is a family of \( 2^n \)-additive two-valued measures on \( \kappa \), then there exists an \( X \subseteq \kappa \) non-measurable with respect to any of them.

0. Terminology and preliminaries. We shall use standard set-theoretical notation and terminology. Letters \( \kappa, \lambda, \mu \) will always denote uncountable cardinals. "\( I \) is an ideal on \( X \)" will mean "\( I \) is a \( \sigma \)-complete proper ideal of subsets of \( X \) such that \( \{ x \} \in I \) for all \( x \in X \)."

An ideal \( I \) is \( \lambda \)-complete iff \( \{ x : \exists \eta < \lambda \} \in I \) implies \( \{ \bigcup \{ x : \xi < \eta \} \} \in I \) for \( \forall \eta < \lambda \). A cardinal \( \lambda \) is called the character of an ideal \( I \) on \( \kappa \) iff \( \lambda \) is the least cardinal such that \( \exists X \subseteq \kappa \) such that \( |X| = \lambda \) \( X \not\in I \). An ideal \( I \) on \( \kappa \) is uniform iff \( \forall I = \kappa \).

If \( I \) is an ideal on \( \kappa \), then \( I^* \) will denote the dual filter.

Ideals \( I_1 \) and \( I_2 \) on \( \kappa \) are called compatible iff there exists an ideal \( I_3 \) on \( \kappa \) such that \( I_1 \cup I_2 \subseteq I_3 \). It is easy to see that \( I_1, I_2 \) are compatible iff \( I_1 \cap I_2 = \emptyset \) iff \( I_1 \cap I_2 = \emptyset \). Otherwise \( I_1, I_2 \) are incompatible.

MA will denote Martin's Axiom. We shall use the following consequence of MA (see [4]):

1. The union of \( < 2^n \) closed nowhere dense subsets of a metric complete separable space is nowhere dense.

A subset \( A \) of the reals is called strongly Lusin if for every Lebesgue measurable set \( A \cap B \subseteq \mathbb{R} \) if \( A \) has Lebesgue measure 0, it is also a consequence of MA (see [5], cf. also [1], [4], [6] that

2. A strongly Lusin set exists.

We use the following notation:

\( U(\kappa, \lambda, \mu) \) --- For every family \( \{ I_\alpha : \alpha < \lambda \} \) of \( \mu \)-complete ideals on \( \kappa \) we have \( \bigcup \{ I_\alpha \cup I_\alpha^* \} \neq P(\kappa) \).

\( U^*(\kappa, \lambda, \mu) \) --- For every family \( \{ I_\alpha : \alpha < \lambda \} \) of \( \mu \)-complete uniform ideals on \( \kappa \) we have \( \bigcup \{ I_\alpha \cup I_\alpha^* \} \neq P(\kappa) \).
Using this notation, we can formulate the classical problem of Ulam on sets of measures as follows:

Let $\kappa$ be less than the first measurable cardinal. What is the minimal cardinal $\lambda$ such that non $U(\kappa, \lambda, \omega_1)$?

A particularly interesting case is $\kappa = 2^\omega$. If $2^\omega$ is less than the first cardinal carrying a $\sigma$-complete $\sigma$-saturated ideal, then the Erdős–Alaoglu theorem (cf. e.g. [7]) gives $U(2^\omega, \omega_1, \omega_1)$.

On the other hand, the second author proved in [5] that $U(2^\omega, \omega_1, 2^\omega)$. In [8] A. Taylor strengthened this result to $U(2^\omega, \omega_1, \omega_1)$.

In the present paper we investigate the case when the family of ideals is uncountable. It turns out that under some additional set-theoretical assumptions it is possible to get information in this case as well.

1. Main results. We begin with the following

**Lemma 1.1.** Let $\kappa$ be an uncountable cardinal. Let $f$ be an atomless measure defined on a $\sigma$-algebra $\mathcal{S} = \mathcal{P}(\kappa)$ and $I_1$ the ideal of f-null sets. Then

(i) For every family $\{I_\alpha : \alpha < \lambda\}$ of ideals on $\kappa$ which are compatible with $I_1$ we have $\bigcup_{\alpha < \lambda} (I_\alpha \cup I_1) = \mathcal{P}(\kappa)$.

(ii) If, in addition, $f$ is such that the metric space $\mathcal{S}/I_1$ with the metric $d([A], [B]) = f(A,B)$ is a separable space, then MA implies that for every family $\{I_\alpha : \alpha < \lambda\}$, $\lambda < 2^\kappa$, of ideals on $\kappa$ which are compatible with $I_1$ we have $\bigcup_{\alpha < \lambda} (I_\alpha \cup I_1) = \mathcal{P}(\kappa)$.

**Proof.** (i) Since $I_1$ is compatible with $I_1$, there exist ideals $I_\alpha \supseteq I_\alpha \cup I_1$ and it suffices to show that $\bigcup_{\alpha < \lambda} (I_\alpha \cup I_1) = \mathcal{P}(\kappa)$.

Let $S = \mathcal{S}/I_1$. For $S_\alpha = S/I_\alpha \cap S$, $S_\alpha$ are of course $\sigma$-algebras. Let $S = S/I_1$.

Write $S_{\alpha \in \lambda} = (I_\alpha \cup I_1)$. Then $S_{\alpha \in \lambda}$ are of course $\sigma$-algebras. Let $S = S/I_1$. For $S_{\alpha \in \lambda}$, $S_{\alpha \in \lambda}$ is of course $\sigma$-algebras.

Write $S_{\alpha \in \lambda} = (I_\alpha \cup I_1)$. Then $S_{\alpha \in \lambda}$ are of course $\sigma$-algebras. Let $S = S/I_1$. For $S_{\alpha \in \lambda}$, $S_{\alpha \in \lambda}$ are of course $\sigma$-algebras.

The proof now follows from the existence of such a family.

2. Let $S = \mathcal{S}/I_1$. Then $S$ is a complete metric space and $S$ is a closed subset of $S$.

CLAIM. $S_{\alpha \in \lambda}$ is a nowhere dense subset of $S$.

Assume that it is not. Then there exist a set $A$ and a positive real $\varepsilon$ such that $K \setminus \{A\}$, $\varepsilon \subseteq S_{\alpha \in \lambda}$. Consider any $B \subseteq S$ such that $[A] \cap [B] = 0$. Since $B$ is atomless, it is a disjoint union of $A_i$ such that $f(A_i) = 2^{-\infty}$. For every $i$ we have $A_i \subseteq I_1$ and $A_i \subseteq I_1$. But it follows from the construction that $f([A], [B]) = 0$ for every $i$, contradicting the assumption that $f(A,B) = 0$.

Let us now show that $S$ is a complete metric space. $S$ is also a complete metric space with the same distance and hence it is a closed subset of $S$.

Assume that it is not. Then there exist a set $A$ and a positive real $\varepsilon$ such that $K \setminus \{A\}$, $\varepsilon \subseteq S_{\alpha \in \lambda}$. Consider any $B \subseteq S$ such that $[A] \cap [B] = 0$. Since $B$ is atomless, there exists a partition $\{B_i : i < k\}$ of $B$ such that $f(B_i) < \varepsilon$. Hence $[A] \subseteq B_i \subseteq S_{\alpha \in \lambda}$ and since $S_{\alpha \in \lambda}$ is a $\sigma$-algebra, we get $[A] \subseteq [B_i] \subseteq S_{\alpha \in \lambda}$ and $[B_i] \subseteq S_{\alpha \in \lambda}$. In a similar way (using $\varepsilon - \varepsilon$ instead of $\varepsilon$) we get $[B_i] \subseteq S_{\alpha \in \lambda}$ for every $B_i \subseteq A$. It follows that $S_{\alpha \in \lambda}$ is a contradiction.

We have shown that $S_{\alpha \in \lambda}$ is a nowhere dense subset of $S$. Using the Baire Category Theorem, we get $\bigcup_{\alpha \in \lambda} S_{\alpha \in \lambda}$ and hence there exists an $x \subseteq S$ such that $X \subseteq S_{\alpha \in \lambda}$.

(ii) In this case we use the Strong Baire Category Theorem. It is possible since we have MA and the space $S$ is separable.

Before stating the next lemma, let us fix the following notational convention:

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be families of ideals on $\kappa$. Then $I_1 = \bigcap \mathcal{F}_1$ is also an ideal on $\kappa$. We say that the families $\mathcal{F}_1$, $\mathcal{F}_2$ are compatible if $I_1$, $I_2$ are compatible.

If $\mathcal{F}$ is a family of ideals, then $\mathcal{F}^* = \{I^* : I \in \mathcal{F}\}$.

**Lemma 1.2.** Let $\lambda < \mu \leq \kappa$ be uncountable cardinals. For $\alpha < \lambda$ let $\mathcal{F}_\alpha$ be a family of $\mu$-complete ideals on $\kappa$ such that $\bigcup \mathcal{F}_\alpha \cup \bigcup \mathcal{F}_\alpha^* = \mathcal{P}(\kappa)$. Assume moreover that the families $\mathcal{F}_\alpha$: $\alpha < \lambda$, are pairwise incompatible. Then $\bigcup \mathcal{F}_\alpha \cup \bigcup \mathcal{F}_\alpha^* = \mathcal{P}(\kappa)$.

**Proof.** Let $I_\gamma = \bigcap \mathcal{F}_\lambda$ for $\alpha < \lambda$. We shall construct a family $\{A_{\beta, \gamma} : \beta \in \mathcal{F}_\lambda\}$ such that the following conditions hold:

(i) $A_{\beta, \gamma} \cap I_{\gamma} = I_{\gamma}$.

(ii) $A_{\beta, \gamma} \cap I_{\gamma} = I_{\gamma}$ for $A_{\beta, \gamma}$ for $\gamma < \alpha$.

(iii) $A_{\beta, \gamma} \cap I_{\gamma} = I_{\gamma}$ for $A_{\beta, \gamma}$ for $\gamma < \beta$.

First we show how our lemma follows from the existence of such a family.

Let $A_{\beta, \gamma} = \bigcap A_{\beta, \gamma} \cap I_{\gamma}$. Condition (ii) implies that $A_{\beta, \gamma} \cap I_{\gamma} = I_{\gamma}$ for $\alpha < \beta$. Since $I_{\gamma}$ is $\mu$-complete, it follows from (i) and (ii) that $A_{\beta, \gamma} \cap I_{\gamma} = I_{\gamma}$.

**Claim.** For $\alpha < \lambda$ there exists a $B_\alpha \subseteq A_{\beta, \gamma}$ such that $B_\alpha \neq A_{\beta, \gamma} \cup B_{\beta, \gamma}$. Using the assumption of our lemma, we take $B_{\beta, \gamma} \neq A_{\beta, \gamma} \cup B_{\beta, \gamma}$. Since $A_{\beta, \gamma}$ is $\mu$-complete, it is easy to see that $B \cap A_{\beta, \gamma} \neq \emptyset$ and $B \cap B_{\beta, \gamma} \neq \emptyset$ and we put $B_\alpha = A_{\beta, \gamma} - B_{\beta, \gamma}$. Now $A_{\beta, \gamma} \neq A_{\beta, \gamma} \cup B_{\beta, \gamma}$ and $B_{\beta, \gamma} \neq B_{\beta, \gamma} - A_{\beta, \gamma}$. Hence it suffices to construct the sets $A_{\beta, \gamma}$. We proceed by induction. As $A_{\beta, \gamma}$ is a $\kappa$-complete ideal, it is easy to see that $A_{\beta, \gamma}$ is a $\kappa$-complete ideal. We put $U_{\beta, \gamma} = A_{\beta, \gamma} \cap I_{\gamma}$ for $\gamma < \beta$.

We are now ready to prove

**Theorem 1.3.** Assume MA. Then $U(2^\omega, \lambda, \omega_1, \omega_1)$ for $\lambda < 2^\omega$.

**Proof.** Let $\mathcal{F}$ be a strongly Suslin set. Denote by $\mathcal{B}$ the family $\{B \cap \mathcal{F} : B$ is a Borel subset of the reals}. Clearly $\mathcal{B}$ is a $\sigma$-algebra on $\mathcal{F}$, Denote by $m$ the Lebesgue measure. We define for $A \subseteq \mathcal{F}$

$$f(A) = m(B)$$

if $A = B \cap \mathcal{F}$. \(\square\)
Since $\mathcal{P}$ is a strongly Lusin set, the function $f$ is well defined. Indeed, take $B_1, B_2$ s.t. $A_1 = B_1 \cap \mathcal{P} = B_2 \cap \mathcal{P}$. Then $(B_1 \backslash B_2) \cap \mathcal{P} = \emptyset$, hence $m(B_1 \backslash B_2) = 0$ and finally $m(B_1) = m(B_2)$.

The function $f$ is an atomless measure on $\mathcal{B}$ and $\mathcal{B}f$ with the usual distance is a separable space. Applying Lemma 1.1, we conclude that for every family $\{I_\xi: \xi < \lambda\}, \lambda \leq 2^{\kappa}$, of ideals on $\mathcal{P}$ compatible with $I_\lambda$, we have $\bigcup_{\xi < \lambda} (I_\xi \cup I_\lambda^c) \neq P(\mathcal{P})$.

Since $|\mathcal{P}| = 2^\kappa$, it is sufficient to show that every uniform ideal on $\mathcal{P}$ is compatible with $I_\lambda$.

It turns out that every uniform ideal $I$ on $\mathcal{P}$ contains $I_\lambda$. Indeed, if $f(X) = 0$ for $X \subseteq \mathcal{P}$, then $|X \cap \mathcal{P}| = |X| < 2^\kappa$. Since $I$ is uniform, $X \subseteq \mathcal{P}$. This completes the proof.

Our next theorem gives some information about sets of ideals on cardinals greater than $2^\kappa$.

**Theorem 1.4.** Let $\kappa$ be the first cardinal carrying a $2^\kappa$-complete $2^\kappa$-saturated ideal. Assume MA. Then $2^\kappa < \kappa^+ < \kappa^{++}$ for $\lambda > 2^\kappa$.

Proof. Let $I$ be a $2^\kappa$-complete ideal in $\kappa$. Since $\kappa^+ < \kappa$, there exists a pairwise disjoint partition $\{A_\alpha: \alpha \in 2^\kappa\}$ of $\kappa$ such that $A_\alpha \notin I$. Let $A_\alpha = \bigcup \mathcal{B} \alpha$ for Borel $\mathcal{B} \subseteq 2^\kappa$. Then $\mathcal{B} = \{X \subseteq 2^\kappa: \kappa \in X \in \mathcal{B}\} \kappa$ is a $\sigma$-algebra on $\kappa$. Let $\mathcal{B} = \{X \subseteq X: X \in \mathcal{B}\}$. We define for $Y \in \mathcal{B}$, $Y = \bigcup \mathcal{B} Y$.

$f(Y) = m(Y)$.

It is easy to see that $f$ is an atomless measure on $\mathcal{B}$ and $\mathcal{B}f$ is separable.

For $\lambda < 2^\kappa$, let $I_\lambda = \{\xi: \xi < \lambda\}$ be an arbitrary family of $2^\kappa$-complete ideals on $\kappa$. For every $I_\lambda$, let $I_\lambda$ be the above atomless measure. It suffices to show that $\bigcup_{\xi < \lambda} (I_\xi \cup I_\lambda^c) \neq P(\kappa)$.

We construct the following sequence of families of ideals:

Consider the sequence $\{I_\xi: \xi < \lambda\}$. Let $I_\lambda = \{I_\xi: I_\xi \text{ is compatible with } I_\lambda\}$. If $I_\lambda, I_\lambda$ are defined for $\xi < \lambda$, let $I_\lambda$ be the first $I_\lambda$ which is incompatible with any $I^\lambda = I_{\xi 0} \bigcup \mathcal{B} I_\lambda$ and $I_\lambda$ is compatible with $I^\lambda$.

We proceed in this way for all $\xi < \lambda$. Then we define $I_\lambda = \{I_\xi \cup I_\lambda: I_\xi \in I_\lambda\}$ for $\xi < \lambda$. Clearly, $I_\lambda \cap I_\lambda^c$ is pairwise compatible. Now it follows from the construction that the families $I_\lambda I_\lambda, I_\lambda^c$ are pairwise compatible. Hence it follows from Lemma 1.1 that, for every $\xi < \lambda$, $(I_\lambda \cup I_\lambda^c) \neq P(\kappa)$. The assumptions of Lemma 1.2 are fulfilled and hence $\bigcup_{\xi < \lambda} (I_\xi \cup I_\lambda) \neq P(\kappa)$ and we get $\bigcup_{\xi < \lambda} (I_\lambda \cup I_\lambda^c) \neq P(\kappa)$.

2. Applications to the countable case. We begin with the following fact connecting properties $U$ and $U^*$:

**Proposition 2.1.** Let $\lambda < \mu < \kappa$ be uncountable cardinals. Then $U(\kappa, \lambda, \mu) \iff \forall\mu < \kappa < \kappa \Rightarrow U^*(\kappa, \lambda, \mu)$.

Proof. In both cases we argue by contradiction:

Assume that $\pi < \kappa_0$ such that $(I_\xi: \xi < \lambda)$ are uniform $\mu$-complete ideals on $\kappa$ such that $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$. Let $I_\xi = \{A \in \kappa: A \in I_\xi\}$. Clearly, $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$ and $I_\lambda$ are $\mu$-complete, a contradiction.

Let $(I_\xi: \xi < \lambda)$ be $\mu$-complete ideals on $\kappa$ such that $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$.

Let $\mathcal{A} = \{A_\xi: \xi < \lambda\}$ be the family of those ideals which have character $a_\xi$. We enumerate the set $\mathcal{A}$: $\{a_\xi: \xi < \lambda\}$ for $\xi < \lambda$. Let $\{I_\xi: \xi < \lambda\}$ be the family of those ideals which have character $a_\xi$. It is possible that some of them appear in the enumeration several times.

For $I_\xi$ let $I_\xi^c$ be a set of cardinality $a_\xi$ such that $I_\xi \notin I_\xi^c$ and let $I_\xi^c = \bigcup I_\xi^c$.

Hence $|I_\xi^c| = a_\xi$ and $I_\lambda \notin I_\lambda^c$ for $\lambda < \lambda$.

Consider $J_\lambda = \{X \subseteq \kappa: X \notin I_\lambda^c\}$. $J_\lambda$ are $\mu$-complete ideals. Write $J^\mu = \{J_\lambda: \lambda < \lambda\}$ for the families $J^\mu$. $J^\mu$ is pairwise compatible and by the assumption $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$.

Hence by Lemma 1.2 we get $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$, and thus $\bigcup (I_\xi \cup I_\lambda^c) = P(\kappa)$, contrary to our assumption.

**Theorem 2.2.** Assume that $2^\kappa$ is the first cardinal carrying a $\kappa$-complete $\sigma$-saturated ideal. Then $U(2^\kappa, \omega_1, \omega_1)$.

Proof. The theorem follows from the Erdős-Alaoglu theorem, and $U(2^\kappa, \omega_1, \omega_1)$ by the above proposition where $\kappa = 2^\kappa$, $\lambda = \omega_1$, $\mu = \omega_1$.

Ulam's problem in the countable case (i.e., $U(\kappa, \omega_1, \omega_1)$) is closely connected with the existence of ideals on $\kappa$ such that $P(\kappa)/I$ has a countable dense set. Such ideals are called separable. Actually it is proved in [7] that $U(\kappa, \omega_1, \omega_1)$ iff no uniform ideal on $\kappa$ is separable. A closer inspection of this proof gives, for every $\mu < \kappa$, $U(\kappa, \omega_1, \omega_1)$ iff no $\mu$-complete ideal on $\kappa$ is separable. Thus the investigation of ideals $I$ such that $P(\kappa)/I$ has a dense set of a given cardinality seems to be interesting.

**Proposition 2.3.** Let $2^\kappa < \kappa^{+1}$ measurable cardinal. Then $P(\kappa)$ does not have dense sets of cardinality $\kappa$ for $(2^\kappa)^*$-complete ideal $I$ on $\kappa$.

Proof. Let $I$ be a $(2^\kappa)^*$-complete ideal on $\kappa$. Let $s: \kappa \rightarrow P(\kappa)$ be a function $t: \kappa \rightarrow P(\kappa)$ which will be called a flip of $s$, iff, for all $a < \kappa$, $t(a) = s(a)$ or $t(a) = \kappa - s(a)$.

Let $F(s)$ denote the family of all flips of $s$. Clearly, $|F(s)| = 2^\kappa$. Define $\bigcup (I_\xi \cup I_\lambda^c) = \kappa$; hence there exists $u \in F(s)$ such that $\bigcup u I \notin J$. Assume that $s = (\pi(\xi): \xi < \lambda)$ is such that $(\pi(\xi)): \xi < \lambda$ is dense in $P(\kappa)/I$.

Take a flip $u$ of $s$ such that $\bigcup u I \notin J$. Hence there is an $s$ such that $s(\pi(\xi)) \in \bigcup (I_\xi \cup I_\lambda^c) \kappa$ for (mod $I$), but $P(\kappa)/I$ is atomless, a contradiction.

**Corollary 2.4.** Let $2^\kappa < \kappa^{+1}$ measurable cardinal. Then $U(\kappa, \omega_1, (2^\kappa)^*)$.

We conclude this section with the following proposition, pointed out by R. Szczech.

$\kappa$—Fundamenta Mathematicae G65
PROPOSITION 2.5. Let \( \langle f_n : n \in \omega \rangle \) be a family of atomless measures on \( 2^\omega \) such that \( \text{Dom} f_n \neq 2^\omega \). Then there exists a subset of \( 2^\omega \) non-measurable with respect to any of them.

Proof. Use the proof of Lemma 1.1. Notice that each atomless measure \( f_\infty \) can be extended to an outer atomless measure \( f_\infty' \). Consider \( h = \sum_{n \in \omega} \left( \frac{1}{2^n} \right) f_{n} \). Then \( h \) is an atomless outer measure and if \( I = \{ A \subseteq 2^\omega : h(A) = 0 \} \), then \( P(2^\omega) / I \) with the metric \( \rho(I, [B]) = h(A \Delta B) \) forms a complete space.

3. Some remarks on consistency. Our Theorems 1.3 and 1.4 give some information about the consistency of sentences \( U(x, \alpha, \mu) \) for uncountable \( \alpha \).

PROPOSITION 3.1. (i) Let \( x \) be a regular uncountable cardinal and \( \lambda < x \). Then \( \text{Con}(ZFC) \rightarrow \text{Con}(ZFC + U^*(x, \lambda, \alpha_1)) \).

(ii) Let \( \theta \) be the first cardinal carrying a \( 2^\omega \)-complete \( 2^\alpha \)-saturated ideal and \( \lambda < \mu = \text{cf} \mu \leq \kappa \). Then \( \text{Con}(ZFC) \rightarrow \text{Con}(ZFC + U(\kappa, \lambda, \mu)) \).

Proof. (i) We force \( MA + 2^\omega = \kappa \) and apply Theorem 1.3.

(ii) We force \( MA + 2^\kappa = \mu \) and apply Theorem 1.4.

Our next remark refers to the countable case. It is a consequence of a result of R. Laver (cf. [1]), namely, that \( \text{Con}(ZFC + \text{a measurable cardinal exists}) \) implies \( \text{Con}(ZFC + 2^\omega \text{-complete } \sigma \text{-saturated ideal } + U(2^\omega, \lambda, \alpha_1) \text{ for } \lambda < 2^\omega) \). The proof, however, does not seem to generalize so as to allow the real-valued measurability of \( 2^\omega \) even if \( \lambda = \omega_1 \). In view of that, consider

PROPOSITION 3.2. \( \text{Con}(ZFC + \text{a measurable cardinal exists}) \rightarrow \text{Con}(ZFC + 2^\omega \text{-real-valued measurable} + U(2^\omega, \omega_1, \alpha_1)) \).

Proof. It is possible to make \( 2^\omega \) real-valued measurable without any cardinal carrying a \( \sigma \)-complete \( \sigma \)-saturated ideal below. Hence our proposition follows from Theorem 2.2.

4. Problems. We close our paper with a list of open problems.

A. Is it possible to prove in ZFC that \( U(2^\omega, \alpha, \omega_1) \)?

In view of Proposition 2.1 Problem A is equivalent to the question whether \( \forall \alpha < 2^\omega \left( U^*(\alpha, \alpha, \alpha_1) \right) \). On the other hand, by Taylor's result and Corollary 2.4, we have \( U(\alpha, \omega_1, \alpha) \) for all \( \alpha \) s.t. \( 2^\omega < \alpha < 1 \text{st measurable cardinal} \). This yields the following problem, less general than A;

B. Is it possible to prove in ZFC that \( U(\alpha, \alpha, \alpha) \) for all \( \alpha < 2^\omega \)?

Of course, in view of the Erdős–Alaoglu theorem, problems A and B are interesting only in the case where \( 2^\omega \) is large.

What about results of the type: non \( U^*(\alpha, \alpha, \alpha) \)? The only one known is due to Magidor (cf. [7]); if there exists a huge cardinal, then \( \text{Con}(\text{non } U^*(\alpha_1, \omega_1, \omega_1)) \).

C. Is \( U(2^\omega, 2^\omega, \omega_1) \) consistent with ZFC?

Notice that if we change non \( U^*(2^\omega, 2^\omega, \omega_1) \) into non \( U(2^\omega, 2^\omega, \omega_1) \), problem C has an easy affirmative answer. Finally, notice that for all \( \alpha \), non \( U(\alpha, 2^\omega, \omega_1) \). This yields the following problem:

D. Is \( U(\alpha, x^*, \omega_1) \) consistent with ZFC for some \( x \) (e.g. \( x = 2^\omega \))?

References

[8] On the cardinality of reduced products and the Boolean Algebra \( \mathbb{B}(\alpha)/I \), to appear.
[9] On the cardinality of \( \mathbb{B}(\alpha)/I \), handwritten paper.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW

Accepted by the Redaction le 23. 7. 1979