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Dimension of non-normal spaces

by

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Abstract. Let X be a general topological space and $\dim X$ the covering dimension of X due to Katetov defined by means of finite cozero covers. If V is a cozero set of X, then $\dim V \leqslant \dim X$. If $\{V_i\}$ is a countable cozero cover of X, then $\dim X = \sup \dim V_i$. Several applications of the subset and sum theorems thus stated are also given.

0. Introduction. Let X be a topological space. Then $\dim X \leqslant n$ if each finite cozero cover of X is refined by a finite cozero cover of order $\leqslant n+1$. This definition of covering dimension for general topological spaces stems from Katětov [4] and coincides with the usual definition of covering dimension for normal spaces. There has been a great amount of studies for the dimension of normal spaces in many aspects. On the contrary we have only a few for non-normal case. Especially, concerning subset and sum theorems we have had nothing with the exception of those due to Katětov [4]. Sections 1 and 2 below constitute the body of the paper where subset and sum theorems for non-normal spaces will respectively be given. In Sections 3 and 4 we give product and inverse limiting theorems for non-normal or normal spaces which will refine known results. In this paper all spaces are non-empty topological spaces and maps are continuous.

1. Subset theorem.

1.1. THEOREM. *Let V be a cozero set of a space X. Then* $\dim V \leqslant \dim X$.

Proof. When $\dim X$ is infinite the inequality is clear. Consider the case when $\dim X = n$. Let $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ be an arbitrary finite cozero cover of V. It is to be noted that each U_α is cozero in X since V is cozero in X. Let f be an element of $C(X, I)$ with $V = \{x \in X\colon f(x) > 0\}$. Set

$$V_i = \{x \in X\colon f(x) > 1/i\}, \quad F_i = \{x \in X\colon f(x) \geqslant 1/i\}.$$

Then $V = \bigcup_{i=1}^{\infty} V_i$ and $V_i \subset F_i \subset V_{i+1}$ for each i. Set

$$\mathcal{W}_1 = \{W_{1\alpha} = U_\alpha \cup (X - F_2)\colon \alpha \in A\}.$$

Then $\mathcal{W}_1$ is a finite cozero cover of X. Let $\mathcal{U}_1 = \{U_{1\alpha}\colon \alpha \in A\}$ be a cozero cover of X such that $U_{1\alpha} \subset W_{1\alpha}$ for each $\alpha \in A$ and order $\mathcal{U}_1 \leqslant n+1$. Set

$$\mathcal{W}_2 = \{W_{2\alpha} = (U_{1\alpha} \cap V_2) \cup (U_\alpha - F_1) \cup (X - F_3)\colon \alpha \in A\}.$$

Then $\mathscr{W}_2$ is a finite cozero cover of X. Let $\mathscr{U}_2 = \{U_{2\alpha}\colon \alpha\in A\}$ be a cozero cover of X such that $U_{2\alpha}\subset W_{2\alpha}$ for each $\alpha\in A$ and order $\mathscr{U}_2\leqslant n+1$. Continuing in this fashion, we get sequences $\mathscr{W}_i = \{W_{i\alpha}\colon \alpha\in A\}$ and $\mathscr{U}_i = \{U_{i\alpha}\colon \alpha\in A\}$ of cozero covers of X satisfying the following three conditions:

(1) $U_{i\alpha}\subset W_{i\alpha}$, $\alpha\in A$.

(2) $W_{i\alpha} = (U_{i-1,\alpha}\cap V_i)\cup(U_\alpha - F_{i-1})\cup(X-F_{i+1})$, $\alpha\in A$.

(3) order $\mathscr{U}_i\leqslant n+1$.

Set

$$\mathscr{D} = \{D_\alpha = \bigcup_{i=1}^{\infty}(U_{i\alpha}\cap V_i)\colon \alpha\in A\}.$$

Then each D_α is cozero in X. Let x be an arbitrary point of V. Pick k with $x\in V_k$. Since $\mathscr{U}_k$ covers X, there exists $\beta\in A$ with $x\in U_{k\beta}$. Then $x\in U_{k\beta}\cap V_k\subset D_\beta$. Thus $\mathscr{D}$ covers V.

Let us prove that $U_{i\alpha}\cap V_{i+1}\subset U_\alpha$ by induction on i. Since

$$U_{1\alpha}\cap V_2\subset W_{1\alpha}\cap V_2 = (U_\alpha\cup(X-F_2))\cap V_2 = U_\alpha\cap V_2\subset U_\alpha,$$

the assertion is true for $i=1$. Assume that the assertion is true for $i\leqslant m$. Then by (1) and (2),

$$\begin{aligned}U_{m+1,\alpha}\cap V_{m+2}&\subset W_{m+1,\alpha}\cap V_{m+2}\\ &= ((U_{m\alpha}\cap V_{m+1})\cup(U_\alpha-F_m)\cup(X-F_{m+2}))\cap V_{m+2}\\ &= ((U_{m\alpha}\cap V_{m+1})\cap V_{m+2})\cup((U_\alpha-F_m)\cap V_{m+2})\\ &\subset(U_\alpha\cap V_{m+2})\cup U_\alpha\subset U_\alpha.\end{aligned}$$

Thus the induction is completed. Hence

$$D_\alpha = \bigcup_{i=1}^{\infty}(U_{i\alpha}\cap V_i)\subset\bigcup_{i=1}^{\infty}(U_{i\alpha}\cap V_{i+1})\subset U_\alpha.$$

To prove order $\mathscr{D}\leqslant n+1$ let x be an arbitrary point of V and j the minimum with $x\in V_j$. Let $j<i$. Then

$$\begin{aligned}U_{i\alpha}\cap V_j&\subset W_{i\alpha}\cap V_j\\ &= ((U_{i-1,\alpha}\cap V_i)\cup(U_\alpha-F_{i-1})\cup(X-F_{i+1}))\cap V_j\\ &= U_{i-1,\alpha}\cap V_j.\end{aligned}$$

Hence $U_{i\alpha}\cap V_j\subset U_{j\alpha}\cap V_j$, which implies that the order of $\mathscr{D}$ at x is the order of $\mathscr{U}_j$ at x. Thus the order of $\mathscr{D}$ at x is at most $n+1$. That completes the proof.

1.2. Definition. Let X be a space and S a subset of X. S is said to be *cozero-embedded* in X if for each cozero subset U of S there exists a cozero set V of X with $V\cap S = U$.

1.3. Theorem. *Let X be a space and S a cozero-embedded subset of X. Then* $\dim S\leqslant\dim X$.

Proof. Let $\mathscr{U} = \{U_\alpha\colon \alpha\in A\}$ be a finite cozero cover of S. Let V_α, $\alpha\in A$, be cozero sets of X with $U_\alpha = V_\alpha\cap S$. Set $V = \bigcup V_\alpha$. Then by Theorem 1.1 there exist cozero sets W_α of X such that $W_\alpha\subset V_\alpha$ for each $\alpha\in A$, $V = \bigcup W_\alpha$, and the order of $\{W_\alpha\colon \alpha\in A\}$ is at most $\dim X+1$. Then $\mathscr{W} = \{W_\alpha\cap S\colon \alpha\in A\}$ is a cozero cover of S such that $W_\alpha\cap S\subset U_\alpha$ for each $\alpha\in A$ and order $\mathscr{W}\leqslant\dim X+1$. That completes the proof.

This is a generalization of Katětov's subset theorem [4] where S is assumed to be C^*-embedded in X.

1.4. Remark. If S is C^*-embedded in X, then S is clearly cozero-embedded in X. However the converse is not true in general. Let Q be the rationals in the reals R. Then Q is clearly cozero-embedded in R. Choose $f\in C(Q, I)$ such that $f(x) = 1$ for $x<\sqrt{2}$ and $f(x) = 0$ for $x>\sqrt{2}$. This f cannot be extended over R.

It can easily be seen that a subset S of a space X is cozero-embedded in X if and only if for each finite cozero cover $\mathscr{U}$ of S there exists a finite cozero collection $\mathscr{V}$ of X such that $\mathscr{V}|S<\mathscr{U}$ and $S\subset\mathscr{V}^\#(=\bigcup\{V\colon V\in\mathscr{V}\})$.

If each subset of a space X is cozero-embedded in X, then X is hereditarily normal. However the converse of this assertion is not true as follows. Let $X = [0,\omega_1)$ be the space of countable ordinals. Then X is hereditarily normal. Let G be the subset of isolated ordinals in X. Let U be the subset of elements of G whose predecessors are limit ordinals. Then U is a cozero set of G. If V is a cozero set of X with $V\supset U$, then $V\supset[\alpha,\omega_1)$ for some $\alpha<\omega_1$ and hence $\alpha+2\in V\cap G$. Thus $V\cap G\neq U$, which proves that G is not cozero-embedded in X.

2. Sum theorem.

2.1. Definition. Let U, V be subsets of a space X. U is said to be an *exact subset* of V if for some zero set F of X, $U\subset F\subset V$. A sequence U_i, $i\in N$, of subsets of X is said to be *exactly decreasing* if U_{i+1} is an exact subset of U_i for each i.

2.2. Lemma. *Let X be a space, $\mathscr{U} = \{U_\alpha\colon \alpha\in A\}$ a finite cozero cover of X, V a cozero set of X with $\dim V\leqslant n$, W a cozero set of X having a cozero cover $\{W_\alpha\colon \alpha\in A\}$ of order $\leqslant n+1$ with $W_\alpha\subset U_\alpha$, $\alpha\in A$, and W' an exact cozero subset of W. Then there exists a cozero cover $\mathscr{V} = \{V_\alpha\colon \alpha\in A\}$ of $V\cup W'$ such that $V_\alpha\subset U_\alpha$ and $V_\alpha\cap W' = W_\alpha\cap W'$ for each $\alpha\in A$, and order $\mathscr{V}\leqslant n+1$.*

Proof. Let F be a zero set of X with $W'\subset F\subset W$. Set

$$U'_\alpha = ((U_\alpha - F)\cup W_\alpha)\cap V,\quad \alpha\in A.$$

Then $\{U'_\alpha\colon \alpha\in A\}$ is a cozero cover of V. Let $\{V'_\alpha\colon \alpha\in A\}$ be a cozero cover of V of order $\leqslant n+1$ such that $V'_\alpha\subset U'_\alpha$ for each $\alpha\in A$. Set $V_\alpha = V'_\alpha\cup(W_\alpha\cap W')$. Then $\mathscr{V} = \{V_\alpha\colon \alpha\in A\}$ satisfies the desired condition. That completes the proof.

Let us say in the sequel that a collection $\mathscr{V} = \{V_\alpha\colon \alpha\in A\}$ is *special* (with respect to $\mathscr{U} = \{U_\alpha\colon \alpha\in A\}$) if $V_\alpha\subset U_\alpha$ for each $\alpha\in A$, order $\mathscr{V}\leqslant n+1$, and each V_α is cozero in X. $\mathscr{V}$ as in Lemma 2.2 is said an *extension* of $\{W_\alpha\colon \alpha\in A\}|W'$.

2.3. Lemma. *Let X be a space, $\mathscr{U} = \{U_\alpha\colon \alpha\in A\}$ a finite cozero cover of X, V_i,*

$i \in N$, a sequence of cozero sets of X with $\dim V_i \leqslant n$ for each i, W a cozero set of X, and $\mathscr{W} = \{W_\alpha\colon \alpha \in A\}$ a special (with respect to $\mathscr{U}$) cover of W. Let W' be an exact cozero subset of W, and V_i' an exact cozero subset of V_i for each i. Then there exists a special cover $\mathscr{D} = \{D_\alpha\colon \alpha \in A\}$ of $W' \cup (\bigcup_{i=1}^{\infty} V_i')$ which is an extension of $\mathscr{W}|W'$.

Proof. Let V_{ij}, $j \in N$, be an exactly decreasing sequence of cozero sets of X such that $V_{i1} = V_i$ and $V_i' \subset \bigcap_{j=1}^{\infty} V_{ij}$. Let W_j, $j \in N$, be an exactly decreasing sequence of cozero sets of X such that $W_1 = W$ and $W' \subset \bigcap_{j=1}^{\infty} W_j$. By Lemma 2.2 there exists a special cover $\mathscr{W}_1 = \{W_{1\alpha}\colon \alpha \in A\}$ of $W_2 \cup V_{11}$ which is an extension of $\mathscr{W}|W_2$. By repeated application of Lemma 2.2 there exists for each i a special cover $\mathscr{W}_i = \{W_{i\alpha}\colon \alpha \in A\}$ of $W_{i+1} \cup (\bigcup \{V_{jk}\colon j+k = i+1\})$ which is an extension of the restriction of $\mathscr{W}_{i-1}$ to $W_i \cup (\bigcup \{V_{jk}\colon j+k = i\})$. Set

$$\mathscr{D} = \{D_\alpha = (W_{1\alpha} \cap W') \cup (\bigcup_{i=1}^{\infty} (W_{i\alpha} \cap V_i'))\colon \alpha \in A\}.$$

Then $\mathscr{D}$ is the desired. That completes the proof.

Let us say that $\mathscr{D}$ thus constructed is a special cover generated by the system:

$$\{\{\mathscr{W}, (W, W')\}, (V_1, V_1'), (V_2, V_2'), \ldots\}.$$

2.4. Theorem. *Let X be a space and V_i, $i \in N$, a sequence of cozero sets of X with $X = \bigcup V_i$ and $\dim V_i \leqslant n$ for each i. Then $\dim X \leqslant n$.*

Proof. Since each countable cozero cover can be refined by a locally finite countable cozero cover, we assume without loss of generality that $\{V_i\}$ itself is locally finite by virtue of Theorem 1.1. Let $\mathscr{U} = \{U_\alpha\colon \alpha \in A\}$ be an arbitrary finite cozero cover of X. Let f_i be an element of $C(X, I)$ with $V_i = \{x \in X\colon f_i(x) > 0\}$. Set

$$V_{ij} = \{x \in X\colon f_i(x) > 1/j\}, \quad j \in N,$$

$$D_j = \bigcup_{i=1}^{\infty} V_{ij}, \quad j \in N.$$

Let $\mathscr{W}_1 = \{W_{1\alpha}\colon \alpha \in A\}$ be a special cover (with respect to $\mathscr{U}$) of V_{13}. First consider the system:

$$\{\{\mathscr{W}_1, (V_{13}, V_{12})\}, (V_{i3}, V_{i2})\colon i = 2, 3, \ldots\}.$$

By Lemma 2.3 the system generates a special cover $\mathscr{D}_1 = \{D_{1\alpha}\colon \alpha \in A\}$ of D_2. Next consider the system:

$$\{\{\mathscr{D}_1, (D_2, D_1)\}, (V_{i4}, V_{i3})\colon i = 1, 2, \ldots\}.$$

Again by Lemma 2.3 the system generates a special cover $\mathscr{D}_2 = \{D_{2\alpha}\colon \alpha \in A\}$ of D_3 which is an extension of $\mathscr{D}_1|D_1$.

Continuing in this fashion, we get for each i with $i \geqslant 2$ a special cover $\mathscr{D}_i = \{D_{i\alpha}\colon \alpha \in A\}$ of D_{i+1} which is an extension of $\mathscr{D}_{i-1}|D_{i-1}$ for $i = 2, 3, \ldots$ Set

$$\mathscr{E} = \{E_\alpha = \bigcup_{i=1}^{\infty} (D_{i\alpha} \cap D_i)\colon \alpha \in A\}.$$

Then as can easily be seen $\mathscr{E}$ is a cozero cover of X of order $\leqslant n+1$ which refines $\mathscr{U}$. That completes the proof.

It is to be noted that Pol [13], Example 1, constructed a Tychonoff space X with $\dim X > 0$ which is the sum of two zero sets F and H with $\dim F = \dim H = 0$.

2.5. Theorem. *Let X be a space which admits a locally finite cozero cover $\mathscr{U} = \{U_\alpha\}$ with $\dim U_\alpha \leqslant n$ for each α. Then $\dim X \leqslant n$.*

Proof. Since $\mathscr{U}$ is normal, $\mathscr{U}$ can be refined by a σ-discrete cozero cover $\bigcup \mathscr{V}_i$, with each $\mathscr{V}_i$ discrete. Set $\mathscr{V}_i = \{V_{i\lambda}\colon \lambda \in \Lambda_i\}$. Then for each $\lambda \in \Lambda_i$, $\dim V_{i\lambda} \leqslant n$ by Theorem 1.1. Therefore $\dim \mathscr{V}_i^{\#} \leqslant n$ by the discreteness of $\mathscr{V}_i$. Since each $\mathscr{V}_i^{\#}$ is cozero and $X = \bigcup \mathscr{V}_i^{\#}$, $\dim X \leqslant n$ by Theorem 2.4. That completes the proof.

2.6. Corollary. *Let X be a space with $\dim X \leqslant n$ and U a D-open set of X, i.e U admits a locally finite (in U) cover $\mathscr{V}$ each element of which is cozero in X. Then $\dim U \leqslant n$.*

This is a direct consequence of Theorems 1.1 and 2.5. The concept of D-open sets stems from Dowker [1] and recently considered by Nishiura [9].

3. Product theorem. According to Ishiwata [3], a map $f\colon X \to Y$ is said to be a Z-map if the image of each zero set of X under f is closed in Y. This notion is a generalization of closed maps. When a space X is a Tychonoff space, βX denotes the Stone–Čech compactification of X.

3.1. Lemma. *Let X be a Tychonoff space, Y a space, and $\pi\colon X \times Y \to Y$ the projection. If π is a Z-map, then $X \times Y$ is C^*-embedded in $\beta X \times Y$.*

Proof. Let f be an element of $C(X \times Y, I)$. Set for y, $y' \in Y$,

$$d(y, y') = \sup\{|f(x, y) - f(x, y')|\colon x \in X\}.$$

To see d is a pseudo-metric on Y, choose an arbitrary point y_0 of Y and an arbitrary positive number a. Set

$$g(x, y) = |f(x, y) - f(x, y_0)|, \quad (x, y) \in X \times Y,$$

$$F = \{(x, y) \in X \times Y\colon g(x, y) \geqslant a\}.$$

Then $g \in C(X \times Y, I)$ and hence F is a zero set of $X \times Y$. Since $g(x, y_0) = 0$, $(X \times \{y_0\}) \cap F = \emptyset$ and hence $\{y_0\} \cap \pi(F) = \emptyset$. Since $\pi(F)$ is closed, $Y - \pi(F)$ is an open neighborhood of y_0. Since $(X \times (Y - \pi(F))) \cap F = \emptyset$, $g(x, y) = |f(x, y) - f(x, y_0)| < a$ for each $x \in X$ and each $y \in Y - \pi(F)$ and hence $d(y, y_0) \leqslant a$ whenever $y \in Y - \pi(F)$. Thus d is a pseudo-metric on Y.

For each $y \in Y$ let $h_y: \beta X \times \{y\} \to I$ be an extension of $f|X \times \{y\}$. Define a transformation $h: \beta X \times Y \to I$ by:

$$h|\beta X \times \{y\} = h_y, \quad y \in Y.$$

Then $h|X \times Y = f$. To prove the continuity of h let (p, q) be an arbitrary point of $\beta X \times Y$ and ε an arbitrary positive number. By the continuity of $h|\beta X \times \{q\}$ there exists an open neighborhood V of p in βX such that $|h(p, q) - h(x, q)| < \frac{1}{3}\varepsilon$ for each $x \in V$. Set $W = \{y \in Y: d(q, y) < \frac{1}{3}\varepsilon\}$. Then W is an open neighborhood of q. Let (x, y) be an arbitrary point of $V \times W$. By the continuity of $h|\beta X \times \{y\}$ there exists a point $x' \in V \cap X$ such that $|h(x, y) - h(x', y)| < \frac{1}{3}\varepsilon$. Then

$$|h(x, y) - h(p, q)| \leqslant |h(x, y) - h(x', y)| + |h(x', y) - h(x', q)| + |h(x', q) - h(p, q)| < \tfrac{1}{3}\varepsilon + d(y, q) + \tfrac{1}{3}\varepsilon = \varepsilon.$$

Thus h is continuous and the proof is completed.

3.2. THEOREM. *Let X be a Tychonoff space, Y a space, and $\pi: X \times Y \to Y$ the projection. If π is a Z-map,* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Proof. Since $X \times Y$ is C^*-embedded in $\beta X \times Y$ by Lemma 3.1, $\dim(X \times Y) \leqslant \dim(\beta X \times Y)$ by Theorem 1.3. Since $\dim(\beta X \times Y) \leqslant \dim \beta X + \dim Y$ by Morita [7], Theorem 5.5, $\dim(X \times Y) \leqslant \dim \beta X + \dim Y = \dim X + \dim Y$. That completes the proof.

This generalizes Filippov [2], Theorem 1.

3.3. COROLLARY. *Let X be a paracompact Hausdorff space, Y a space, and $\pi: X \times Y \to Y$ the projection. If each point $x \in X$ has a closed neighborhood U_x such that $\pi|U_x \times Y$ is a Z-map, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Proof. Let V_x be the interior of U_x. Let $\{W_\alpha \neq \emptyset: \alpha \in A\}$ be a locally finite cozero cover of X refining $\{V_x: x \in X\}$. For each W_α choose $V_{x(\alpha)}$ with $W_\alpha \subset V_{x(\alpha)}$. By Theorem 3.2, $\dim(U_{x(\alpha)} \times Y) \leqslant \dim U_{x(\alpha)} + \dim Y$. Since X is normal and $U_{x(\alpha)}$ is closed, $\dim U_{x(\alpha)} \leqslant \dim X$. Since $W_\alpha \times Y$ is cozero in $U_{x(\alpha)} \times Y$, $\dim(W_\alpha \times Y) \leqslant \dim U_{x(\alpha)} + \dim Y$ by Theorem 1.1. Thus $\dim(W_\alpha \times Y) \leqslant \dim X + \dim Y$. Since $\{W_\alpha \times Y: \alpha \in A\}$ is a locally finite cozero cover of $X \times Y$,

$$\dim(X \times Y) \leqslant \sup\{\dim(W_\alpha \times Y): \alpha \in A\}$$

by Theorem 2.5 and hence $\dim(X \times Y) \leqslant \dim X + \dim Y$. That completes the proof.

3.4. LEMMA (Katětov [4]). *Let X be a space and F_i, $i \in N$, a sequence of C^*-embedded subsets of X with $X = \bigcup F_i$. Then* $\dim X = \sup \dim F_i$.

3.5. LEMMA. *Let Z be a normal P-space due to Morita [6], X a closed subset of Z, Y a space, and $\pi: X \times Y \to Y$ the projection. If π is a Z-map, then $X \times Y$ is C^*-embedded in $Z \times Y$.*

Proof. Let f be an element of $C(X \times Y, I)$ and d the pseudometric on Y defined in the proof of Lemma 3.1. Let Y^* be the metric space Y/d and $\varphi: Y \to Y^*$ the natural map. For $y \in Y$ denote $f(y)$ by y^*. Let $f^*: X \times Y^* \to I$ be a transformation defined by: $f^*(x, y^*) = f(x, y)$. Then f^* is continuous. Since $X \times Y^*$ is a closed subset of a normal space $Z \times Y^*$, f^* has an extension $g^* \in C(Z \times Y^*, I)$. Define $g: Z \times Y \to I$ by:

$$g(z, y) = g^*(z, y^*), \quad (z, y) \in Z \times Y.$$

Then g is an extension of f over $Z \times Y$. That completes the proof.

3.6. THEOREM. *Let X be a normal P-space, X_i, $i \in N$, a sequence of closed subsets of X with $X = \bigcup X_i$, Y a space, and $\pi: X \times Y \to Y$ the projection. If $\pi|X_i \times Y$ is a Z-map for each i, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Proof. Since each $X_i \times Y$ is C^*-embedded in $X \times Y$ by Lemma 3.5, $\dim(X \times Y) = \sup \dim(X_i \times Y)$ by Lemma 3.4. On the other hand $\dim(X_i \times Y) \leqslant \dim X_i + \dim Y$ by Theorem 3.2. Thus $\dim(X \times Y) \leqslant \sup \dim X_i + \dim Y = \dim X + \dim Y$. That completes the proof.

As for the case when the inequality $\dim(X \times Y) \leqslant \dim X + \dim Y$ is no longer true, we have now beautiful examples due to Wage [15] and to Przymusiński [14].

4. Inverse limits.

4.1. THEOREM. *Let $\{X_i, \pi_j^i\}$ be an inverse system of a sequence of normal spaces X_i with $\dim X_i \leqslant n$ with the onto bonding maps $\pi_j^i: X_i \to X_j$ $(i \geqslant j)$. Let its inverse limit X satisfy the condition:*

(*) *An arbitrary countable cover of X consisting of monotonically increasing open sets can be refined by a countable cover consisting of cylindrical closed sets.*

Then $\dim X \leqslant n$. (A set of type $\pi_i^{-1}(S)$ is said to be a cylindrical closed set if S is a closed set of X_i, where $\pi_i: X \to X_i$ is the projection.)

Proof. By Nagami [8], Theorems 1.2 and 1.3, X is countably paracompact normal. Let $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ be an arbitrary finite open cover of X. Let $U_{i\alpha}$ be the maximal open set of X_i with $\pi_i^{-1}(U_{i\alpha}) \subset U_\alpha$. Set

$$U_i = \bigcup\{U_{i\alpha}: \alpha \in A\},$$
$$\mathcal{V} = \{\pi_i^{-1}(U_i): i \in N\}.$$

Then $\mathcal{V}$ is a cover of X consisting of monotonically increasing open sets. Let S_i be a closed set of X_i with $S_i \subset U_i$ such that $\{\pi_i^{-1}(S_i): i \in N\}$ covers X and $\pi_i^{-1}(S_i) \subset \pi_{i+1}^{-1}(S_{i+1})$ for each i. Let V_i and W_i be cozero sets of X_i such that

$$S_i \subset W_i \subset \overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i,$$
$$(\pi_i^{i+1})^{-1}(\overline{V}_i) \subset W_{i+1}.$$

Let $\mathcal{V}_1 = \{V_{1\alpha}: \alpha \in A\}$ be an open cover of V_1 of order $\leqslant n+1$ such that $V_{1\alpha} \subset U_{1\alpha}$, $\alpha \in A$. Let $\mathcal{V}_2 = \{V_{2\alpha}: \alpha \in A\}$ be an open cover of V_2 of order $\leqslant n+1$ such that

$$V_{2\alpha} \subset (\pi_1^2)^{-1}(V_{1\alpha}) \cup (U_{2\alpha} - (\pi_1^2)^{-1}(\overline{W}_1)), \quad \alpha \in A.$$

Continuing in this fashion, we get, for each $i \geq 2$, an open cover $\mathscr{V}_i = \{V_{i\alpha}: \alpha \in A\}$ of V_i of order $\leq n+1$ such that

$$(**) \qquad V_{i\alpha} \subset (\pi_{i-1}^i)^{-1}(V_{i-1,\alpha}) \cup (U_{i\alpha} - (\pi_{i-1}^i)^{-1}(\overline{W}_{i-1})), \quad \alpha \in A .$$

Set

$$\mathscr{D} = \{D_\alpha = \bigcup_{i=1}^{\infty} \pi_i^{-1}(V_{i\alpha} \cap W_i): \alpha \in A\} .$$

Then D_α is open and $D_\alpha \subset U_\alpha$.

To see that $\mathscr{D}$ is a cover of X of order $\leq n+1$ let $x = (x_i)$ be an arbitrary point of X. Since $\{\pi_i^{-1}(W_i): i \in N\}$ covers X, there exists the minimal j with $x \in \pi_j^{-1}(W_j)$ and hence with $x_j \in W_j$. Since $W_j \subset V_j$, $x_j \in V_{j\beta}$ for some β. Thus $x_j \in V_{j\beta} \cap W_j$, $x \in \pi_j^{-1}(V_{j\beta} \cap W_j) \subset D_\beta$, and hence $\mathscr{D}$ covers X. From the inequality (**) it can easily be seen that, for each $i \in N$,

$$D_\alpha \cap (\pi_i^{-1}(W_i) - \pi_{i-1}^{-1}(W_{i-1})) = \pi_i^{-1}(V_{i\alpha}) \cap (\pi_i^{-1}(W_i) - \pi_{i-1}^{-1}(W_{i-1})), \quad \alpha \in A ,$$

where we set $W_{-1} = \emptyset$. Thus the order of $\mathscr{D}$ at x is the order of $\mathscr{V}_j$ at x_j which is at most $n+1$. $\mathscr{U}$ is now refined by a finite open cover $\mathscr{D}$ of order at most $n+1$. That proves $\dim X \leq n$ and the proof is completed.

4.2. Remark. If one of the following conditions is satisfied, then X satisfies the condition (*).

(1) Each open set of each X_i is F_σ.

(2) X is countably paracompact and each π_j is open.

(3) X is countably paracompact and each π_j^i is perfect.

Thus Theorem 4.1 generalizes Nagami [8], Theorem 1.7, and Pasynkov [11] Theorem 3, at the same time, where they considered the case when X is countably paracompact and each π_j^i is open or perfect.

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