It follows that the carrier $\gamma_P$ of $W$ is a pinched surface which satisfies the condition

$$p_1(\gamma_P) \leq 2k.$$ 

Hence $p_1(\gamma_P) \leq 2k$. We infer by (5.3) that the collection of all topological types of carriers $\gamma_P$ is finite. Since also the number $l = \left| \ell(P) \right| \leq p_1(P) \leq k$ is finite, we infer that there exists only a finite number of shapes of polyhedra $P$ with $p_1(P) \leq k$ and $n_0(P) \leq k$. Thus the proof of Theorem (11.1) is finished.

The limitation of the values of $p_1(P)$ and of $p_3(P)$ does not suffice for the finiteness of the collection of shapes of $P$. In fact, consider the 3-dimensional Poincaré sphere $M$ (i.e., a polyhedron which is a closed 3-manifold with $p_1(M) = p_3(M) = 0$ and $p_2(M) = 1$, but with a non-trivial fundamental group). Let $A$ be a 3-dimensional simplex of a triangulation of $M$. Then $N = M \setminus A$ is an acyclic 3-dimensional polyhedron with a non-trivial fundamental group. Using the operation of collapsing, one gets from $N$ a 2-dimensional acyclic polyhedron $P$ (of dimension $\leq 2$) with a non-trivial shape.

Consider now a system $P_1, \ldots, P_r$ of polyhedra homeomorphic to $P$ and constituting a bouquet with center $c$. Setting $P_k^* = P_1 \cup \ldots \cup P_k$, one gets for every $k = 1, 2, \ldots$ an acyclic polyhedron $P_k^*$ and one sees easily that $\text{Sh}(P_k^*) \neq \text{Sh}(P_{k-1}^*)$ for $k \neq k'$.

Let us add that the values of $n_0(P)$ and of $n_1(P)$ remain unknown. The following problem remains open:

(11.4) Problem. Does there exist a connected polyhedron such that for every standard representative of its shape not all carriers of wings are surfaces?

12. Non-connected polyhedra. If $P_1, \ldots, P_r$ are components of a polyhedron $P$, then every set $P_k$ is a connected polyhedron and we can assign to every standard representative of it its scheme $\xi$. The system consisting of those schemes is a finite numerical scheme which can be considered as the scheme of the polyhedron $P$. It is clear that Theorem (9.2) implies that this scheme determines the homotopy type of $P$.

References


Accepted by la Redaction le 23. 3. 1978

On a question of H. H. Corson and some related problems

by

Roman Pol (Warszawa)

Abstract. In this paper we investigate a property of a Banach space defined by Corson [7] which is a convex counterpart to the Lindelöf property of weak topology.

1. Introduction. H. H. Corson defined in [7] the following property of a Banach space $E$ (which we shall call "the property (C)""): every collection of closed convex subsets of $E$ with empty intersection contains a countable subcollection with empty intersection.

If a Banach space $E$ is Lindelöf in the weak topology then it has the property (C) (as the closed convex sets are the same in the norm or weak topology) and Corson asked [7, Remark (1), p. 7], whether the converse is true. It turns out that many familiar function spaces $C(K)$ have the property (C) while their weak topology fails to have the Lindelöf property (1), for example, this is the case, when $K$ is the lexicographic square. In fact, we show that the Banach spaces with the property (C) form a rather wide class, closed under some standard operations.

We prove that the property (C) of a Banach space $E$ is equivalent to a property of the unit ball in the dual space $E^{*}$ endowed with the weak-star topology, which is a convex analogue to the countable tightness (2). We show further that for a compact scattered space $K$ the property (C) of the function space $C(K)$ is equivalent to the countable tightness of $K$; in general, the property (C) of the function space $C(K)$ being, as stated above, related to a kind of the countable tightness in the space of Radon measures on $K$, seems essentially stronger than the countable tightness of $K$—however, we do not know a corresponding example. Another result about function spaces is that if $K$ is an I-hereditarily compact and $E$ has the property (C), then so does the space $C(K \times E)$, but we do not know, for example, if $C(S \times S)$ has the property (C), provided that $(S \times S)$ does. We discuss these and related, questions in the last paragraph.

(1) Thus the counterpart to the classical characterization of compactness for the weak topology [16, Theorem 11.2(c)] fails for the Lindelöf property.

(2) The terminology is explained in the next paragraph.

--- Fundamenta Mathematicae 65
2. Terminology and notation. Our terminology (and, if possible, notation) related to general topology, function spaces, or topological vector spaces follows Engelking [10], Semadeni [19], or Schaefer [18], respectively. Let us agree on the notation. The cardinality of a set $A$ is denoted by $|A|$: the $n$th derived set of a space $A$ is denoted by $A^{(n)}$; if a closed set $A \subseteq E$ we denote by $A$ the space obtained from $A$ by plucking $A$ from a point. If $K$ is a compact space and $E$ is a Banach space, then $C(K, E)$ is the Banach space of all continuous mappings from $K$ to $E$ endowed with the sup norm and we write $C(K)$ when $E$ is the real line; if $A \subseteq K$ is closed, $C_0(K\setminus A, E)$ (resp. $C_0(K\setminus{a})$) is the subspace of all continuous mappings vanishing on $A$ — it can be identified with the subspace of $C(K, E)$ [19, Ch. II, §4]. Given a Banach space $E$, sets $A, B \subseteq E$ and a scalar $\lambda$, we put

$$\text{dist}(A, B) = \inf \{|a-b|: a \in A, b \in B\}$$

and

$$A + \lambda \cdot B = \{a + \lambda b: a \in A, b \in B\}.$$  

We denote by $E'$ the dual space of the Banach space $E$, $\langle x, x' \rangle$ stand for the value of the functional $x' \in E'$ at the point $x \in E$, conv $\mathcal{A}$ is the convex hull of the set $\mathcal{A} \subseteq E$ and in $\sigma(E', E)$ denote the weak-star topology in $E'$.

Recall, that a space $K$ has the countable tightness if, whenever $x \in K$, then $x \in C$ for a countable $C \subseteq K$, and that an Eberlein compact is a space homeomorphic to a compact subspace of a Banach space endowed with the weak topology, for the last notion as well as for the notion of a WCG Banach space the reader is referred to [5, Ch. VI].

3. The property (C).

3.1. The following property was defined by Corson (see Introduction): a convex subset $M$ of a Banach space $E$ (*) has the property (C) if every collection of closed convex subsets of $M$ with empty intersection contains a countable subcollection with empty intersection.

A base for the sequel is the following lemma. Let $E$ be a Banach space. If $E$ does not have the property (C), then there exists a collection $\mathcal{U}$ of nonempty convex subsets of the unit ball, closed under countable intersections (*) and an $\varepsilon > 0$ such that for every convex subset $M$ of $E$ with the property (C) there is a $C \in \mathcal{U}$ with $\text{dist}(M, C) < \varepsilon$.

Proof. At first let us verify that if $\text{dist}(a, C) < \varepsilon$ for every collection $\mathcal{U}$ of nonempty convex subsets of $E$ closed under countable intersections and every $\varepsilon > 0$ there is a point $a \in E$ with $\text{dist}(a, C) < \varepsilon$ for $C \in \mathcal{U}$.

(*) In fact, we are only interested in the case when $M$ is itself a Banach space.

(**) It means that $\cap A \cap C$, if $A \subseteq C$ is countable.

then $E$ has the property (C). Let $\mathcal{U}$ be a collection of nonempty, closed, convex subsets of $E$ closed under countable intersections; a standard reformulation of the property (C) cf. [10, 3.8] shows that it is enough to check that $\cap \mathcal{U} \neq \emptyset$. For this purpose, let us define inductively for every $C \in \mathcal{U}$ nonempty convex sets $C^{(0)}$ and choose points $a_0 \in E$ in such a way that

1. $C = C^{(0)} \supseteq C^{(1)} \supseteq \ldots$, $\text{dist}(C^{(n)}, C^{(n+1)}) < 2^{-n}$,
2. every collection $\mathcal{U}^{(0)} = \{C^{(0)}: C \in \mathcal{U}\}$ is closed under countable intersections,
3. $\text{dist}(a_n, C^{(n)}) < 2^{-n}.$

We choose $a_0$ using $(*)$ with $\mathcal{U} = \emptyset$ and $\varepsilon = 2^{-3}$. Assume that $\mathcal{U}^{(n)}$ and $a_n$ are defined. For every $C \in \mathcal{U}$ put $C^{(n+1)} = C^{(n)} \cap (a_n + 2^{-n+1} B)$ where $B$ stand (also in the sequel) for the unit open ball in $E$. The sets $C^{(n+1)}$ are convex and nonempty, by (3), the condition (1) is satisfied obviously and, finally, the family $\mathcal{U}^{(n)}$ satisfies (2), as $\mathcal{U}^{(n)}$ does. We complete the inductive step choosing a point $a_{n+1}$ by means of $(*)$, where $\mathcal{U} = \mathcal{U}^{(n)}$ and $\varepsilon = 2^{-n}.$

Now, the points $a_0, a_1, \ldots$ form a Cauchy sequence, as by (1) and (3) we have $\|a_j - a_{n+1}\| < 2^{-n+1} + 2^{-n+2} + 2^{-n+3} + \ldots < 3^{-n}$ for $j > n$.

Let $a$ be the limit point of this sequence. By (3) we have $\text{dist}(a, C) = 0$ for $C \in \mathcal{U}$ and hence $a \in \cap \mathcal{U}$ because each $C \in \mathcal{U}$ is closed.

We pass to the proof of the lemma. Since $E$ does not have the property (C), the property $(*)$ fails for some $\mathcal{U}$ and $a$. Since $\mathcal{U}$ is closed under countable intersections there exists a natural number $n$ such that each member of $\mathcal{U}$ intersects the ball $B$. Put $\mathcal{U}' = \{B \cap n \cdot B: C \in \mathcal{U}\}$ and $\varepsilon = \frac{a}{n}$.

The family $\mathcal{U}'$ consists of nonempty convex subsets of the unit ball, it is closed under countable intersections and for every $x \in E$ there exists $C_x \in \mathcal{U}'$ with $\text{dist}(x, C_x) < \varepsilon$.

We claim that $\mathcal{U}'$ and $a$ are the family and the number we are looking for. For, let $M$ be a convex subset of $E$ with the property (C) and suppose that $\text{dist}(M, C) < \varepsilon$ if $C \in \mathcal{U}$. Let us put $C^* = (C \setminus a B) \cap M$ for every $C \in \mathcal{U}$. The sets $C^*$ are convex and nonempty and given a countable collection $\mathcal{U} \subseteq \mathcal{U}$ we have $\cap C \subseteq \cap C^*$ (recall, that $\cap C = \emptyset$). Thus by the property (C) of $M$ there exists $x \in \cap C \subseteq \cap C^*$.

But then $\text{dist}(x, C) < \varepsilon$ for every $C \in \mathcal{U}$, a contradiction which completes the proof.

3.2. The lemma yields easily the following propositions.

Proposition 1. Let $T: E \to F$ be a continuous linear mapping of a Banach space $E$ onto a Banach space $F$. If both $F$ and the kernel of the operator $T$ have the property (C) then so does the space $E$.

Proof. Suppose the contrary and let $\mathcal{U}$ and $\varepsilon > 0$ be as in the lemma. Since each $T^{-1}(y)$ has the property (C), there exists for every $y \in F \cap C_x \subseteq \mathcal{U}$ with $\text{dist}(T^{-1}(y), C_x)$
By the Banach open mapping principle we have $y \in TC_{\gamma}$ for $y \in F$ and thus $\bigcap \{ TC_{\gamma} : C \in \mathcal{G} \} = \emptyset$, a contradiction with the property (C) of $F$.

**Proposition 2.** Let $M_1, M_2, \ldots$ be convex sets with the property (C) in a Banach space $E$. If the union $\bigcup M_i$ is dense in $E$, then $E$ has the property (C).

**Proof.** For, in the opposite case there would exist a collection $W$ and an $\varepsilon > 0$ as in the lemma. But then $\text{dist}(M_i, C) > \varepsilon$ for some $C \in \mathcal{G}$ and we would have $\text{dist}(\bigcup M_i, \bigcap C) > \varepsilon$, which is impossible, as $\bigcap C \neq \emptyset$ and $\bigcup M_i$ is dense in $E$.

3.3. As was pointed out by Corson (see Introduction) the class of Banach spaces which have the property (C) includes those which are Lindelöf in the weak topology. The last class contains all Banach subspaces of WCG-spaces, by Talagrand [21], but not exclusively — for the examples see Pol [16], Talagrand [22], [23] (cf. also Alaoglu and Pol [2] and Gulko [11] and Pol [18, Example 2]). Here we give some function spaces (all of them provide "standard examples" in topology) with the property (C), which are not Lindelöf in the weak topology, thus answering a question of Corson mentioned in Introduction.

**Example 1.** Let, as in Corson [7, Example 2] C, be the space of all bounded real-valued functions on the unit interval $I$ which are continuous on the right and which have a finite limit on the left, endowed with the sup norm. Corson showed that $C_I$ is not Lindelöf in the weak topology, but it admits a linear continuous operator whose range is the space $C(I)$, and the kernel — the space $C(I)$, are Lindelöf in the weak topology. By Proposition 1 the space $C_I$ has the property (C).

Observe, that the space $C_I$ may be identified in a natural way with the function space $C(K)$ on the "two arrows space" $K$ of Alexandrov and Urysohn [10, 3.10, C].

**Example 2.** Let $L$ be the lexicographic square [10, 3.12.1 (d)], i.e.

$$L = [0, 1] \times [0, 1]$$

is endowed with the order topology induced by the lexicographic order. The subspace $K = \{(x, y) \in L : y = 0$ or $y = 1\}$ is the two arrows space.

Thus $C(L)$ is not Lindelöf in the weak topology. However, $C(L)$ has the property (C), by Proposition 1, as the restriction operator $T : C(L) \rightarrow C(K)$ has the range with the property (C), by Example 1, while the kernel $C(L)/K$ has the property (C) by [7, Corollary 1] (since $K/L$ is a continuous image of the one-point compactification of the free union of continuum copies of the circle).

**Example 3.** If $S$ is an uncountable compact separable space with the $c_0$th derived set empty (\footnote{For such examples see [19, 5.10(G)] and [20, 5.10(G)]}) then $C(S)$ is not Lindelöf in the weak topology [17, Theorem], but it has the property (C). The last assertion is a particular case of Corollary 4.1.1, but it can be also easily verified step by step by means of Propositions 1 and 2.

**Example 4.** Let $H$ be the Helly space, i.e. the space of all nondecreasing functions from the unit interval into itself, endowed with the pointwise topology [10, 3.2.16]; $H$ is a compact convex set. Let $H' = (H')^*$ be the Banach space of all continuous affine functions on $H$ [19, 23.1]. The set $K$ of the extremal points of $H'$ consisting of those functions from $H$ which take only the values 0 or 1, is in fact the two arrows space. Thus, by the maximum principle [19, 23.1.10], $A(H')$ can be considered as a subspace of $C(K)$ and so, by Example 1, it has the property (C). But, it is easy to see, that $A(H')$ is not Lindelöf in the weak topology.

We do not know whether $A(H')$ has the property (C) (cf. Added in proof).

3.4. In this section we characterize the property (C) of a Banach space in terms of the dual space. The characterization bases on the following lemma, which is a counterpart to Lemma 3.1; we shall exploit this lemma also in the next paragraph.

**Lemma.** Assume that a Banach space $E$ does not have the property (C). Then there exists a subset $A'$ of the unit ball in the dual space $E'$ and an $\varepsilon > 0$ such that (a) for every linear subspace $M$ of $E$ with the property (C) there is an $x' \in A'$ vanishing on $M$, but (b) for every countable $C' \subset A'$ there is a $u \in E$ with $\|u\| \leq 1$ and $\langle \xi, x' \rangle > 0$ for every $x' \in C'$.

**Proof.** Let a collection $W$ and an $\varepsilon > 0$ be as in Lemma 3.1 and let $A'$ be the family of all linear subspaces of $E$ which have the property (C). So, given an $M \in A'$ there exists $C_M \in W$ such that $\text{dist}(M, C_M) > \varepsilon$. Let $B$ be the open unit ball in $E$. Since $(M + \varepsilon B) \cap C_M = \emptyset$, there exists by the separation theorem [18, Ch. II, 9.1] a linear functional $x_M' \in E'$ such that $\|x_M'\| = 1$ and

$$\sup_{x \in (M + \varepsilon B)} \inf\{\langle \xi, x_M' \rangle : x \in C_M\} = 0$$

Since $M$ is a linear space and since $\|x_M'\| = 1$ we have

$$(1) \quad x_M'|M = 0 \quad \text{and} \quad \langle \xi, x_M' \rangle > \varepsilon \quad \text{for every} \quad x \in C_M.$$

Let us put $A' = \{x_M' : M \in A'\}$; we claim that this set and the $e$ from (1) are the objects we are looking for. We have only to check (b). But, if $C' = \{x_M', x_{M'}', \ldots\}$ is a countable subset of $A'$, we take a point $e \in \bigcap C_M$ and (b) follows from the second part of (1).

**Theorem.** For a Banach space $E$ the following conditions are equivalent:

(i) $E$ has the property (C);

(ii) the dual space $E'$ (equivalently — the unit ball $B'$ in $E'$) endowed with $\sigma(E', E)$-topology has the property (C); if $x' \in A' \subset E'$ (resp. $A' \subset B'$) then $x' \varepsilon$ conv $C'$ for a countable $C' \subset A'$.

Proof. The fact that the property (C) of $B'$ implies the property (C) of $E$ follows immediately from the lemma, as $0$ is in the $\sigma(E', E)$-closure of the set $A'$ defined in this lemma, by (a), but not in the $\sigma(E', E)$-closure of any conv $C'$ with $C' \subset A'$ countable, by (b).

The fact that (ii) implies the property (C) of $E'$ can be proved by general duality
arguments as follows. Let $a' \in A \subseteq E'$ (we consider $\sigma(E', E)$-topology in $E'$). For every $x' \in A'$ put $C_{x'} = \{x \in E': (x, x') \supseteq (x, a') + 1\}$. The sets $C_{x'}$ are closed, convex and $\cap (C_{x'}, x' \in A') = \emptyset$. So there is a countable set $C \subseteq A$ with $\cap (C_{x'}, x' \in C') = \emptyset$. Then $a' \in \text{conv} C = H'$. Indeed, in the opposite case there was an $x \in E$ such that $(x, H') \supseteq (x, a') + 1$ for $k' \in H'$, [18, Ch. III, 9.1 and Ch. IV, 1.2], but then $x \in \cap (C_{x'}, x' \in C')$, a contradiction. 

3.5. The following corollary to the theorem extends a result of Corson [7, Corollary 1] that function a space $C(X)$, where $X$ is the one-point compactification of a locally compact metrizable space, has the property (C); a particular case of this corollary is a step in the proof of Theorem 4.1.

**Corollary.** Let $(E_{x})_{x \in X} = (E_{x})_{x \in X}$ be a family of Banach spaces and let $F$ be the $c_{0}(S)$-product of the family, i.e. $F$ is the space $X \in \prod E_{x} = (E_{x})_{x \in X}$ endowed with the sup norm. If each $E_{x}$ has the property (C), then so does the space $F$.

**Proof.** The dual space $F'$ of $F$ is the $l_{1}(S)$-product of the duals of the family $(E_{x})_{x \in X}$ [8, Ch. II, §2, (1)] and thus the unit ball $B'$ in $F'$ endowed with the $\sigma(F', F)$-topology is the subspace $B' = \{x \in \prod B_{x}: \sum \|x_{x}\| \leq 1\}$ of the Tykhonoff product $\prod B_{x}$ of the unit balls $B_{x} = E_{x}$ endowed with the $\sigma(E_{x}, E_{x})$-topology. By standard reasonings related to such subspaces of products (cf. [10, X-products, p. 158]) one can easily verify that since each $E_{x}$ has the property (C), then so does $B'$ (the needed fact that each product $B_{x} \times \cdots \times B_{x}$ has the property (C) follows from the identity $(E_{x} \times \cdots \times E_{x}) = E_{x} \times \cdots \times E_{x}$) by Proposition 3.2.1 (i).

A straightforward proof of the corollary is sketched in Section 5.5.

4. Function spaces with the property (C).

4.1. The main result of this paragraph is the following.

**Theorem.** Let $K = X_{1} \times X_{2} \times \cdots$ be the countable product of compact scattered spaces of countable tightness and let $E$ be a Banach space with the property (C). Then the function space $C(K, E)$ has the property (C).

The particular case, when $K$ is itself scattered and $E$ is the real line, seems to be worth while noticing separately; we shall do it explicitly to emphasize the connections between the topology of $K$ and the linear-topological structure of $C(K)$.

**Corollary 1.** For a compact scattered space $K$ the following statements are equivalent:

(i) if $x \in A \subseteq K$, then $x \in C = \text{C} \subseteq A$;

(ii) if $\mathcal{A}$ is a collection of closed convex sets in $C(K)$ and $\cap \mathcal{A} = \emptyset$ then $\cap \mathcal{A} = \emptyset$ for a countable $\mathcal{A} \subseteq \mathcal{A}$.

Here, the implication (ii) $\Rightarrow$ (i) (for arbitrary $K$) is contained in Theorem 3.4, but it can be easily proved directly — see Section 5.1.

(?) One can also follow reasonings of Mal'cev [13, Theorem 4].

Of course, the assertion of the theorem is also true for continuous images of closed subsets of such products $X_{1} \times X_{2} \times \cdots$ so in particular we have (cf. [5, Lemma 1-1], [19, 5.3.4] and [9, Corollary 3]).

**Corollary 2.** If $K$ is an Ebrel-compact and a Banach space $E$ has the property (C), then so does $C(K, E)$; in particular, if $C(S)$ has the property (C) for a compact space $S$ then so does $C(K \times S)$, as well as $C(S, F)$, provided that $F$ is a WCG space.

The rest of this paragraph is devoted to the proof of the theorem.

4.2. We shall need the following two simple facts. The first is a corollary to a Mal'cev's result [15, Theorem 4].

**Lemma 1.** Let $X$ be a compact space of countable tightness, let $p \in X$ and let $\mathcal{F}$ be a family of finite subsets of $X$ such that each $\mathcal{F}$-set containing $p$ contains a member of $\mathcal{F}$. Then there exists a countable subfamily $\mathcal{F}^{*}$ of $\mathcal{F}$ such that each neighbourhood of the $\mathcal{F}^{*}$ contains a member of $\mathcal{F}^{*}$.

**Proof.** Put $\mathcal{F}_{p} = \{p \in \mathcal{F} : |F| < n\}$. By the assumption there is an $n$ such that each neighbourhood of $p$ contains a member of $\mathcal{F}_{p}$. It remains to use the fact that the product $X^{n}$ is of countable tightness [10, 3.12.8(e)] to choose the family $\mathcal{F}^{*} \subset \mathcal{F}_{p}$.

The next fact, closely related to tensor products, is certainly well-known, cf. [18, Ch. IV, 9.2], [12, Corollary]; we indicate here briefly a direct (and fairly standard) proof.

**Lemma 2.** Let $K$ be a compact scattered space and let $E$ be a Banach space. For every linear functional $f \in C(K, E)$ there exist points $a_{1}, a_{2}, \ldots$ from $K$ and functionals $u_{1}, u_{2}, \ldots$ from $E$ such that

\[
\|f\| = \sum_{i} |u_{i}|, \quad a_{i} \neq 0,
\]

\[
\langle f, a \rangle = \sum_{i} \langle f(a_{i}), u_{i} \rangle \quad \text{for every } f \in C(K, E).
\]

**Proof.** One can assume that $E = C(S)$ [19, 6.1.9]. Identifying $C(K, C(S))$ with $C(K \times S)$ we consider $f$ as a Radon measure $\mu$ on $K \times S$ [19, 18.4.7]. Let $X$ be the image of the total variation $|\mu|$ under the projection $p: K \times S \to K$ [19, 18.3.1 and 17.2.1]. Since $K$ is scattered, $X$ is purely atomic $[19, 18.7.7]$; let $a_{1}, a_{2}, \ldots$ be the atoms and let $\mu_{i} = \mu_{z}^{-1}(a)$. Each $\mu_{i}$ can be considered as an $u_{i} \in C(S)$, again by [19, 18.4.1]. The points $a_{i}$ and the functionals $u_{i}$ satisfy (a) and (b).

4.3. At first we shall prove the theorem in the case, when $K$ is a compact scattered space of countable tightness. The proof goes by the induction with respect to the ordinal $\alpha$ such that the $\alpha$th derived set of $K$ is finite and nonempty. The case $\alpha = 0$ follows from Corollary 3.5 (with $S$ finite), let us assume that the assertion holds for all $\beta < \alpha$ and let $K^{(\beta)}$ be finite and nonempty.

**Assume, that $\alpha = \beta + 1$.** Let $T: C(K, E) \to C(K^{(\beta)}, E)$ be the restriction operator. The kernel $C(K)(K^{(\beta)}, E)$ of $T$ has the property (C) by the inductive assumption, while the range $C(K^{(\beta)}, E)$ of $T$ has the property (C) by Corollary 3.5. It remains to apply Proposition 3.2.1.
Assume that \( a \) is a limit ordinal. It is easy to see that this case reduces to the proof of the

Addition Lemma. Let \( p \) be a point in our space \( K \) such that for every compact space \( T = K \) disjoint in \( p \) the function space \( C(T, E) \) has the property \( C \). Then \( C(K, E) \) has the property \( C \).

It is enough to verify that the space \( C_0(K, E) \) of all functions from \( C(K, E) \) vanishing at \( p \) has the property \( C \) (see Proposition 3.2.1). Assume the contrary and choose a set \( A' \) from the unit ball in \( C_0(K, E) \) and an \( \varepsilon > 0 \) as in Lemma 3.4.

Let \( \mathcal{F} \) be the family of all closed \( G_\beta \)-sets in \( K \) containing the point \( p \). Fix an \( S \in \mathcal{F} \) and let \( G_1 \) be open sets such that \( S = \bigcap G_1 \) and \( G_1 \in \mathcal{F} \). By the assumption each \( M_f = C(K) \cdot G_1, E) \) has the property \( C \) and so does the space \( C(K, E) \) by Proposition 3.2.2, as the union \( \bigcup M_f \) is dense in this space. It follows (see Lemma 3.4) that there exists a \( f_\beta \in A' \) vanishing on \( C(K, E) \). It means exactly that, representing \( f_\beta \) as in Lemma 4.2.2 by the points \( a_{1, \alpha}, a_{2, \alpha}, \ldots \) and the functionals \( u_{1, \alpha}, u_{2, \alpha}, \ldots \), we have \( a_{i, \alpha} \in S \) for every \( i \). By the property \( (i) \) of this lemma, there exists a \( k_\beta \) such that

\[
\sum_{n \leq k_\beta} ||u_{n, \beta}|| < \frac{1}{2^n}.
\]

Let \( F_\beta = \{a_{i, \beta} : i \leq k_\beta\} \in S \). We can exploit Lemma 4.2.1 to choose a countable family \( \mathcal{G} \subset \mathcal{F} \) such that every neighbourhood of \( p \) contains some \( F_\beta \) with \( S \in \mathcal{G} \).

The set \( A' \) was chosen (see Lemma 3.4) in such a way that we can pick now a \( \tau \in C_0(K, E) \) with

\[
||\tau|| \leq 1 \quad \text{and} \quad \langle \tau, f_\beta \rangle > \frac{1}{\mathcal{G}} \quad \text{for every} \quad S \in \mathcal{G}.
\]

Put \( V = \{x \in K : ||\tau(x)|| < \frac{1}{3}e\} \); then \( V \) is a neighbourhood of \( p \) and hence there exists \( S_0 \in \mathcal{G} \) with \( F_\beta = V \). So we have by \( (1) \), the first part of \( (2) \) and Lemma 4.2.2(b):

\[
\sum_{n \leq k_\beta} ||u_{n, \beta}|| < \frac{1}{2^n} \sum_{n \leq k_\beta} ||u_{n, \beta}|| < \frac{1}{2^n} \sum_{n \leq k_\beta} ||u_{n, \beta}|| < \varepsilon,
\]

a contradiction with the second part of \( (2) \).

4.4. Now, let \( K = X_1 \times X_2 \times \ldots \) be as in the theorem and let \( p_i : K \rightarrow K_i = X_i \times X_{i+1} \times \ldots \) be the projection. Each \( C(K_i, E) \) has the property \( C \), by the case just proved in 4.3 (cf. [11, 31.3(5)]) and so does the isomorphic space

\[
M_i = \{f \in p_i : f \in C(K_i, E)\}.
\]

Finally, the union \( \bigcup M_i \) is dense in \( C(K, E) \) [6, Ch. X, § 4,1] and an application of Proposition 3.2.2 completes the proof of the theorem.

5. Comments.

5.1. Let \( K = X_1 \times X_2 \times \ldots \) be a compact space such that the function space \( C(K, E) \) has the property \( C \). Then

\[(t_3) \text{ the space } P(K) \text{ of probability Radon measures on } K \text{ has the property } C \text{ from Theorem 3.4;}
\]

\[(t_4) \text{ if } A \text{ is a closed subset of } K \text{ and } \mathcal{P} \text{ is a collection of probability Radon measures on } K \text{ such that for every neighbourhood } V \text{ of } A \text{ there is a measure } \mu \text{ from } \mathcal{P} \text{ concentrated on } V \text{; then there exists a countable collection } \mathcal{P}' \subset \mathcal{P} \text{ such that given a neighbourhood } V \text{ of } A \text{ and } \varepsilon > 0 \text{ there is a } \mu \in \mathcal{P}' \text{ with } \mu(K \setminus V) < \varepsilon;}
\]

\[(t_5) \text{ if } p \in K \text{ and } A \text{ is a family of finite sets in } K \text{ such that each neighbourhood of } p \text{ contains some member of } A, \text{ then there is a countable subfamily } \mathcal{P}' \subset \mathcal{P} \text{ such that given a neighbourhood } V \text{ of } p \text{ and an } \varepsilon > 0 \text{ there is a } \mu \in \mathcal{P}' \text{ with } \mu(V) > ||V|| - \varepsilon > 0;}
\]

\[(t_6) K \text{ has the countable tightness.}
\]

The condition \( t_3 \) is in fact equivalent to the property \( C \), by Theorem 3.4; indeed, in notation of the theorem, \( P(K) = B' \), while \( B' \) is the image of \( P(K) 	imes [0, 1] \) under the mapping \( (\mu, s) \mapsto s \cdot \mu - \tau \cdot \varepsilon \), and such mappings preserve the property \( C \) (cf. [15, 17.2.5] and [10, 31.3(8)]). A direct proof that \( C \Rightarrow t_3 \) runs as follows: put

\[
C_0 = \{f \in C(K) : \{|f| > \varepsilon \} \text{ for } \mu \in \mathcal{P} ;
\]

then \( \bigcap C_0 = \emptyset \) and there exists a countable set \( \mathcal{P}' \subset \mathcal{P} \) with \( \bigcap C_0 = \emptyset \). To show that \( t_3 \Rightarrow t_3 \) it is enough to consider for each \( F \in \mathcal{P} \) the measure \( \mu_F = \frac{1}{|F|} \sum_{x \in F} \delta_x \), where \( \delta_x \) is the point mass at \( x \). Finally, we have the obvious implication \( t_3 \Rightarrow t_3 \).

It seems interesting to clarify what are the exact relationships between the conditions \( t_3 \) (for scattered \( K \) they are simply equivalent, by Corollary 4.1.1); we conjecture that \( t_3 \) is an essentially stronger property of a compact space than the countable tightness.

Another natural question is, whether the property \( C \) from Theorem 3.4 (or, equivalently, \( t_3 \)) is exactly the countable tightness of \( B' \) (or \( P(K) \)).

5.2. Let us consider the following property of a compact space \( K \) following from the property \( C \) of the function space \( C(K) \) (cf. Telgárna [22, Theorem 6.6]):

\[(M) \text{ every Radon measure on } K \text{ has the separable support. That } (C) \text{ implies } (M) \text{ can be justified as follows; let } \mu \text{ be a Radon measure supported exactly on } K, \text{ fix a natural } i \text{ and for } x \in K \text{ put}
\]

\[
C_x = \{f \in C(K) : \{|f| > \varepsilon \} \text{ and } \langle f, x \rangle = 0 \},
\]

then \( \bigcap C_x = \emptyset \) and so \( \bigcap C_x = \emptyset \) for a countable \( A \subset K \); now \( K = \bigcup A \).

This observation, a result of Sapirvovskii [20] and a result obtained independently by Alster and Pol [2] and Gulko [11] yield the equivalence of the following state-

(\*) We consider \( P(K) \) with the weak-star topology.
On a question of H. H. Corson and some related problems

bidual Banach spaces" (preprint), where the following result was proved (Theorem 3.1):

assuming the continuum hypothesis, there exists a compact space S which is first-countable and non-separable, but it is the carrier of a Radon measure.

Since S was constructed for other purpose, the properties are not explicitly stated in Haydon's paper, however they follow easily from the facts proved there.

Now, let us observe that:

(a) C(S) does not have the property (C), by 5.2, and so, under the continuum hypothesis, the condition (t) in 5.1 does not imply the condition (t) (cf. also Introduction);

(b) since S does not have the property (M) formulated in 5.2, being first-countable, it follows that each of the equivalent statements (a) and (b) in 5.2 is independent of the ZFC-axioms for set theory.

Added in proof.


b) G. Godefroy proved in Pacific J. Math. (to appear) that if S is a compact set in the space B(P) of the first Baire class functions on the irrationals with the pointwise topology then C(S) has property (C), and moreover that P(S) embeds in B(P).

c) Since the bidual space P(S) of the first Baire class functions on the irrationals with the pointwise topology then C(S) has property (C), and moreover that P(S) embeds in B(P).

d) In "Note on the space P(S) of regular probability measures whose topology is determined by countable subsets" (preprint) we show that if C(S) has property (C) then also C(P(S)) has property (C), and in particular, P(S) has countable tightness.

References


On families of \(\sigma\)-complete ideals

by

Adam Krawczyk and Andrzej Pelc (Warszawa)

Abstract. Our main results are the following: Assume Martin's Axiom. Then
1. For every \(\lambda < 2^\omega\) and every family \(\{\mu_\alpha : \alpha < \lambda\}\) of two-valued uniform measures on \(2^\omega\) there exists an \(X \subseteq 2^\omega\) non-measurable with respect to any of them.
2. For every cardinal \(\kappa\) such that \(2^\kappa < \kappa^+\) there is a cardinal carrying a \(2^\omega\)-complete \(\kappa^+\)-saturated ideal the following holds: if \(\lambda < 2^\omega\) and \(\mu_\alpha : \alpha < \lambda\) is a family of \(2^\omega\)-additive two-valued measures on \(\kappa\), then there exists an \(X \subseteq \kappa\) non-measurable with respect to any of them.

6. Terminology and preliminaries. We shall use standard set-theoretical notation and terminology. Letters \(\kappa, \lambda, \mu\) will always denote uncountable cardinals. "\(I\) is an ideal on \(X\)" will mean "\(I\) is a \(\sigma\)-complete proper ideal of subsets of \(X\) such that \(\{x\} \in I\) for all \(x \in X\). An ideal \(I\) is \(\lambda\)-complete iff \(\{x : \kappa < \eta < \lambda\} \in I\) implies \(\bigcup \{x : \xi < \kappa\} \in I\) for \(\eta < \lambda\). A cardinal \(\lambda\) is called the character of an ideal \(I\) on \(\kappa\) iff \(\chi(I) = \lambda\) is the least cardinal such that \(\exists X \subseteq \kappa, |X| = \lambda, X \notin I\). An ideal \(I\) on \(\kappa\) is uniform iff \(\forall \mu < \kappa\), if \(I\) is an ideal on \(\kappa\), then \(\{x\} \notin I\) will denote the dual filter.

Ideals \(I_1\) and \(I_2\) on \(\kappa\) are called compatible iff there is an ideal \(I_0\) on \(\kappa\) such that \(I_1 \subseteq I_2 < I_0\). It is easy to see that \(I_1, I_2\) are compatible iff \(I_1 \cap I_2^0 = \emptyset\) iff \(I_2 \cap I_1^0 = \emptyset\).

Otherwise \(I_1, I_2\) are incompatible.

MA will denote Martin's Axiom. We shall use the following consequence of MA (see [4]):

1. The union of \(< 2^\omega\) closed nowhere dense subsets of a metric complete separable space is nowhere dense.
2. A subset \(A\) of the reals is called strongly Lusin if for every Lebesgue measurable set \(A \cap [x, y) = \emptyset\) iff \(A\) has Lebesgue measure 0. It is also a consequence of MA (see [2], cf. also [1], [4], [6]) that

A strongly Lusin set exists.

We use the following notation:

\(U_\kappa(\lambda, \mu) : \) — For every family \(\{\mu_\alpha : \alpha < \lambda\}\) of \(\mu\)-complete ideals on \(\kappa\) we have \(\bigcup \{\mu_\alpha \cup I^\alpha\} \in \mu(\kappa)\).

\(U(\lambda, \mu) : \) — For every family \(\{I_\alpha : \alpha < \lambda\}\) of \(\mu\)-complete ideals on \(\kappa\) we have \(\bigcup \{I_\alpha \cup I^\alpha\} \notin P(\kappa)\)