Dimension of non-normal spaces

by

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Abstract. Let $X$ be a general topological space and $\dim X$ the covering dimension of $X$ due to Katetov defined by means of finite cozero covers. If $V$ is a cozero set of $X$, then $\dim V \leq \dim X$. If $(V_1)$ is a countable cozero cover of $X$, then $\dim X = \sup \dim V_1$. Several applications of the subset and sum theorems thus stated are also given.

0. Introduction. Let $X$ be a topological space. Then $\dim X < n$ if each finite cozero cover of $X$ is refined by a finite cozero cover of order $\leq n+1$. This definition of covering dimension for general topological spaces stems from Katetov [4] and coincides with the usual definition of covering dimension for normal spaces. There has been a great amount of studies for the dimension of normal spaces in many aspects. On the contrary we have only a few for non-normal case. Especially, concerning subset and sum theorems we have had nothing with the exception of those due to Katetov [4]. Sections 1 and 2 below constitute the body of the paper where subset and sum theorems for non-normal spaces will respectively be given. In Sections 3 and 4 we give product and inverse limiting theorems for non-normal or normal spaces which will refine known results. In this paper all spaces are non-empty topological spaces and maps are continuous.

1. Subset theorem.

1.1. Theorem. Let $V$ be a cozero set of a space $X$. Then $\dim V \leq \dim X$.

Proof. When $\dim X = n$. Let $\mathcal{F} = \{U_\alpha : \alpha \in A\}$ be an arbitrary finite cozero cover of $V$. It is to be noted that each $U_\alpha$ is cozero in $X$ since $V$ is cozero in $X$. Let $f$ be an element of $C(X, I)$ with $V = \{x \in X : f(x) > 0\}$. Set

$$V_1 = \{x \in X : f(x) > 1/\ell\}, \quad F_1 = \{x \in X : f(x) > 1/\ell\}.$$

Then $V = \bigcup_{\ell=1}^{\infty} V_\ell$ and $V_\ell \subseteq F_{\ell+1}$. Then for each $i$, set

$$\mathcal{W}_i = \{W_\alpha = U_\alpha \cup (X - F_{\ell}) : \alpha \in A\}.$$

Then $\mathcal{W}_i$ is a finite cozero cover of $X$. Let $\mathcal{W}_1 = \{U_\alpha : \alpha \in A\}$ be a cozero cover of $X$ such that $U_i \subseteq \mathcal{W}_i$ for each $\alpha \in A$ and order $\mathcal{W}_i \leq \mathcal{W}_{i+1}$. Set

$$\mathcal{W}_2 = \{U_\alpha \cap V_2 \cup (U_\alpha - F_1) \cup (X - F_1) : \alpha \in A\}.$$
Then $\mathcal{W}_2$ is a finite cozero cover of $X$. Let $\mathcal{W}_2 = \{U_x; x \in A\}$ be a cozero cover of $X$ such that $U_x \subseteq W_x$ for each $x \in A$ and order $\mathcal{W}_2 \leq n+1$. Continuing in this fashion, we get sequences $\mathcal{W}_i = \{W_x; x \in A\}$ and $\mathcal{W}_i = \{U_x; x \in A\}$ of cozero covers of $X$ satisfying the following three conditions:

1. $U_x \subseteq W_x, x \in A$.
2. $W_x = (U_{x,i} \cap V_j) \cup (U_{x,i} - F_{i,j-1}) \cup (X - F_{i,j})$, $x \in A$.
3. order $\mathcal{W}_i \leq i+1$.

Set $\mathcal{D} = \bigcup_{i=1}^m \{U_x \cap V_j; x \in A\}.

Then each $U_x$ is cozero in $X$. Let $x$ be an arbitrary point of $V$. Pick $k$ with $x \in U_k$. Since $\mathcal{W}_k$ covers $X$, there exists $y \in A$ with $x \in U_y$. Then $x \in U_k \cap V_j \subseteq D_x$. Thus $\mathcal{D}$ covers $V$.

Let us prove that $U_x \cap V_{i+1} \subseteq U_x$ by induction on $i$. Since $U_x \cap V_{i+1} \subseteq W_x \cap V_{i+1}$, the assertion is true for $i = 1$. Assume that the assertion is true for $i \leq m$. Then by (1) and (2),

$\begin{align*}
U_{n+1} &\cap V_{m} = W_{n+1} \cap V_{m} \\
&= (U_{n+1} \cap V_{m+1}) \cup (U_{n+1} - F_{n+1}) \cup (X - F_{n+1}) \\
&= (U_{n+1} \cap V_{m+1}) \cup (U_{n+1} - F_{n+1}) \cup (U_{n+1} - F_{n+1}) \cup (X - F_{n+1}) \\
&= (U_{n+1} \cap V_{m+1}) \cup U_{n+1} \subseteq U_{n+1}.
\end{align*}$

Thus the induction is completed. Hence

$\mathcal{D} = \bigcup_{i=1}^m \{U_x \cap V_j; x \in A\} \subseteq U_x$.

To prove order $\mathcal{D} \leq n+1$ let $x$ be an arbitrary point of $V$ and $j$ the minimum with $x \in U_j$. Let $j < l$. Then

$U_x \cap V_j \subseteq W_x \cap V_j$,

$= ((U_x - F_j) \cup (U_x - F_{j-1}) \cup (X - F_{j+1})) \cap V_j$

$= U_{x,1} \cap V_j$.

Hence $U_x \cap V_j \subseteq U_x \cap V_j$, which implies that the order of $\mathcal{D}$ at $x$ is the order of $\mathcal{F}$ at $x$. Thus the order of $\mathcal{D}$ at $x$ is at most $n+1$. That completes the proof.

1.2. Definition. Let $X$ be a space and $S$ a subset of $X$. $S$ is said to be cozero-embedded in $X$ if for each cozero subset $U$ of $S$ there exists a cozero set $V$ of $X$ with $V \cap S = U$.

1.3. Theorem. Let $X$ be a space and $S$ a cozero-embedded subset of $X$. Then dim $S \leq$ dim $X$.

Proof. Let $\mathcal{W} = \{U_x; x \in A\}$ be a finite cozero cover of $S$. Let $V_x, x \in A$, be cozero sets of $X$ with $U_x = V_x \cap S$. Set $V = \bigcup V_x$. Then by Theorem 1.1 there exist cozero sets $W_x$ of $X$ such that $W_x \subseteq V_x$ for each $x \in A$, $V = \bigcup W_x$, and the order of $\{W_x; x \in A\}$ is at most dim $X + 1$. Then $\mathcal{W} = \{W_x \cap S; x \in A\}$ is a cozero cover of $S$ such that $W_x \cap S \subseteq U_x$ for each $x \in A$ and order $\mathcal{W} \leq \text{dim} X + 1$. That completes the proof.

This is a generalization of Kakutani's subset theorem [4] where $S$ is assumed to be $C^\kappa$-embedded in $X$.

1.4. Remark. If $S$ is $C^\kappa$-embedded in $X$, then $S$ is clearly cozero-embedded in $X$. However the converse is not true in general. Let $\mathcal{Q}$ be the rationals in the reals $R$. Then $\mathcal{Q}$ is clearly cozero-embedded in $R$. Choose $f_\epsilon \in C(R, R)$ such that $f(\epsilon) = 1$ for $\epsilon < \sqrt{2}$ and $f(\epsilon) = 0$ for $\epsilon \geq \sqrt{2}$. This $f$ cannot be extended over $R$.

It can easily be seen that a subset $S$ of a space $X$ is cozero-embedded in $X$ if and only if for each finite cozero collection $\mathcal{C}$ of $S$ there exists a finite cozero collection $\mathcal{D}$ of $X$ such that $\mathcal{D}|S \subseteq \mathcal{C}$ and $S \subseteq \mathcal{D}|S = \bigcup \{V; V \in \mathcal{D}\}$.

If each subset of a space $X$ is cozero-embedded in $X$, then $X$ is hereditarily normal.

However the converse of this assertion is not true as follows. Let $X = [0, a_x]$ be the space of countable ordinals. Then $X$ is hereditarily normal. Let $G$ be the subset of isolated ordinals in $X$. Let $U$ be the subset of elements of $G$ whose predecessors are limit ordinals. Then $U$ is a cozero set of $G$. If $V$ is a cozero set of $X$ with $V \cap G = U$, then $V = [a_x, a_y]$ for some $a_x < a_y$ and hence $a_x < 0$ in $V \cap G$. Thus $V \cap G \neq U$, which proves that $G$ is not cozero-embedded in $X$.

2. Sum theorem.

2.1. Definition. Let $U, V$ be subsets of a space $X$. $U$ is said to be a exact subset of $V$ if for some zero set $F$ of $X$, $U \cap F = \emptyset$. For sequence $U_i$, $i \in N$, of subsets of $X$ is said to be exactly decreasing if $U_{i+1}$ is an exact subset of $U_i$ for each $i$.

2.2. Lemma. Let $X$ be a space, $\mathcal{W} = \{U_x; x \in A\}$ a finite cozero cover of $X$, $V$ a cozero set of $X$ with dim $V \leq n$, $W$ a cozero set of $X$ having a cozero cover $\{W_x; x \in A\}$ of order $\leq n+1$ with $W_x \subseteq U_x$, $x \in A$, and $W'$ an exact cozero subset of $W$. Then there exists a cozero cover $\mathcal{V} = \{V_x; x \in A\}$ of $V \cup W'$ such that $V_x \subseteq U_x$ and $V_x \cap W = W_x \cap W'$ for each $x \in A$, and order $\mathcal{V} \leq n+1$.

Proof. Let $F$ be a zero set of $X$ with $W' \cap F = \emptyset$. Set

$U_x = (U_x - F_x) \cup W_x, x \in A$.

Then $\{U_x; x \in A\}$ is a cozero cover of $V$. Let $V'_x = \{V_x; x \in A\}$ be a cozero cover of $V$ of order $\leq n+1$ such that $V'_x \subseteq U'_x$ for each $x \in A$. Set $V_x = V'_x \cup (W_x \cap W')$. Then $\mathcal{V} = \{V_x; x \in A\}$ satisfies the desired condition. That completes the proof.

Let us say in the sequel that a collection $\mathcal{V} = \{V'_x; x \in A\}$ is special (with respect to $\mathcal{W} = \{U_x; x \in A\}$) if $V'_x \subseteq U_x$ for each $x \in A$, order $\mathcal{V} \leq n+1$, and each $V_x$ is cozero in $X$. $\mathcal{V}$ as in Lemma 2.2 is said an extension of $\{W_x; x \in A\}$.

2.3. Lemma. Let $X$ be a space, $\mathcal{W} = \{U_x; x \in A\}$ a finite cozero cover of $X$, $V_x$
to be a $\gamma$-cozero cover of $X$ of order $\leq \alpha + 1$ which refines $\mathfrak{G}$. That completes the proof.

It is to be noted that Pol [13], Example 1, constructed a Tychonoff space $X$ with $\dim X > 0$ which is the union of two zero sets $F$ and $H$ with $\dim F = \dim H = 0$.

2.5. Theorem. Let $X$ be a space which admits a locally finite cozero cover $\mathfrak{G} = \{U_\alpha\}$ with $\dim U_\alpha \leq \alpha$ for each $\alpha$. Then $\dim X \leq \alpha$.

Proof. Since $\mathfrak{G}$ is normal, $\mathfrak{G}$ can be refined by a $\sigma$-discrete cozero cover $\{\mathcal{V}_\alpha\}$, with each $\mathcal{V}_\alpha$ discrete. Set $\mathfrak{F}_1 = \{V_\alpha : \alpha \in A\}$, where each $\mathcal{V}_\alpha$ is $\alpha$-discrete, so $\dim \mathfrak{F}_1 \leq \alpha$. Then $\dim X \leq \alpha$ by Theorem 1.1. Therefore $\dim \mathfrak{F}_1 \leq \alpha$ by the discreteness of $\mathfrak{F}_1$, since each $\mathfrak{F}_1$ is cozero and $X = \bigcup \mathfrak{F}_1$, $\dim X \leq \alpha$ by Theorem 2.4. That completes the proof.

2.6. Corollary. Let $X$ be a space with $\dim X \leq \alpha$ and $U$ a $D$-open set of $X$, i.e., $U$ admits a locally finite (in $U$) cover $\{\mathcal{V}_\alpha\}$ of each element of which is cozero in $X$. Then $\dim U \leq \alpha$.

This is a direct consequence of Theorems 1.1 and 2.5. The concept of $D$-open sets stems from Dowker [1] and recently considered by Nishimura [9].

3. Product theorem. According to Ishiwata [3], a map $f : X \to Y$ is said to be a $Z$-map if the image of each zero set of $X$ under $f$ is closed in $Y$. This notion is a generalization of closed maps. When a space $X$ is a Tychonoff space, $\beta X$ denotes the Stone–Čech compactification of $X$.

3.1. Lemma. Let $X$ be a Tychonoff space, $X$ and $Y$ be spaces, and $\pi : X \times Y \to Y$ the projection. If $\pi$ is a $Z$-map, then $X \times Y$ is $C^*$-embedded in $\beta X \times Y$.

Proof. Let $f$ be an element of $C(X \times Y)$. Set for $x, y \in X$,

$$
\Delta(y, y') = \sup \{|f(x, y) - f(x, y')| : x \in X\}.
$$

To see if $\Delta$ is a pseudo-metric on $Y$, choose an arbitrary point $y_0$ of $Y$ and an arbitrary positive number $\alpha$. Set

$$
g(x, y) = |f(x, y) - f(x, y_0)|, \quad (x, y) \in X \times Y,
F = \{(x, y) \in X \times Y : g(x, y) \geq \alpha\}.
$$

Then $g \in C(X \times Y)$ and hence $F$ is a zero set of $X \times Y$. Since $g(x, y_0) = 0$, $(X \times \{y_0\}) \cap F = \emptyset$ and hence $\pi(x_0) \cap \pi(F) = \emptyset$. Since $\pi$ is a $Z$-map, $Y - \pi(F)$ is an open neighborhood of $y_0$. Since $(X \times (Y - \pi(F))) \cap F = \emptyset$, $g(x, y) = |f(x, y) - f(x, y_0)| < \alpha$ for each $x \in X$ and each $y \in Y - \pi(F)$ and hence $d(y, y_0) < \alpha$ whenever $y \in Y - \pi(F)$. Thus $d$ is a pseudo-metric on $Y$. 

For each \( y \in Y \) let \( h_y: \beta X \times \{y\} \to I \) be an extension of \( f|\{x \times \{y\}| \). Define a transformation \( h: \beta X \times Y \to I \) by:

\[
h(h_x \times \{y\}) = h_y, \quad y \in Y.
\]

Then \( h: \beta X \times Y = f \). To prove the continuity of \( h \) let \( p, q \in \beta X \times Y \) be an arbitrary point of \( \beta X \times Y \) and \( e \) an arbitrary positive number. By the continuity of \( h|\beta X \times \{y\} \) there exists an open neighborhood \( V \) of \( p \) in \( \beta X \times Y \) such that \( h(p, q) = h(x, y) < e \) for each \( x \in V \). Set \( W = \{ y \in Y : d(q, y) < e \} \). Then \( W \) is an open neighborhood of \( q \).

Let \( (x, y) \) be an arbitrary point of \( V \times W \). By the continuity of \( h|\beta X \times \{y\} \) there exists a point \( x' \in V \cap X \) such that \( h(x, y) = h(x', y') < e \). Then

\[
|h(x, y) - h(p, q)| = |h(x', y') - h(x', y)| + |h(x', y) - h(x', y')| + |h(x', y') - h(p, q)| < \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e.
\]

Thus \( h \) is continuous and the proof is completed.

3.2. Theorem. Let \( X \) be a Tychonoff space, \( Y \) a space, and \( \pi: X \times Y \to Y \) the projection. If \( \pi \) is a \( Z \)-map, then \( \dim(X \times Y) \leq \dim X + \dim Y \).

Proof. Since \( X \times Y \) is \( C^* \)-embedded in \( \beta X \times Y \) by Lemma 3.1, \( \dim(X \times Y) \leq \dim(\beta X \times Y) \) by Theorem 1.3. Since \( \dim(\beta X \times Y) \leq \dim(\beta X) + \dim Y \) by Morita [7], Theorem 5.5, \( \dim(X \times Y) \leq \dim X + \dim Y = \dim X + \dim Y \). That completes the proof.

This generalizes Filippov [2], Theorem 1.

3.3. Corollary. Let \( X \) be a paracompact Hausdorff space, \( Y \) a space, and \( \pi: X \times Y \to Y \) the projection. If each point \( x \in X \) has a closed neighborhood \( U_x \) such that \( \pi(U_x \times Y) \) is a \( Z \)-map, then \( \dim(X \times Y) \leq \dim X + \dim Y \).

Proof. Let \( V_x \) be the interior of \( U_x \). Let \( (W_{x} \neq \emptyset : \pi \in A) \) be a locally finite cozero cover of \( X \) refining \( (V_x : x \in X) \). For each \( W_x \), choose \( V_{x, x} \subseteq V_{x, x} \). By Theorem 3.2, \( \dim(U_{x, x} \times Y) \leq \dim U_{x, x} + \dim Y \). Since \( X \) is normal and \( U_{x, x} \subseteq X \) is closed, \( \dim U_{x, x} \subseteq X \). Since \( W_x \subseteq Y \) is cozero in \( U_{x, x} \times Y \), \( \dim(W_x \times Y) \leq \dim(W_x \times Y) \) by Theorem 1.1. Thus \( \dim(W_x \times Y) \leq \dim X + \dim Y \). Since \( (W_x \times Y : x \in A) \) is a locally finite cozero cover of \( X \times Y \),

\[
\dim(X \times Y) \leq \sup \{ \dim(W_x \times Y) : x \in A \}
\]

by Theorem 2.5 and hence \( \dim(X \times Y) \leq \dim X + \dim Y \). That completes the proof.

3.4. Lemma (Katelhov [4]). Let \( X \) be a space and \( F_1, F_2 \subseteq X \), \( i \in N \), a sequence of \( C^* \)-embedded subsets of \( X \) with \( X = \bigcup_{i \in N} F_i \), then \( \dim(X \times Y) \leq \dim X + \dim Y \).

3.5. Lemma. Let \( Z \) be a normal \( P \)-space due to Morita [6], \( X \) a closed subset of \( Z \), \( Y \) a space, and \( \pi: X \times Y \to Y \) the projection. If \( \pi \) is a \( Z \)-map, then \( X \times Y \) is \( C^* \)-embedded in \( Z \times Y \).

Proof. Let \( f \) be an element of \( C(X \times Y, f) \) and \( d \) the pseudometric on \( Y \) defined in the proof of Lemma 3.1. Let \( y^* \) be the metric space \( Y_d \) and \( \varphi: Y \to Y^* \) the natural map. For \( y \in Y \) denote \( f(y) \) by \( y^* \). Let \( f^*: X \times Y^* \to I \) be a transformation defined by:

\[
f^*(x, y^*) = f(x, y).
\]

Then \( f^* \) is continuous. Since \( X \times Y \) is a closed subset of \( Z \times Y^* \), \( f^* \) has an extension \( \varphi^* \in C(Z \times Y^*, I) \). Define

\[
g: Z \times Y \to I \text{ by:}
\]

\[
g(z, y) = \varphi^*(z, y^*) \in Z \times Y.
\]

Then \( g \) is an extension of \( f \) over \( Z \times Y \). That completes the proof.

3.6. Theorem. Let \( X \) be a normal \( P \)-space, \( X_i, i \in N \), a sequence of closed subsets of \( X \) with \( X = \bigcup_{i \in N} X_i \), \( Y \) a space, and \( \pi: X \times Y \to Y \) the projection. If \( \pi|X_i \times Y \) is a \( Z \)-map for each \( i \), then \( \dim(X \times Y) \leq \dim X + \dim Y \).

Proof. Since each \( X_i \times Y \) is \( C^* \)-embedded in \( X \times Y \) by Lemma 3.5, \( \dim(X \times Y) \leq \dim(X_i \times Y) \) by Lemma 3.4. On the other hand \( \dim(X_i \times Y) \leq \dim X_i + \dim Y \) by Theorem 3.2. Thus \( \dim(X \times Y) \leq \dim X_i + \dim Y \leq \dim X + \dim Y \). That completes the proof.

As for the case when the inequality \( \dim(X \times Y) \leq \dim X + \dim Y \) is no longer true, we have now beautiful examples due to Wage [15] and to Przymusinski [14].

4. Inverse limits.

4.1. Theorem. Let \( (X_i, \pi_i) \) be an inverse system of a sequence of normal spaces \( X_i \) with \( \pi_i \subseteq n \) with the onto bonding maps \( \pi_i : X_i \to X_{i+1} \) (for \( i \)). Let \( \pi \) be an inverse limit of \( \pi_i \). Let its inverse limit satisfy the condition:

\[
(\ast) \text{ An arbitrary countable cover of } X \text{ consisting of monotonically increasing open sets can be refined by a countable cover consisting of cylindrical closed sets.}
\]

Then \( \dim X \leq n. \) (A set of type \( \pi^{\ast}_i(S) \) is said to be a cylindrical closed set if \( S \) is a closed set of \( X_i \), where \( \pi_i : X_i \to X_{i+1} \) is the projection.)

Proof. By Nagami [8], Theorems 1.2 and 1.3, \( X \) is countably paracompact normal. Let \( \mu = (U_\alpha : \alpha \in A) \) be an arbitrary finite open cover of \( X \). Let \( U_\alpha \) be the maximal open set of \( X_i \) with \( \pi^{\ast}_i(U_\alpha) \subseteq U_\alpha \). Set

\[
U_\alpha = \bigcup (U_\alpha : \alpha \in A),
\]

\[
\varphi = (\pi^{-1}_i(U_\alpha) : \alpha \in A).
\]

Then \( \varphi \) is a cover of \( X \) consisting of monotonically increasing open sets. Let \( S_i \) be a closed set of \( X_i \) with \( S_i \subseteq U_\alpha \) such that \( \{\pi^{\ast}_i(S_i) : i \in N \} \) covers \( X \) and \( \pi^{\ast}_i(S_i) \subseteq U_\alpha^{\ast}_i(X_i) \) for each \( i \). Let \( V_i \) and \( W_i \) be cozero sets of \( X_i \) such that

\[
S_i \subseteq W_i \subseteq U_\alpha \subseteq V_i \subseteq U_\alpha, \quad (\pi^{\ast}_i)^{-1}(V_i) = W_i.
\]

Let \( \varphi = (V_\alpha : \alpha \in A) \) be an open cover of \( V_i \) of order \( \leq \pi_i + 1 \) such that \( V_\alpha \subseteq U_\alpha \), \( \alpha \in A \). Let \( \varphi = (V_\alpha : \alpha \in A) \) be an open cover of \( V_\alpha \) of order \( \leq \pi_n + 1 \) such that

\[
V_\alpha \subseteq (\pi^{\ast}_i)^{-1}(V_i) \cup (U_\alpha - (\pi^{\ast}_i)^{-1}(W_i)), \quad \alpha \in A.
\]
Continuing in this fashion, we get, for each \( j \geq 2 \), an open cover \( \mathcal{V}_j = \{ V_{j, i} : x \in B \} \) of \( X \), of order \( \leq n + 1 \) such that

\[
(\star) \quad V_{j, i} \in (\sigma_{i-1}^{-1}(V_{i-1, 1}) \cup (U_{i-1} \setminus (\sigma_{i-1}^{-1}(W_{i-1, 1}))), \quad x \in B.
\]

Set

\[
\mathcal{B} = \{ D_x = \bigcup_{i=1}^{\infty} \sigma_{i-1}^{-1}(V_{i, x} \cap W_j) : x \in A \}.
\]

Then \( D_x \) is open and \( D_x \subseteq U_x \).

To see that \( \mathcal{B} \) is a cover of \( X \) of order \( \leq n + 1 \) let \( x = (\chi) \) be an arbitrary point of \( X \). Since \( \{ \sigma_{i-1}^{-1}(W_j) : i \in N \} \) covers \( X \), there exists the minimal \( j \) with \( x \in \sigma_{j-1}^{-1}(W_j) \) and hence with \( x_j \in W_j \). Since \( W_j \in \mathcal{V}_j \), \( x_j \in V_{j, j} \) for some \( j \). Thus \( x_j \in V_{j, j} \cap W_j \), \( x \in \sigma_j^{-1}(V_{j, j} \cap W_j) \subseteq D_x \), and hence \( \mathcal{B} \) covers \( X \). From the inequality (\( \star \)) it can easily be seen that, for each \( i \in N \),

\[
D_x \cap (\sigma_{i-1}^{-1}(W_j) - \sigma_{i-1}^{-1}(W_{j-1})) = \sigma_{i-1}^{-1}(V_{i, x} \cap W_j) - \sigma_{i-1}^{-1}(W_{j-1}),( \quad x \in A,
\]

where we set \( W_{-1} = \emptyset \). Thus the order of \( \mathcal{B} \) at \( x \) is the order of \( \mathcal{V}_j \) at \( x_j \) which is at most \( n + 1 \). \( \mathcal{B} \) is now refined by a finite open cover \( \mathcal{B} \) of order at most \( n + 1 \). That proves \( \dim X \leq n \) and the proof is completed.

4.2. Remark. If one of the following conditions is satisfied, then \( X \) satisfies the condition (\( \star \)).

1. Each open set of each \( X \) is \( F_x \).
2. \( X \) is countably paracompact and each \( x_j \) is open.
3. \( X \) is countably paracompact and each \( x_j \) is perfect.

Thus Theorem 4.1 generalizes Nagami [8], Theorem 1.7, and Pasyukov [11] Theorem 3, at the same time, where they considered the case when \( X \) is countably paracompact and each \( x_j \) is open or perfect.

References