Homotopy equivalences and mapping torus projections

by

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Abstract. Let $f\colon X\to X$ be a mapping on a connected ANR. There is a natural projection $p\colon T_f\to S^1$ from the mapping torus to the circle which has the property that each point inverse is a copy of X. The purpose of this note is to show that there is a relationship between the strength of f as a homotopy equivalence and the strength of p as a fibration. For example, f is a homotopy equivalence if and only if p is an approximate fibration; and f is a CE mapping if and only if p is a Hurewicz fibration.

0. Introduction. If $f\colon X\to X$ is a map, then there is a natural projection $p\colon T_f\to S^1$, where T_f is the mapping torus of f. The map p has the property that $p^{-1}(x)\cong X$ for all $x\in X$ and that p is a locally trivial fibration over the complement of a single point of S^1.

Clearly if f is a homeomorphism (topological equivalence), then p is a locally trivial fibration. The purpose of this note is to show that there is a further relationship between the strength of f as an equivalence (e.g., homotopy) and the strength of p as a fibration. We believe this relationship clarifies the differences between some kinds of fibrations and provides insight into the question of how a fibration can go "bad" at a single point. The following theorems are our main results. The terminology is defined in the next section.

THEOREM A. *Let $f\colon X\to X$ be a map on a connected* ANR *and let $p\colon T_f\to S^1$ be the natural projection from the mapping torus of f. The following are equivalent:*
1) *f is a homotopy equivalence, and*
2) *p is an approximate fibration.*

THEOREM B. *Let $f\colon X\to X$ be a map on a connected* ANR *and let $p\colon T_f\to S^1$ be the natural projection from the mapping torus of f. The following are equivalent:*
1) *f is a homotopy equivalence and $\tau(f)\in(1-f_*)\mathrm{Wh}(X)$,*
2) *p can be stably approximated by Hurewicz fibrations, and*
3) *p can be stably approximated by locally trivial fibrations.*

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THEOREM C. *Let $f\colon X \to X$ be a map on a connected* ANR *and let $p\colon T_f \to S^1$ be a natural projection from the mapping torus. The following are equivalent*:

1) *f is a* CE *mapping*,

2) *p is a Serre fibration, and*

3) *p is a Hurewicz fibration.*

Sections 2, 3, and 4 contain the proofs of these theorems. The last section contains several examples. Our point of view there is that the above theorems make it easy to construct certain kinds of fibrations.

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1. Notation and definitions. Throughout this paper X denotes a connected ANR; that is, a connected, compact, metric, absolute neighborhood retract; and $f\colon X \to X$ denotes a map. The *mapping torus*, T_f, of f is the quotient space $X \times I/\sim$, where $\sim$ is the equivalence relation generated by $(x, 1) \sim (f(x), 0)$. We use the notations $[x, t]$ for the equivalence class of (x, t) in T_f, and $\eta\colon X \times I \to T_f$ for the quotient map. We let $I = [0, 1]$, $S^1 = I/_{\{0,1\}}$, $\nu\colon I \to S^1$ the quotient map, and $[t] = \nu(t)$. Define $p\colon T_f \to S^1$ by $p([x, t]) = \nu(t)$. It follows from [B2, Ch. V, Th. 9.1] that T_f is an ANR. Similarly, we denote the mapping cylinder of f by M_f.

A *homotopy lifting problem* (Z, h, H) for a map $q\colon E \to B$ is a space Z, a map $h\colon Z \times \{0\} \to E$, and a homotopy $H\colon Z \times I \to B$ such that $H|Z \times \{0\} = qh$. A map $q\colon E \to B$ has the *homotopy lifting property* for a space Z if for every homotopy lifting problem (Z, h, H) there is a homotopy $\tilde{H}\colon Z \times I \to E$ such that $\tilde{H}|Z \times \{0\} = h$ and $q\tilde{H} = H$. Also $q\colon E \to B$ has the *approximate homotopy lifting property* for a space Z if for every homotopy lifting problem (Z, h, H) and for every open cover ε of B there is a homotopy $\tilde{H}\colon Z \times I \to E$ such that $\tilde{H}|Z \times \{0\} = h$ and $q\tilde{H}$ is ε-close to H.

If $q\colon E \to B$ has the homotopy lifting property for all spaces (all polyhedra), then q is a *Hurewicz* (*Serre*) *fibration*. If $q\colon E \to B$ has the approximate homotopy lifting property for all spaces, q is an *approximate fibration*. Next, we say that a map $q\colon E \to B$ is a *locally trivial fibration* with fiber F if for each $b \in B$ there is a neighborhood U of b and a homeomorphism $h\colon U \times F \to q^{-1}(U)$ such that $qh = \pi$ where π is projection onto the first factor. We say $q\colon E \to B$ can be *stably approximated* by locally trivial (Hurewicz) fibrations provided the composition $q\pi\colon E \times Q \to B$, where π is projection onto the first factor and Q is the Hilbert cube, can be approximated by locally trivial (Hurewicz) fibrations. The reader is referred to [CD1], [CD2], [Du], [S], for properties of approximate, Hurewicz, or Serre fibrations.

We shall use simple homotopy theory for ANR's as developed by Chapman in [C2]. A map $g\colon Y \to Z$ of ANR's is a CE *mapping* provided g is surjective and each point inverse has trivial shape in the sense of Borsuk [B1]. The symbol $\cong$ is to be read "is homeomorphic to".

2. Proof of Theorem A. To show that 1) $\Rightarrow$ 2), note that if f is a homotopy equivalence, $V \subset S^1$ is any open interval, and $c \in V$, then the inclusion $p^{-1}(c) \to p^{-1}(V)$

is a homotopy equivalence. Then, since every fiber of p is an ANR, it is easy to show that p is completely movable [CD2], so p is an approximate fibration.

To show 2) $\Rightarrow$ 1), assume $p\colon T_f \to S^1$ is an approximate fibration. We will prove that f is a homotopy equivalence.

Let $A = p^{-1}\nu([\frac{3}{4}, 1])$, $B = p^{-1}\nu([\frac{1}{2}, 1] \cup [0, \frac{1}{4}])$, and $J = \nu([\frac{3}{4}, 1])$. Note that A is homeomorphic to the mapping cylinder of f with $p^{-1}[\frac{3}{4}]$ corresponding to the domain and $p^{-1}[1]$ the range, and that there is a retraction $\psi\colon B \to A$. Let $\varphi\colon J \times I \to J$ be a homotopy such that $\varphi_0 = 1_J$, $\varphi_1(J) = [\frac{3}{4}]$, and $\varphi_t[\frac{3}{4}] = [\frac{3}{4}]$ for all t. Let $H\colon A \times I \to S^1$ be defined by $H_t = \varphi_t p$. By the stationary lifting property of approximate fibrations [CD1, Prop. 1.5], there a homotopy $\tilde{H}\colon A \times I \to B$ such that $\tilde{H}_0 = 1_A$, $\psi\tilde{H}_1(A) \subset A - p^{-1}[1]$, and $H_t[x, \frac{3}{4}] = [x, \frac{3}{4}]$ for all $t \in I$, $x \in X$. Composing $\psi\tilde{H}$ with a strong deformation retraction of $A - p^{-1}[1]$ onto $p^{-1}[\frac{3}{4}]$ gives a deformation retraction of A onto $p^{-1}[\frac{3}{4}]$, which implies that f is a homotopy equivalence.

3. Proof of Theorem B. We first show that 2) is equivalent to 3). Clearly 3) $\Rightarrow$ 2) and we now prove that 2) $\Rightarrow$ 3). Let $f\colon X \to X$ and $p\colon T_f \to S^1$ have the meaning given in Section 1, and let $\pi\colon T_f \times Q \to T_f$ be projection. Assume $p\pi$ can be approximated by Hurewicz fibrations. Given $\varepsilon > 0$, there is a Hurewicz fibration $g\colon T_f \times Q \to S^1$ such that $d(g, p\pi) < \frac{1}{2}\varepsilon$. By [C1] there is a homeomorphism $\alpha\colon T_f \times Q \to T_f \times Q \times Q$ so that $d(gq\alpha, g) < \frac{1}{2}\varepsilon$ where q is the projection of $T_f \times Q \times Q$ onto the first two factors. By [F] the fibers of g are ANR's, so by [E] the fibers of $gq\alpha$ are Q-manifolds. Hence $gq\alpha$ is locally trivial fibration [CF], and $d(gq\alpha, p\pi) < \varepsilon$.

Before proving that 1) $\Rightarrow$ 3), we need some notation and a proposition due to Ferry [Fe1]. First when there is more than one mapping we will denote the natural map $p\colon T_f \to S^1$ by p_f. If $k\colon Y \to Y$ and $l\colon Y \to Y$ are maps, then $T(k, l)$ will denote the quotient space $(Y \times [0, \frac{1}{2}] \cup Y \times [\frac{1}{2}, 1])/\sim$ where $\sim$ is the equivalence relation generated by $(x, \frac{1}{2}) \sim (k(x), \frac{1}{2})$ and $(x, 1) \sim (l(x), 0)$. Also $p(k, l)\colon T(k, l) \to S^1$ will denote the natural projection. Finally, if $k\colon Y \to Y$, then

$$\bar{k} = k \times 1\colon Y \times Q \to Y \times Q.$$

Note that if Y is an ANR, then $T_{\bar{k}}$ is a Q-manifold.

PROPOSITION 3.1 (Ferry). *Let $k, l\colon Y \to Y$ be maps on a locally compact* ANR.

a) *If k is properly homotopic to l, there is a homeomorphism $\theta\colon T_{\bar{k}} \to T_{\bar{l}}$ such that $p_k\pi_k$ is homotopic to $p_l\pi_l\theta$, where $\pi_k\colon T_{\bar{k}} \to T_k$ is the obvious projection.*

b) *There is a homeomorphism $\eta\colon T_{\overline{kl}} \to T(k, l)$ such that $p_{kl}\pi_{kl}$ is homotopic to $p_{(k,l)}\pi_{(k,l)}\eta$; and*

c) *There is a homeomorphism $\xi\colon T(\bar{k}, \bar{l}) \to T(\bar{l}, \bar{k})$ such that $p_{(k,l)}\pi_{(k,l)}$ is homotopic to $p_{(l,k)}\pi_{(l,k)}\xi$.*

Parts a) and b) follow directly from [Fel, 3.2–3.4]. Part c) is proved by a 180° rotation.

Now to prove that 1) $\Rightarrow$ 3). Suppose that $f\colon X\to X$ is a homotopy equivalence such that $\tau(f)\in(1_*-f_*)\mathrm{Wh}(X)$. Then there is a homotopy equivalence $\varphi\colon Y\to X$ with homotopy inverse $\psi\colon X\to Y$ such that $\tau(f) = (1_*-f_*)\tau(\varphi) = \tau(\varphi)-f_*\tau(\varphi)$. Consider $\psi f\varphi\colon Y\to Y$. Using [C2, §5], we have

$$\tau(\psi f\varphi) = \tau(\psi)+\psi_*\tau(f)+\psi_*f_*\tau(\varphi) = \psi_*[-\tau(\varphi)+\tau(f)+f_*\tau(\varphi)] = 0\,.$$

Hence $\bar\psi\bar f\bar\varphi\colon Y\times Q\to Y\times Q$ is homotopic to a homeomorphism h [C1]. Now let λ be the following composition of homeomorphisms given by 3.1.

$$T_{\bar f}\to T_{\bar f\bar\varphi\bar\psi}\to T(\bar f\bar\varphi, \bar\psi)\to T(\bar\psi, \bar f\bar\varphi)\to T_{\bar\psi\bar f\bar\varphi}\to T_h\,.$$

Then $p_h\lambda$ is a locally trivial fibration homotopic to $p_f\pi_f$. It follows from [Hu, Theorem B] that $p_f\pi_f$ can be approximated by locally trivial fibrations.

Finally, to show 3) $\Rightarrow$ 1), suppose that $p\pi\colon T_{\bar f}\to S^1$ can be approximated by a locally trivial fibration q chosen so that $q^{-1}([\frac13])\subset\pi^{-1}p^{-1}v(0,\frac23)$. Let $d\colon D\to T_f\times Q$ be the covering map induced from $v\colon R\to S^1$ (here we extend v periodically) by $p\pi$, let $r\colon D\to R$ be a map such that $vr = p\pi d$. Let $t\colon D\to D$ be a covering translation such that $r(tx) = r(x)+1$ for $x\in D$. Identify X with $r^{-1}(\frac23)$ and M_f with $r^{-1}([\frac23, 1])$. Now define $Z = r^{-1}(0)$, $Y = d^{-1}q^{-1}([\frac13])\cap r^{-1}[0,1]$, K = closure of the component of $r^{-1}[0,1]-Y$ which contains Z, and L = closure of $(r^{-1}[0,1]-(K\cup M_f))$. Let $Z_1 = t(Z)$, $X_1 = t(X)$, $Y_1 = t(Y)$, and $K_1 = t(K)$. Finally, let $i\colon Z\to K$, $j\colon Y\to K$, $k\colon Y\to L$, $l\colon X\to L$, $m\colon X\to M_f$, and $n\colon Z_1\to M_f$ be inclusion maps. Each of these inclusions is a homotopy equivalence. For example to see that $j\colon Y\to K$ is a homotopy equivalence, construct a strong deformation retraction from K to Y by lifting a contraction of $dq(K)$ to $[\frac13]$.

Since q is a locally trivial fibration with fiber Y, $L\cup M_f\cup K_1$ is homeomorphic to the product $Y\times I$ with Y corresponding to $Y\times\{0\}$, and Y_1 to $Y\times\{1\}$.

Hence $\tau(L\cup M_f\cup K_1, Y) = 0$. Also, $K\cup L$ is homeomorphic to $X\times I$ with Z corresponding to $X\times\{0\}$ and X to $X\times\{1\}$. Hence $\tau(K\cup L, Z) = 0$. But we have

$$\tau(L\cup M_f\cup K_1, Y) = \tau(L, Y)+k_*^{-1}l_*\tau(M_f, X)+k_*^{-1}l_*m_*^{-1}n_*\tau(K_1, Z_1)\,;$$

$$\tau(K\cup L, Z) = \tau(K, Z)+l_*^{-1}j_*\tau(L, Y)\,,$$

and

$$\tau(K_1, Z_1) = t_*\tau(K, Z)\,.$$

Combining these facts, we conclude that

$$\tau(L, Y)+k_*^{-1}l_*\tau(M_f, X)-k_*^{-1}l_*m_*^{-1}n_*t_*i_*^{-1}j_*\tau(L, Y) = 0\,.$$

Applying $f_*l_*^{-1}k_*$ and rearranging yields

$$\tau(f) = (f_*m_*^{-1}n_*t_*i_*^{-1}j_*k_*^{-1}l_*-f_*)l_*^{-1}k_*\tau(L, Y)\,.$$

But $f_*m_*^{-1}n_*t_*i_*^{-1}j_*k_*^{-1}l_* = 1_*$ on $\mathrm{Wh}\,X$ since sliding down the mapping cylinder is the same as performing f. Hence $\tau(f)\in(1_*-f_*)\mathrm{Wh}\,X$.

4. Proof of Theorem C. Before proving C, we need some definitions and several propositions. A map $q\colon E\to B$ is *n-regular* if q is open for each $b\in B$, $e\in q^{-1}(b)$, and neighborhood U of e in E there is a neighborhood V of e in U such that every singular k-sphere ($k\leqslant n$) in $q^{-1}(c)\cap V$, $c\in q(V)$, is null-homotopic in $q^{-1}(c)\cap U$ [Ha]. Also $q\colon E\to B$ is a *UV^n-mapping* if q is surjective and for each $b\in B$ and neighborhood U of $q^{-1}(b)$ in E there is a neighborhood V of $q^{-1}(b)$ in U such that every singular k-sphere ($k\leqslant n$) in V is null-homotopic in U [L]. We say that q is *strongly regular* provided that for each $b\in B$ and $\varepsilon>0$ there is $\delta>0$ such that if $c\in B$ and $d(b, c)<\delta$ then there are maps $g_{cb}\colon q^{-1}(c)\to q^{-1}(b)$ and $g_{bc}\colon q^{-1}(b)\to q^{-1}(c)$ where $g_{cb}g_{bc}$ and $g_{bc}g_{cb}$ are ε-homotopic to the identity.

PROPOSITION 4.1. *$f\colon X\to X$ is surjective if and only if $p\colon T_f\to S^1$ is open.*

Proof. Suppose f is not surjective. Then if V is any open set in $X-f(X)$, $\eta(V\times[0,\frac12))$ is open in T_f and $p\eta(V\times[0,\frac12)) = v([0,\frac12))$ is not open in S^1. If f is surjective, T_f has a basis consisting of sets of the form $\eta(V\times(t-\varepsilon, t+\varepsilon))$ for $t\in(0,1)$ and V open in X, together with sets of the form $\eta(f^{-1}(v)\times(1-\varepsilon, 1]\cup V\times[0,\varepsilon))$. Clearly the images of these basic elements are open under p.

PROPOSITION 4.2. *If $p\colon T_f\to S^1$ is open and satisfies the homotopy lifting property for cells of dimension $\leqslant n$, then $f\colon X\to X$ is a UV^{n-1}-mapping.*

Proof. Let $x\in X$ be given and let U be a neighborhood of x in X. We may assume that $U = f^{-1}(U_0)$ for some neighborhood U_0 of $f(x)$ in X. Let V_0 be a neighborhood of $f(x)$ in U_0 such that V_0 is contractible in U_0, let $V = f^{-1}(V_0)$. Given $k\leqslant n-1$ and $\alpha\colon S^k\to V$ there is a map $\beta\colon B^{k+1}\to U_0$ such that $\beta|S^k = f\alpha$. Define $g\colon B^{k+1}\to T_f$ by

$$g(y) = \begin{cases}[\alpha(y/|y|), 2-2|y|] & \text{if}\quad 1\geqslant|y|\geqslant\frac12,\\ [\beta(2y), 1] & \text{if}\quad \frac12\geqslant|y|\geqslant0\end{cases}$$

and $G\colon B^{k+1}\times I\to S^1$ by

$$G(y, t) = \begin{cases}[(2-2|y|)(1-t)] & \text{if}\quad 1\geqslant|y|\geqslant\frac12,\\ [1-t] & \text{if}\quad \frac12\geqslant|y|>0\,.\end{cases}$$

By hypothesis, there is a $\tilde G\colon B^{k+1}\times I\to T_f$ such that $\tilde G_0 = g$, $p\tilde G = G$, and $\tilde G$ is stationary on S^k. Choose $s>0$ small enough that $\tilde G_s(B^{k+1})$ is contained in $\eta(U\times[0, 1-s])$. There is retraction r of $\eta(U\times[0, 1-s])$ onto $\eta(U\times\{0\})$, so that $r\tilde G_0(B^{k+1})$ is an extension of α into U, and the proof is complete.

PROPOSITION 4.3. *If $f\colon X\to X$ is a CE-mapping, then $p\colon T_f\to S^1$ is strongly regular.*

Proof. We need only check strong regularity at $b = [1]$. Let $\varepsilon>0$ be given. The check is trivial for $c = [t]$ where $0\leqslant t<\varepsilon$. On the other hand, for $1-\varepsilon<t<1$ let $f_t\colon p^{-1}[t]\to p^{-1}[1]$ be defined by $f_t([x, t]) = [x, 1]$. It is not hard, using the fact that each f_t is CE, to find a $\delta>0$ and a cover α of $p^{-1}[1]$ such that α and the cover $f_t^{-1}(\alpha)$ have mesh less than ε for $1-\delta<t<1$. Now for $b = [1]$ and $c = [t]$ with

$1-\delta<t<1$, the maps g_{cb} and g_{bc} are guaranteed by the fact that each f_t is a fine homotopy equivalence [Hav].

Now for the proof of Theorem C. To show 1) ⇒ 3), note that p is strongly regular by 4.3. Now apply Theorem 1 of [Fe2]. Clearly 3) ⇒ 2). To show 2) ⇒ 3), use 4.2 to see that f is a UV^n map for all n. Therefore, f is a homotopy equivalence over each open set by Theorem 2 of [K] and it follows that each $f^{-1}(x)$ has property UV^∞.

5. Examples.

EXAMPLE 1. Let $f: [-1, 1] \to [-1, 1]$ be the folding map $f(x) = 2|x|-1$. By 4.1, $p: T_f \to S^1$ is an open map, which is easily seen to be a Dold fibration [Do]. By Theorem C, p is not a Hurewicz fibration. This gives a negative answer to a question of Addis [A].

EXAMPLE 2. Let M be a PL manifold, X and Y be ANR's, and $f: X\times \mathrm{Bd}\, M \to Y\times \mathrm{Bd}\, M$. Let $E_f = (X\times M) \cup (Y\times \mathrm{Bd}\, M)/\sim$ where $\sim$ is the equivalence relation generated by $(x, m) \sim f(x, m)$, and define $p: E_f \to M$ by $p[x, m] = m$. Then p is an approximate fibration if and only if f is a homotopy equivalence and p is a Hurewicz fibration if and only if f is CE. Both of these follow from the arguments already given and the observation that the image of a collar on $\mathrm{Bd}\, M$ in E_f is the mapping cylinder of f.

EXAMPLE 3. Let $f: X \to X$ be a surjection which is not a homotopy equivalence. In particular one could take $f: S^1 \to S^1$ to be the double cover. Then $p: T_f \to S^1$ is an open mapping each of whose point inverses are homeomorphic to X, but p is not an approximate fibration.

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