It follows that the carrier c_W of W is a pinched surface which satisfies the condition

$$p_1(C_W) \leqslant p_1(W) .$$

Hence $p_1(C_W) \leqslant 2k$. We infer by (5.3) that the collection of all topological types of carriers C_W is finite. Since also the number $l = l(P) \leqslant p_1(P) \leqslant k$ is finite, we infer that there exists only a finite number of shapes of polyhedra P with $p_1(P) \leqslant k$ and $n_0(P) \leqslant k$. Thus the proof of Theorem (11.1) is finished.

The limitation of the values of $p_1(P)$ and of $p_2(P)$ does not suffice for the finity of the collection of shapes of P. In fact, consider the 3-dimensional Poincaré sphere M (i.e. a polyhedron which is a closed 3-manifold with $p_1(M) = p_2(M) = 0$ and $p_3(M) = 1$, but with a non-trivial fundamental group). Let $\varDelta$ be a 3-dimensional simplex of a triangulation of M. Then $N = \overline{M \setminus \varDelta}$ is an acyclic 3-dimensional polyhedron with a non-trivial fundamental group. Using the operation of collapsing, one gets from N a 2-dimensional acyclic polyhedron P (of dimension $\leqslant 2$) with a non-trivial shape.

Consider now a system $P_1, \ldots, P_k$ of polyhedra homeomorphic to P and constituting a bouquet with center c. Setting $P_k^* = P_1 \cup \ldots \cup P_k$, one gets for every $k = 1, 2, \ldots$ an acyclic polyhedron P_k^* and one sees easily that $\mathrm{Sh}(P_k^*) \neq \mathrm{Sh}(P_{k'}^*)$ for $k \neq k'$.

Let us add that the values of $m_0(P)$ and of $n_0(P)$ remain unknown. The following problem remains open:

(11.4) Problem. *Does there exist a connected polyhedron such that for every standard representative of its shape not all carriers of wings are surfaces?*

12. Non connected polyhedra. If $P_1, \ldots, P_k$ are components of a polyhedron P, then every set P_i is a connected polyhedron and we can assign to every standard representative of it its scheme $\mathfrak{S}_i$. The system consisting of those schemes is a finite numerical system which can be considered as the scheme of the polyhedron P. It is clear that Theorem (9.2) implies that this scheme determines the homotopy type of P.

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On a question of H. H. Corson and some related problems

by

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Abstract. In this paper we investigate a property of a Banach space defined by Corson [7] which is a convex counterpart to the Lindelöf property of weak topology.

1. Introduction. H. H. Corson defined in [7] the following property of a Banach space E (which we shall call "*the property* (C)"): *every collection of closed convex subsets of E with empty intersection contains a countable subcollection with empty intersection.*

If a Banach space E is Lindelöf in the weak topology then it has the property (C) (as the closed convex sets are the same in the norm or weak topology) and Corson asked [7, Remark (1), p. 7], whether the converse is true. It turns out that many familiar function spaces $C(K)$ have the property (C) while their weak topology fails to have the Lindelöf property (1), for example, this is the case, when K is the lexicographic square. In fact, we show that the Banach spaces with the property (C) form a rather wide class, closed under some standard operations.

We prove that the property (C) of a Banach space E is equivalent to a property of the unit ball in the dual space E' endowed with the weak-star topology, which is a convex analogue to the countable tightness (2). We show further that for a compact scattered space K the property (C) of the function space $C(K)$ is equivalent to the countable tightness of K; in general, the property (C) of the function space $C(K)$ being, as stated above, related to a kind of the countable tightness in the space of Radon measures on K, seems essentially stronger than the countable tightness of K — however, we do not know a correspondent example. Another result about function spaces is that if K is an Eberlein compact and E has the property (C), then so does the space $C(K, E)$, but we do not know, for example, if $C(S \times S)$ has the property (C), provided that $C(S)$ does. We discuss these, and related, questions in the last paragraph.

(1) Thus the counterpart to the classical characterization of compactness for the weak topology [18, Theorem 11.2 (c)] fails for the Lindelöf property.

(2) The terminology is explained in the next paragraph.

2. Terminology and notation. Our terminology (and, if possible, notation) related to general topology, function spaces, or topological vector spaces follows Engelking [10], Semadeni [19], or Schaefer [18], respectively. Let us agree the notation. The cardinality of a set A is denoted by $|A|$; the αth derived set of a space K is denoted by $K^{(\alpha)}$; given a closed set $A\subset K$ we denote by K/A the space obtained from K by pinching A to a point. If K is a compact space and E is a Banach space, then $C(K, E)$ is the Banach space of all continuous mappings from K to E endowed with the sup norm and we write $C(K)$ when E is the real line; if $A\subset K$ is closed $C_0(K||A, E)$ (resp. $C_0(K||A)$) is the subspace of all continuous mappings vanishing on A — it can be identified with the subspace of $C(K/A, E)$ [19, Ch. II, § 4]. Given a Banach space E, sets $A, B\subset E$ and a scalar λ, we put

$$\operatorname{dist}(A, B) = \inf\{||a-b||\colon a\in A,\ b\in B\}$$

and

$$A+\lambda\cdot B = \{a+\lambda b\colon a\in A,\ b\in B\}.$$

We denote by E' the dual space of the Banach space E, $\langle x, x'\rangle$ stand for the value of the functional $x'\in E'$ at the point $x\in E$, $\operatorname{conv} A$ is the convex hull of the set $A\subset E$ and finally, $\sigma(E', E)$ denote the weak-star topology in E'.

Recall, that a space K has the *countable tightness* if, whenever $x\in \bar{A}\subset K$, then $x\in \bar{C}$ for a countable $C\subset A$ [10] and that an *Eberlein compact* is a space homeomorphic to a compact subspace of a Banach space endowed with the weak topology; for the last notion as well as for the notion of a WCG Banach space the reader is referred to [9, Ch. V].

3. The property (C).

3.1. The following property was defined by Corson (see Introduction): *a convex subset M of a Banach space E* (³) *has the property* (C) *if every collection of closed convex subsets of M with empty intersection contains a countable subcollection with empty intersection.*

A base for the sequel is the following

LEMMA. *Let E be a Banach space. If E does not have the property* (C), *then there exists a collection $\mathscr{C}$ of nonempty convex subsets of the unit ball, closed under countable intersections* (⁴) *and an $\varepsilon>0$ such that for every convex subset M of E with the property* (C) *there is a $C_M\in\mathscr{C}$ with* $\operatorname{dist}(M, C_M)\geqslant\varepsilon$.

Proof. At first let us verify that if

(*) *for every collection $\mathscr{K}$ of nonempty convex sets in E closed under countable intersections and every $\sigma>0$ there is a point $a\in E$ with $\operatorname{dist}(a, C)\leqslant\sigma$ for $C\in\mathscr{K}$*,

(³) In fact, we are only interested in the case when M is itself a Banach space.

(⁴) It means that $\bigcap\mathcal{A}\in\mathcal{C}$, if $\mathcal{A}\subset\mathcal{C}$ is countable.

then E has the property (C). Let $\mathscr{C}$ be a collection of nonempty, closed, convex subsets of E closed under countable intersections; a standard reformulation of the property (C) (cf. [10, 3.8]) shows that it is enough to check that $\bigcap\mathscr{C}\neq\emptyset$. For this purpose, let us define inductively for every $C\in\mathscr{C}$ nonempty convex sets $C^{(i)}$ and choose points $a_i\in E$ in such a way that

(1) $C = C^{(0)}\supset C^{(1)}\supset\ldots$, $\operatorname{diam} C^{(i+1)}\leqslant 2^{-i}$,

(2) every collection $\mathscr{C}^{(i)} = \{C^{(i)}\colon C\in\mathscr{C}\}$ is closed under countable intersections,

(3) $\operatorname{dist}(a_i, C^{(i)})<2^{-i-1}$.

We choose a_0 using (*) with $\mathscr{K} = \mathscr{C}$ and $\sigma = 2^{-2}$. Assume that $\mathscr{C}^{(i)}$ and a_i are defined. For every $C\in\mathscr{C}$ put $C^{(i+1)} = C^{(i)}\cap(a_i+2^{-i-1}\cdot B)$ where B stand (also in the sequel) for the unit open ball in E. The sets $C^{(i+1)}$ are convex and nonempty, by (3), the condition (1) is satisfied obviously and, finally, the family $\mathscr{C}^{(i+1)}$ satisfies (2), as $\mathscr{C}^{(i)}$ does. We complete the inductive step choosing a point a_{i+1} by means of (*), where $\mathscr{K} = \mathscr{C}^{(i+1)}$ and $\sigma = 2^{-i-3}$.

Now, the points $a_0, a_1, \ldots$ form a Cauchy sequence, as by (3) and (1) we have

$$||a_i-a_{i+1}||\leqslant 2^{-i-1}+2^{-i}+2^{-i-2}\leqslant 3\cdot 2^{-i}\quad\text{for}\quad i\geqslant 1.$$

Let a be the limit point of this sequence. By (3) we have $\operatorname{dist}(a, C) = 0$ for $C\in\mathscr{C}$ and hence $a\in\bigcap\mathscr{C}$ because each $C\in\mathscr{C}$ is closed.

We pass to the proof of the lemma. Since E does not have the property (C), the property (*) fails for some $\mathscr{K}$ and σ. Since $\mathscr{K}$ is closed under countable intersections there exists a natural number n such that each member of $\mathscr{K}$ intersects the ball $n\cdot B$. Put

$$\mathscr{C} = \left\{\frac{1}{n}(C\cap n\cdot B)\colon C\in\mathscr{K}\right\}\quad\text{and}\quad \varepsilon = \frac{\sigma}{n}.$$

The family $\mathscr{C}$ consists of nonempty convex subsets of the unit ball, it is closed under countable intersections and for every $x\in E$ there exists $C_x\in\mathscr{C}$ with $\operatorname{dist}(x, C_x)>\varepsilon$.

We claim that $\mathscr{C}$ and ε are the family and the number we are looking for. For, let M be a convex subset of E with the property (C) and suppose that $\operatorname{dist}(M, C)<\varepsilon$ if $C\in\mathscr{C}$. Let us put $C^* = (C+\varepsilon\cdot B)\cap M$ for every $C\in\mathscr{C}$. The sets C^* are convex and nonempty and given a countable collection $\mathscr{A}\subset\mathscr{C}$ we have $(\bigcap\mathscr{A})^*\subset\bigcap\{C^*\colon C\in\mathscr{A}\}$ (recall, that $\bigcap\mathscr{A}\in\mathscr{C}$). Thus by the property (C) of M there exists $x\in\bigcap\{\overline{C^*}\colon C\in\mathscr{C}\}$. But then $\operatorname{dist}(x, C)\leqslant\varepsilon$ for every $C\in\mathscr{C}$, a contradiction which completes the proof.

3.2. The lemma yields easily the following two propositions.

PROPOSITION 1. *Let $T\colon E\to F$ be a continuous linear mapping of a Banach space E onto a Banach space F. If both F and the kernel of the operator T have the property* (C) *then so does the space E.*

Proof. Suppose the contrary and let $\mathscr{C}$ and $\varepsilon>0$ be as in the lemma. Since each $T^{-1}y$ has the property (C), there exists for every $y\in F$ a $C_y\in\mathscr{C}$ with $\operatorname{dist}(T^{-1}y, C_y)$

$\geqslant\varepsilon$. By the Banach open mapping principle we have $y \notin \overline{TC_y}$ for $y \in F$ and thus $\bigcap \{\overline{TC}: C \in \mathscr{C}\} = \emptyset$, a contradiction with the property (C) of F.

PROPOSITION 2. *Let $M_1, M_2, \ldots$ be convex sets with the property* (C) *in a Banach space E. If the union $\bigcup_i M_i$ is dense in E, then E has the property* (C).

Proof. For, in the opposite case there would exist a collection $\mathscr{C}$ and an $\varepsilon>0$ as in the lemma. But then $\operatorname{dist}(M_i, C_i) \geqslant \varepsilon$ for some $C_i \in \mathscr{C}$ and we would have $\operatorname{dist}(\bigcup_i M_i, \bigcap_i C_i) \geqslant \varepsilon$, which is impossible, as $\bigcap_i C_i \neq \emptyset$ and $\bigcup_i M_i$ is dense in E.

3.3. As was pointed out by Corson (see Introduction) the class of Banach spaces which have the property (C) includes those which are Lindelöf in the weak topology. The last class contains all Banach subspaces of WCG-spaces, by Talagrand [21], but not exclusively — for the examples see Pol [16], Talagrand [22], [23] (cf. also Alster and Pol [2] and Gulko [11] and Pol [17, Example 2]). Here we give some function spaces (all of them provide "standard examples" in topology) with the property (C), which are not Lindelöf in the weak topology, thus answering a question of Corson mentioned in Introduction.

EXAMPLE 1. Let, as in Corson [7, Example 2] C_r be the space of all bounded real-valued functions on the unit interval I which are continuous on the right and which have a finite limit on the left, endowed with the sup norm. Corson showed that C_r is not Lindelöf in the weak topology, but it admits a linear continuous operator whose range — the space $c_0(I)$, and the kernel — the space $C(I)$, are Lindelöf in the weak topology. By Proposition 1 the space C_r has the property (C).

Observe, that the space C_r may be identified in a natural way with the function space $C(K)$ on the "two arrows space" K of Aleksandrov and Urysohn [1], [10, 3.10. C].

EXAMPLE 2. Let L be the lexicographic square [1] [10, 3.12.3 (d)], i. e. $L = [0, 1] \times [0, 1]$ is endowed with the order topology induced by the lexicographic order. The subspace $K = \{(x, y) \in L: y = 0 \text{ or } y = 1\}$ is the two arrows space. Thus $C(L)$ is not Lindelöf in the weak topology. However, $C(L)$ has the property (C), by Proposition 1, as the restriction operator $T: C(L) \to C(K)$ has the range with the property (C), by Example 1, while the kernel $C_0(L\|K)$ has the property (C) by [7, Corollary 1] (since K/L is a continuous image of the one-point compactification of the free union of continuum copies of the circle).

EXAMPLE 3. If S is an uncountable compact separable space with the ω_1th derived set empty ([5]) then $C(S)$ is not Lindelöf in the weak topology [17, Theorem], but it has the property (C). The last assertion is a particular case of Corollary 4.1.1, but it can be also easily verified step by step by means of Propositions 1 and 2.

EXAMPLE 4. Let H be the Helly space, i.e. the space of all nondecreasing functions from the unit interval into itself, endowed with the pointwise topology [10, 3.2.E]; H is a compact convex set. Let $A(H)$ be the Banach space of all continuous

([5]) For such examples see [19, 8.5.10(G)] and Mrówka [15].

affine functions on H [19, 23.1]. The set K of the extremal points of H, consisting of those functions from H which take only the values 0 or 1, is in fact the two arrows space. Thus, by the maximum principle [19, 23.1.10], $A(H)$ can be considered as a subspace of $C(K)$ and so, by Example 1, it has the property (C). But, it is easy to see, that $A(H)$ is not Lindelöf in the weak topology.

We do not know whether $C(H)$ has the property (C) (cf. Added in proof).

3.4. In this section we characterize the property (C) of a Banach space in terms of the dual space. The characterization bases on the following lemma, which is a counterpart to Lemma 3.1; we shall exploit this lemma also in the next paragraph.

LEMMA. *Assume that a Banach space E does not have the property* (C). *Then there exists a subset A' of the unit ball in the dual space E' and an $\varepsilon>0$ such that* (a) *for every linear subspace M of E with the property* (C) *there is an $x' \in A'$ vanishing on M, but* (b) *for every countable $C' \subset A'$ there is a $t \in E$ with $\|t\| \leqslant 1$ and $\langle t, x'\rangle \geqslant \varepsilon$ for every $x' \in C'$.*

Proof. Let a collection $\mathscr{C}$ and an $\varepsilon>0$ be as in Lemma 3.1 and let $\mathscr{M}$ be the family of all linear subspaces of E which have the property (C). So, given an $M \in \mathscr{M}$ there exists $C_M \in \mathscr{C}$ such that $\operatorname{dist}(M, C_M) \geqslant \varepsilon$. Let B be the open unit ball in E. Since $(M+\varepsilon\cdot B) \cap C_M = \emptyset$, there exists by the separation theorem [18, Ch. II, 9.1] a linear functional $x'_M \in E'$ such that $\|x'_M\| = 1$ and

$$\sup\{\langle x, x'_M\rangle: x \in M+\varepsilon\cdot B\} \leqslant \inf\{\langle x, x'_M\rangle: x \in C_M\}.$$

Since M is a linear space and since $\|x'_M\| = 1$ we have

$$(1) \qquad x'_M|M \equiv 0 \quad \text{and} \quad \langle x, x'_M\rangle \geqslant \varepsilon \quad \text{for every } x \in C_M.$$

Let us put $A' = \{x'_M: M \in \mathscr{M}\}$; we claim that this set and the ε from (1) are the objects we are looking for. We have only to check (b). But, if $C' = \{x'_{M_1}, x'_{M_2}, \ldots\}$ is a countable subset of A', we take a point $t \in \bigcap_i C_{M_i}$ and (b) follows from the second part of (1).

THEOREM. *For a Banach space E the following conditions are equivalent*:

(i) *E has the property* (C);

(ii) *the dual space E'* (*equivalently — the unit ball B' in E'*) *endowed with $\sigma(E', E)$-topology has the property* (C′): *if $x' \in \bar{A}' \subset E'$* (*resp. $A' \subset B'$*) *then $x' \in \overline{\operatorname{conv} C'}$ for a countable $C' \subset A'$* ([6]).

Proof. The fact that the property (C′) of B' implies the property (C) of E follows immediately from the lemma, as 0 is in the $\sigma(E', E)$-closure of the set A' defined in this lemma, by (a), but not in the $\sigma(E', E)$-closure of any $\operatorname{conv} C'$ with $C' \subset A'$ countable, by (b).

The fact that (i) implies the property (C′) of E' can be proved by general duality

([6]) The equivalence of the property (C′) for E' and B' follows from Krein–Šmulian theorem [18, Ch. IV, 6.4]; however, we obtain this fact easily in the course of the proof.

arguments as follows. Let $a' \in \bar{A}' \subset E'$ (we consider $\sigma(E', E)$-topology in E'). For every $x' \in A'$ put $C_{x'} = \{x \in E: \langle x, x'\rangle \geqslant \langle x, a'\rangle + 1\}$. The sets $C_{x'}$ are closed, convex and $\bigcap \{C_{x'}: x' \in A'\} = \emptyset$, so there is a countable set $C' \subset A'$ with $\bigcap \{C_{x'}: x' \in C'\} = \emptyset$. Then $a' \in \overline{\operatorname{conv} C'} = H'$. Indeed in the opposite case there was an $x \in E$ such that $\langle x, h'\rangle \geqslant \langle x, a'\rangle + 1$ for $h' \in H'$, [18, Ch. II, 9.1 and Ch. IV, 1.2], but then $x \in \bigcap \{C_{x'}: x' \in C'\}$, a contradiction.

3.5. The following corollary to the theorem extends a result of Corson [7, Corollary 1] that function a space $C(X)$, where X is the one-point compactification of a locally compact metrizable space, has the property (C); a particular case of this corollary is a step in the proof of Theorem 4.1.

COROLLARY. *Let $\{E_s\}_{s \in S}$ be a family of Banach spaces and let F be the $c_0(S)$-product of the family, i.e. F is the space $\{x \in \prod_{s \in S} E_s: (\|x_s\|)_s \in c_0(S)\}$ endowed with the sup norm. If each E_s has the property* (C), *then so does the space F.*

Proof. The dual space F' of F is the $l_1(S)$-product of the family $\{E_s'\}_{s \in S}$ [8, Ch. II, § 2, (11)] and thus the unit ball B' in F' endowed with the $\sigma(F', F)$-topology is the subspace $B' = \{x' \in \prod_{s \in S} B_s': \sum_s \|x_s'\| \leqslant 1\}$ of the Tyhonoff product $\prod_{s \in S} B_s'$ of the unit balls $B_s' \subset E_s'$ endowed with the $\sigma(E_s', E_s)$-topology. By standard reasonings related to such subspaces of products (cf. [10, Σ-products, p. 158] one can easily verify that since each B_s' has the property (C′), then so does B' (the needed fact that each product $B_{s_1}' \times \ldots \times B_{s_k}'$ has the property (C′) follows from the identity $(E_{s_1} \times \ldots \times E_{s_k})' = E_{s_1}' \times \ldots \times E_{s_k}'$, by Proposition 3.2.1 (7)).

A straightforward proof of the corollary is sketched in Section 5.5.

4. Function spaces with the property (C).

4.1. The main result of this paragraph is the following

THEOREM. *Let $K = X_1 \times X_2 \times \ldots$ be the countable product of compact scattered spaces of countable tightness and let E be a Banach space with the property* (C). *Then the function space $C(K, E)$ has the property* (C).

The particular case, when K is itself scattered and E is the real line, seems to be worth while noticing separately; we shall do it explicitly to emphasize the connections between the topology of K and the linear-topological structure of $C(K)$.

COROLLARY 1. *For a compact scattered space K the following statements are equivalent*:

(i) *if $x \in \bar{A} \subset K$, then $x \in \bar{C}$ for a countable $C \subset A$,*

(ii) *if $\mathscr{A}$ is a collection of closed convex sets in $C(K)$ and $\bigcap \mathscr{A} = \emptyset$ then $\bigcap \mathscr{C} = \emptyset$ for a countable $\mathscr{C} \subset \mathscr{A}$.*

Here, the implication (ii) ⇒ (i) (for arbitrary K) is contained in Theorem 3.4, but it can be easily proved directly — see Section 5.1.

(7) One can also follow reasonings of Malyhin [13, Theorem 4].

Of course, the assertion of the theorem is also true for continuous images of closed subsets of such products $X_1 \times X_2 \times \ldots$ so in particular we have (cf. [5, Lemma1–1], [19, 5.3.4] and [9, Corollary 3]).

COROLLARY 2. *If K is an Eberlein compact and a Banach space E has the property* (C), *then so does $C(K, E)$; in particular, if $C(S)$ has the property* (C) *for a compact space S then so does $C(K \times S)$, as well as $C(S, F)$, provided that F is a* WCG *space.*

The rest of this paragraph is devoted to the proof of the theorem.

4.2. We shall need the following two simple facts. The first is a corollary to a Malyhin's result [15, Theorem 4].

LEMMA 1. *Let X be a compact space of countable tightness, let $p \in X$ and let $\mathscr{F}$ be a family of finite subsets of X such that each G_δ-set containing p contains a member of $\mathscr{F}$. Then there exists a countable subfamily $\mathscr{F}^*$ of $\mathscr{F}$ such that each neighbourhood of p contains a member of $\mathscr{F}^*$.*

Proof. Put $\mathscr{F}_n = \{F \in \mathscr{F}: |F| \leqslant n\}$. By the assumption there is an n such that each neighbourhood of p contains a member of $\mathscr{F}_n$. It remains to use the fact that the product X^n is of countable tightness [10, 3.12.8(e)] to choose the family $\mathscr{F}^* \subset \mathscr{F}$.

The next fact, closely related to tensor products, is certainly well-known, cf. [18, Ch. IV, 9.2], [12, Corollary]; we indicate here briefly a direct (and fairly standard) proof.

LEMMA 2. *Let K be a compact scattered space and let E be a Banach space. For every linear functional $f' \in C(K, E)'$ there exist points $a_1, a_2, \ldots$ from K and functionals $u_1', u_2', \ldots$ from E' such that*

(a) $\|f'\| = \sum_i \|u_i'\|$, $u_i' \neq 0$,

(b) $\langle f, f'\rangle = \sum_i \langle f(a_i), u_i'\rangle$ *for every* $f \in C(K, E)$.

Proof. One can assume that $E = C(S)$ [19, 6.1.9]. Identifying $C(K, C(S))$ with $C(K \times S)$ we consider f' as a Radon measure μ on $K \times S$ [19, 18.4.1]. Let λ be the image of the total variation $|\mu|$ under the projection $p: K \times S \to K$ [19, 18.3.1 and 17.2.1]. Since K is scattered, λ is purely atomic [19, 19.7.7]; let $a_1, a_2, \ldots$ be the atoms and let $\mu_i = \mu | p^{-1}(a_i)$. Each μ_i can be considered as an $u_i' \in C(S)'$, again by [19, 18.4.1]. The points a_i and the functionals u_i' satisfy (a) and (b).

4.3. At first we shall prove the theorem in the case, when K is a compact scattered space of countable tightness. The proof goes by the induction with respect to the ordinal α such that the αth derived set of K is finite and nonempty. The case $\alpha = 0$ follows from Corollary 3.5 (with S finite), so let us assume that the assertion holds for all $\beta < \alpha$ and let $K^{(\alpha)}$ be finite and nonempty.

Assume, that $\alpha = \beta + 1$. Let T: $C(K, E) \to C(K^{(\beta)}, E)$ be the restriction operator. The kernel $C_0(K \| K^{(\beta)}, E)$ of T has the property (C) by the inductive assumption, while the range $C(K^{(\beta)}, E)$ of T has the property (C) by Corollary 3.5. It remains to apply Proposition 3.2.1.

Assume that α is a limit ordinal. It is easy to see that this case reduces to the proof of the

ADDITION LEMMA. *Let p be a point in our space K such that for every compact space $T\subset K$ disjoint from p the function space $C(T, E)$ has the property* (C). *Then $C(K, E)$ has the property* (C).

It is enough to verify that the space $C_0(K, E)$ of all functions from $C(K, E)$ vanishing at p has the property (C) (see Proposition 3.2.1). Assume the contrary and choose a set A' from the unit ball in $C_0(K, E)'$ and an $\varepsilon>0$ as in Lemma 3.4.

Let $\mathscr{S}$ be the family of all closed G_δ-sets in K containing the point p. Fix an $S\in\mathscr{S}$ and let G_i be open sets such that $S=\bigcap G_i$ and $\overline{G}_{i+1}\subset G_i$. By the assumption each $M_i=C(K||\overline{G}_i, E)$ has the property (C) and so does the space $C(K||S, E)$, by Proposition 3.2.2, as the union $\bigcup\limits_i M_i$ is dense in this space. It follows (see Lemma 3.4) that there exists a $f'_S\in A'$ vanishing on $C(K||S, E)$. It means exactly that, representing f'_S as in Lemma 4.2.2 by the points $a_{1,S}, a_{2,S}, \ldots$ and the functionals $u'_{1,S}, u'_{2,S}, \ldots$, we have $a_{i,S}\in S$ for every i. By the property (a) from this lemma, there exists a k_S such that

$$\sum_{i>k_S}\|u'_{i,S}\|<\tfrac{1}{2}\varepsilon, \tag{1}$$

let $F_S=\{a_{i,S}\colon i\leqslant k_S\}\subset S$. We can exploit Lemma 4.2.1 to choose a countable family $\mathscr{C}\subset\mathscr{S}$ such that every neighbourhood of p contains some F_S with $S\in\mathscr{C}$. The set A' was chosen (see Lemma 3.4) in such a way that we can pick now a $t\in C_0(K, E)$ with

$$\|t\|\leqslant 1 \quad\text{and}\quad \langle t, f'_S\rangle\geqslant\varepsilon \quad\text{for every } S\in\mathscr{C}. \tag{2}$$

Put $V=\{x\in K\colon \|t(x)\|<\frac{1}{2}\varepsilon\}$; then V is a neighbourhood of p and hence there exists $S_0\in\mathscr{C}$ with $F_{S_0}\subset V$. So we have by (1), the first part of (2) and Lemma 4.2.2(b):

$$\langle t, f'_{S_0}\rangle=\sum_i\langle t(a_{i,S_0}), u'_{i,S_0}\rangle=\sum_{i\leqslant k_{S_0}}\langle t(a_{i,S_0}), u'_{i,S_0}\rangle+\sum_{i>k_{S_0}}\langle t(a_{i,S_0}), u'_{i,S_0}\rangle$$
$$\leqslant\tfrac{1}{2}\varepsilon\sum_{i\leqslant k_{S_0}}\|u'_{i,S_0}\|+\sum_{i>k_{S_0}}\|u'_{i,S_0}\|<\varepsilon,$$

a contradiction with the second part of (2).

4.4. Now, let $K=X_1\times X_2\times\ldots$ be as in the theorem and let $p_i\colon K\to K_i=X_1\times\ldots\times X_i$ be the projection. Each $C(K_i, E)$ has the property (C), by the case just proved in 4.3 (cf. [11, 3.12.8(e)]) and so does the isomorphic space

$$M_i=\{f\circ p_i\colon f\in C(K_i, E)\}.$$

Finally, the union $\bigcup\limits_i M_i$ is dense in $C(K, E)$ [6, Ch. X, § 4,4] and an application of Proposition 3.2.2 completes the proof of the theorem.

5. Comments.

5.1. Let K be a compact space such that the function space $C(K)$ has the property (C). Then

(t_3) *the space $P(K)$ of probability Radon measures on K* (8) *has the property* (C′) *from Theorem* 3.4;

(t_2) *if A is a closed subset of K and $\mathscr{P}$ is a collection of probability Radon measures on K such that for every neighbourhood V of A there is a measure from $\mathscr{P}$ concentrated on V, then there exists a countable collection $\mathscr{P}^*\subset\mathscr{P}$ such that given a neighbourhood V of A and $\varepsilon>0$ there is a $\mu\in\mathscr{P}^*$ with $\mu(K\setminus V)<\varepsilon$;*

(t_1) *if $p\in K$ and $\mathscr{F}$ is a family of finite sets in K such that each neighbourhood of p contains some member of $\mathscr{F}$, then there is a countable subfamily $\mathscr{F}^*\subset\mathscr{F}$ such that given a neighbourhood V of p and an $\varepsilon>0$ there is an $F\in\mathscr{F}^*$ with $|F\cap V|/|F|>1-\varepsilon$;*

(t_0) *K has the countable tightness.*

The condition (t_3) is in fact equivalent to the property (C), by Theorem 3.4; indeed, in notation of the theorem, $P(K)\subset B'$, while B' is the image of $(P(K)\times[0,1])^2$ under the mapping $(\mu, s; \nu, t)=s\cdot\mu-t\cdot\nu$ and such mappings preserve the property (C′) (cf. [19, 17.2.5] and [10, 3.12.8]). A direct proof that (C) $\Rightarrow$ (t_2) runs as follows: put

$$C_\mu=\{f\in C(K||A)\colon \textstyle\int f\,d\mu\geqslant 1/i\}\quad\text{for } \mu\in\mathscr{P};$$

then $\bigcap\limits_\mu C_\mu=\emptyset$ and there exists a countable set $\mathscr{P}_i\subset\mathscr{P}$ with $\bigcap\{C_\mu\colon \mu\in\mathscr{P}_i\}=\emptyset$, define $\mathscr{P}^*=\bigcup\limits_i\mathscr{P}_i$. To show that ($t_2$) $\Rightarrow$ (t_1) it is enough to consider for each $F\in\mathscr{F}$ the measure $\mu_F=\dfrac{1}{|F|}\sum\limits_{x\in F}\delta_x$, where δ_x is the point mass at x. Finally, we have the obvious implication (t_1) $\Rightarrow$ (t_0).

It seems interesting to clarify what are the exact relationships between the conditions (t_i) (for scattered K they are simply equivalent, by Corollary 4.1.1); we conjecture that (t_2) is an essentially stronger property of a compact space than the countable tightness (t_0).

Another natural question is, whether the property (C′) from Theorem 3.4 (or, equivalently, (t_3)) is exactly the countable tightness of B' (or $P(K)$).

5.2. Let us consider the following property of a compact space K following from the property (C) of the function space $C(K)$ (cf. Talagrand [23, Theoreme 6.6]): (M) *every Radon measure on K has the separable support.* That (C) implies (M) can be justified as follows: let μ be a Radon measure supported exactly on K, fix a natural i and for $x\in K$ put

$$C_x=\{f\in C(K)\colon \textstyle\int f\,d\mu\geqslant 1/i \quad\text{and}\quad f(x)=0\},$$

then $\bigcap\limits_x C_x=\emptyset$ and so $\bigcap\{C_x\colon x\in A_i\}=\emptyset$ for a countable $A_i\subset K$; now $K=\overline{\bigcup\limits_i A_i}$.

This observation, a result of Šapirovskiĭ [20] and a result obtained independently by Alster and Pol [2] and Gulko [11] yield the equivalence of the following state-

(8) We consider $P(K)$ with the weak-star topology.

ments (cf. [2, 8.3 and footnote (7)]): (a) *every compact space K of countable tightness has the property* (M)(b) *for every compact subspace K of the Σ-product of real lines* [10, 2.7.13] *the function space $C(K)$ has the property* (C).

By a result of Arhangel'skiĭ [3] these statements are consistent with the usual axioms for set theory, but we do not know, whether they can be proved without extra axioms (cf. the note below).

5.3. The following question is related to Corollary 4.1.2 (cf. also Sec. 5.1): if $C(S)$ has the property (C), is it true that $C(S\times S)$ does have this property?

We formulate below a remark, which may be useful in some concrete situations.

Assume that S is a compact space and E is a Banach space admitting a linear continuous operator $T\colon E\to F$ onto a Banach space F such that both $C(S, \ker T)$ and $C(S, F)$ have the property (C). *Then $C(S, E)$ has the property* (C).

For, let $U\colon C(S, E)\to C(S, F)$ be the linear continuous operator defined by $Uf = T\circ f$. Then $\ker U = C(S, \ker T)$ and U is onto, by a theorem of Bartle and Graves [4, Theorem 4] (cf. Michael [14]), so the conclusion follows from Proposition 3.2.1.

As an application we shall show that the space $C(K^N)$ has the property (C), where K is the two arrows space. As in Section 4.4, it is enough to verify this for $C(K^n)$, $n = 1, 2, \ldots$ Assume that $C(K^m)$ has the property (C). Then $C(K^{m+1}) = C(K^m, C(K))$ and the property (C) of $C(K^{m+1})$ follows from Example 3.3.1, the last part of Corollary 4.1.2 and the above remark.

5.4. Note, that *if a Banach space E has the property* (C) *then the weak topology of E is realcompact.* This follows immediately from a characterization given by Corson [7, Lemma 9] and Theorem 3.4.

5.5. We give here briefly a direct proof of Corollary 3.5 which does not appeal to functional-analytic methods. Let us adopt the notation of this corollary. Given a set $T\subset S$ and an $x\in F$ we write $||x||_T = \sup\{||x_s||\colon s\in T\}$.

LEMMA. *Let C be a convex set in the unit ball of F such that* $\operatorname{dist}(0, C)\geqslant\varepsilon$. *Then there exists a finite set $T\subset S$ such that $||x||_T\geqslant\frac{1}{2}\varepsilon$ for all $x\in C$.*

For, if is was not the case we could choose points $x^1, x^2, \ldots, x^n\in C$, where $1/n<\varepsilon/2$, such that the finite sets $\{s\in S\colon ||x_s^i||\geqslant\frac{1}{2}\varepsilon\}$ were disjoint. But then $x = (x^1+\ldots+x^n)/n\in C$ and $||x||<\varepsilon$, a contradiction.

Let us assume that F does not have the property (C) and let a collection $\mathscr{C}$ and an $\varepsilon>0$ be as in Lemma 3.1. Taking into account, that each finite product $E_{s_1}\times\ldots\times E_{s_k}$ has the property (C), by Proposition 3.2.1, and exploiting successively the above lemma, we can find a sequence of disjoint finite subsets $T_1, T_2, \ldots$ of S and members $C_1, C_2, \ldots$ of $\mathscr{C}$ such that $||x||_{T_i}\geqslant\frac{1}{2}\varepsilon$, whenever $x\in C_i$. But then for $x\in\bigcap_i C_i$ we have $||x||_{T_i}\geqslant\frac{1}{2}\varepsilon$ for every i, which is impossible.

Note. After this paper was completed, Professor A. Pełczyński has called to the author's attention a recent paper of R. Haydon "On dual L^1-spaces and injective bidual Banach spaces" (preprint), where the following result was proved (Theorem 3.1):

assuming the continuum hypothesis, there exists a compact space S which is first-countable and non-separable, but it is the carrier of a Radon measure.

Since S was constructed for other purpose, the properties are not explicitely stated in Haydon's paper, however they follow easily from the facts proved there.

Now, let us observe that:

(a) $C(S)$ does not have the property (C), by 5.2, and so, *under the continuum hypothesis, the condition* (t_0) *in* 5.1 *does not imply the condition* (t_3) (cf. also Introduction);

(b) since S does not have the property (M) formulated in 5.2, being first-countable, it follows that *each of the equivalent statements* (a) *and* (b) *in* 5.2 *is independent of the* ZFC-*axioms for set theory.*

Added in proof.

a) The paper of Haydon quoted in the Note appeared in Israel J. Math. 31 (1978), pp. 142–152.

b) G. Godefroy proved in Pacific J. Math. (to appear) that if S is a compact set in the space $B_1(P)$ of the first Baire class functions on the irrationals with the pointwise topology then $C(S)$ has property (C), and morover that $P(S)$ embeds in $B_1(P)$.

c) Since the Helly space H (Example 4) embeds in $B_1(P)$ the result of Godefroy mentioned in b) shows that $C(H)$, and even $C(P(H))$, have property (C). Another proof of these facts was given in the author's paper mentioned below.

d) In "Note on the spaces $P(S)$ of regular probability measures whose topology is determined by countable subsets" (preprint) we show that if $C(S^N)$ has property (C) then also $C(P(S))$ has property (C), and in particular, $P(S)$ has countable tightness.

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On families of σ-complete ideals

by

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Abstract. Our main results are the following: Assume Martin's Axiom. Then

1. For every $\lambda<2^\omega$ and every family $\{\mu_\alpha\colon \alpha<\lambda\}$ of two-valued uniform measures on 2^ω there exists an $X\subset 2^\omega$ non-measurable with respect to any of them.

2. For every cardinal $\varkappa$ such that $2^\omega<\varkappa<$1st cardinal carrying a 2^ω-complete 2^ω-saturated ideal the following holds: if $\lambda<2^\omega$ and $\{\mu_\alpha\colon \alpha<\lambda\}$ is a family of 2^ω-additive two-valued measures on $\varkappa$, then there exists an $X\subset\varkappa$ non-measurable with respect to any of them.

0. Terminology and preliminaries. We shall use standard set-theoretical notation and terminology. Letters $\varkappa$, λ, μ will always denote uncountable cardinals. "I is an ideal on X" will mean "I is a σ-complete proper ideal of subsets of X such that $\{x\}\in I$ for all $x\in X$". An ideal I is *λ-complete* iff $\{x_\xi\colon \xi<\eta\}\subset I$ implies $\bigcup\{X_\xi\colon \xi<\eta\}\in I$ for $\eta<\lambda$. A cardinal λ is called the *character of an ideal I on $\varkappa$* ($\mathrm{ch}\, I=\lambda$) iff λ is the least cardinal such that $\exists X\subset\varkappa$, $|X|=\lambda$, $X\notin I$. An ideal I on $\varkappa$ is *uniform* iff $\mathrm{ch}\, I=\varkappa$. If I is an ideal on $\varkappa$, then I^* will denote the dual filter.

Ideals I_1 and I_2 on $\varkappa$ are called *compatible* iff there exists an ideal I_3 on $\varkappa$ such that $I_1\cup I_2\subset I_3$. It is easy to see that I_1, I_2 are compatible iff $I_1\cap I_2^*=\emptyset$ iff $I_2\cap I_1^*=\emptyset$. Otherwise I_1, I_2 are incompatible.

MA will denote Martin's Axiom. We shall use the following consequence of MA (see [4]):

1. The union of $<2^\omega$ closed nowhere dense subsets of a metric complete separable space is nowhere dense.

A subset $\mathscr{L}$ of the reals is called *strongly Lusin* if for every Lebesgue measurable set A $|\mathscr{L}\cap A|<2^\omega$ iff A has Lebesgue measure 0. It is also a consequence of MA (see [2], cf. also [1], [4], [6]) that

2. A strongly Lusin set exists.

We use the following notation:

$U(\varkappa,\lambda,\mu)$ — For every family $\{I_\alpha\colon \alpha<\lambda\}$ of μ-complete ideals on $\varkappa$ we have $\bigcup_{\alpha<\lambda}(I_\alpha\cup I_\alpha^*)\neq P(\varkappa)$.

$U^*(\varkappa,\lambda,\mu)$ — For every family $\{I_\alpha\colon \alpha<\lambda\}$ of μ-complete uniform ideals on $\varkappa$ we have $\bigcup_{\alpha<\lambda}(I_\alpha\cup I_\alpha^*)\neq P(\varkappa)$.