Analytic sets and Borel isomorphisms

by

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Abstract. We show that if all analytic (i.e. $\Sigma^1_2$) games are determined, then any two analytic non-Borel sets of reals are carried one to the other by a Borel permutation of the reals. A similar theorem concerning homeomorphisms of the Cantor space is proved.

0. Introduction. We shall show that if all analytic (i.e. $\Sigma^1_2$) games are determined, then any two analytic non-Borel sets of reals are Borel isomorphic (carried one to the other by a Borel permutation of the reals). Thus there are, under this determinateness hypothesis, exactly two analytic Borel spaces ([1], p. 46).

The question whether all analytic games are determined has been the subject of much recent investigation. In our opinion, the existing evidence supports an affirmative answer. At any rate, it is shown in [3] that one cannot weaken this determinateness hypothesis here, for it is provable in ZFC that if all analytic non-Borel sets are Borel isomorphic, then all analytic games are determined.

The Borel isomorphisms we produce will actually be class $(1, 1)$ homeomorphisms (cf. [4], p. 374). In §3 we describe some cases in which full homeomorphisms can be obtained.

1. Preliminaries. Let $S_q$ be the set of all finite sequences of zeros and ones, i.e. $S_q = \bigcup_{n=0}^{\omega} \Sigma^n$. The Cantor space is the set $\omega^2$ together with the topology generated by basic open sets of the form

$$[s] = \{x \in \omega^2 | s \equiv x\}$$

where $s \in S_q$. In fact, $[s]$ is clopen in this topology for $s \in S_q$. We work with the Cantor space, but the results of §2 carry over to the real line since there is a Borel isomorphism of low complexity between the Cantor space and the real line.

In general, we use $i, j, k, l, m, n$ to range over $\omega$, $p, q, r, s, t, u$ to range over $S_q$, $v, w, x, y, z$ to range over $\omega^2$, and $A, B, C, D, E$ to range over subsets of $\omega^2$. Let $\sigma : S_q \to S_q$ be order preserving, i.e. $\forall i, \forall j, \forall t (i(t) = \sigma(j) \equiv \sigma(t)).$ Then

* This work formed a part of my Ph. D. thesis [6]. My warm thanks to Professor John Addison, my adviser, for helpful suggestions on this and other problems.
define $f_\alpha$ by $f_\alpha(x) = \bigcup \sigma(x \in \alpha)$. The map $f_\alpha$ is continuous on its domain, which is a $\mathcal{G}_\delta$ set; conversely, if $\text{dom}(f)$ is a $\mathcal{G}_\delta$ subset of $\mathbb{N}$ and $f_\circ \text{dom}(f) \to \mathbb{N}$ is continuous on $\text{dom}(f)$, then $f = f_\alpha$ for some order preserving $\sigma$: $\mathcal{S}_x \to \mathcal{S}_y$.

Our proofs descend from an idea of William Wadge. For $A, B \subseteq \mathbb{N}$, let $A \subseteq^*_\omega B$ iff $\exists f \subseteq^*_\omega \omega (f$ is continuous $\land f^{-1}_\omega(B) = A)$. If $\mathcal{G}$ is a class of subsets of $\omega^\omega$, $\mathcal{G}$ is a pointclass if $\forall A, B \in \mathcal{G}$, $A \subseteq^*_\omega B \Rightarrow A \in \mathcal{G}$. For $\mathcal{G}$ a pointclass, let $\mathcal{I} = \{[\omega^\omega \setminus A] \in \mathcal{G} : A$ is a finite boolean combination of sets in $\mathcal{G}\}$.

Let $\mathcal{G} - \mathcal{AD}$ abbreviate the statement "all games with payoff set in $\mathcal{G}$ are determined" (cf. [5]).

**Lemma 1** (Wadge; cf. [7]). Let $\Gamma$ be a pointclass so that $b \Gamma - \mathcal{AD}$ holds. Then $\forall A, B \in \mathcal{G} (A \subseteq^*_\omega B \mathcal{V} B \subseteq^*_\omega A \mathcal{V} A \subseteq^*_\omega B \mathcal{V} \Gamma(B \mathcal{V} \Gamma(A)) \Rightarrow A \subseteq^*_\omega B$.

The relation $\subseteq^*_\omega$ has been extensively studied; cf. [7] for a survey. The analogous relation for subsets of the real line has not been studied. One obstacle to such a study is that Lemma 1 fails on the real line; let $\Gamma$ be the class of $F_\sigma$ sets, $B$ the class of rationals, and $A$ the closed unit interval for a counterexample.

**2. Borel Isomorphisms.**

For $A, B \subseteq^* \omega^\omega$ let $A \subseteq^*_\omega B$ iff $\exists f (f$ is continuous $\land f^{-1}_\omega(B) = A)$.

**Lemma 2.** $A \subseteq^*_\omega B \Rightarrow \exists h (h$ is a class $(1, 1)$ homeomorphism $\land h^{-1}_\omega(B) = A)$.

**Proof.** One of the usual proofs of the Schroeder-Bernstein Theorem yields this refinement.

In view of Lemma 2, we would like to replace $\subseteq^*_\omega$ by $\subseteq^*_{\omega^\omega}$ in Lemma 1. This can be done if $\mathcal{G}$ is reasonably closed, by which we mean that $\forall A \in \mathcal{G}$, $A \subseteq^*_{\omega^\omega} B$, where $A^*_{\omega^\omega}$ is defined as follows. Fix $x \sigma_\omega$. If $\exists y \in B^*$, then $x \subseteq A^*; \text{ if } \not\exists y \in B^*$. Otherwise $x$ breaks up into infinitely many blocks of zeros separated by blocks of ones. Let $y(0) = 0$ if the $i$th block of zeros in $x$ has even length, and $y(1) = 1$ otherwise. Then $x \subseteq A^* \iff y \in A^*$.

All pointclasses occurring in classical descriptive set theory which properly include the $F_\sigma$ sets are reasonably closed. (A precise description of the ordinals of such classes in the Wadge ordering $\subseteq^*_{\omega^\omega}$ can be abstracted from [6].)

**Lemma 3.** Let $\Gamma$ be reasonably closed and suppose $b \Gamma - \mathcal{AD}$ holds. Then $\forall A, B \in \mathcal{G} \land \Gamma(B \Gamma(A) \Rightarrow A \subseteq^* \omega^\omega B)$.

**Proof.** Let $\Gamma$, $A$, and $B$ be given as above. Then $A^* \subseteq \Gamma$, so by Lemma 1 we have a continuous $f_\circ \subseteq^* \omega^\omega$ so that $f_\circ A \mathcal{V} f_\circ^* B$. Let $f = f_{\langle A, B \rangle}$ where $\sigma$: $\mathcal{S}_x \to \mathcal{S}_y$. We shall define $\tau$: $\mathcal{S}_x \to \mathcal{S}_y$ so that $f_\circ$ is a witness that $A \subseteq^* \omega^\omega B$ and $f_\circ$ is a witness that $A \subseteq^* \omega^\omega B$.

$\tau(\sigma)$ is defined by induction on $\text{dom}(\sigma)$. Let $\tau(\emptyset) = (\emptyset)$. Suppose $\tau(\sigma)$ is defined, where $\text{dom}(\sigma) = \{n \in \mathbb{N} : n \leq i\}$, in such a way that

(i) the last entry in $\sigma(i)$ is 1;
(ii) there are exactly $i$ blocks of zeros in $\tau(\sigma)$, and for $0 \leq n \leq i$, $\tau(n) = 0$ iff the $n$th block of zeros in $\tau(\sigma)$ has even length.

Let $x(0) = y(0) = \tau(\sigma)(0)$. If $n \in \text{dom}(\tau(\sigma))$, let $x(n) = 0$ and $y(n) = 1$ if $n \notin \text{dom}(\tau(\sigma))$. Then $f_{\langle x, A \rangle}$ and $f_{\langle y, B \rangle}$ are incompatible. Let $\tau(\sigma)$ be $\emptyset$ and $\tau(\sigma)(1)$ be the blocks of $x \uparrow k$ and $y \uparrow k$ respectively so that (i) and (ii) remain true; these are easily found. This completes the definition of $\tau$.

By (ii), $\forall x \in A \mathcal{V} f_{\langle x, A \rangle}$. Thus $f_{\langle x, A \rangle}$ is a witness that $A \subseteq^* \omega^\omega B$. But $f_\circ f_{\langle x, A \rangle}$ is one-one by the construction, hence $A \subseteq^* \omega^\omega B$.

**Theorem 1.** Assume $\mathcal{S}_1 - \mathcal{AD}$. Then $\forall A, B \in \mathcal{G} (A \subseteq^* \omega^\omega B \mathcal{V} \mathcal{S}_1(A \subseteq^* \omega^\omega B) \Rightarrow \exists h$: $h$ is a class $(1, 1)$ homeomorphism $\land h^{-1}_\omega(B) = A)$.

**Proof.** $\mathcal{S}_1 - \mathcal{AD}$ implies $\mathcal{S}_2 - \mathcal{AD}$ by [3] and unpublished work of D. Martin. The theorem now follows as once from Lemmas 1, 2 and 3.

**Remark.** Let $2 < \omega < \omega_1$; then $\mathcal{S}_1$ is a reasonably closed pointclass. By [5], $\mathcal{S}_2 - \mathcal{AD}$ holds. Thus any two properly $\mathcal{P}_\omega$ sets are class $(1, 1)$ homeomorphic.

It was known in classical times that any two uncountable Borel sets are Borel isomorphic, however, we conjecture that the refinement just mentioned is not provable by classical methods. That is, we conjecture that $\mathcal{S}_2 - \mathcal{AD}$ is provable in second-order arithmetic together with the statement $\forall x (x < \omega_1$, any two properly $\mathcal{P}_\omega$ sets are class $(1, 1)$ homeomorphic") (cf. [2]).

Lemma 3 is due to L. Harrington. It provides a proof of Theorem 1 much simpler and of more general applicability than our original one.

**3. Homeomorphisms.**

For $A, B \subseteq^* \omega^\omega$ we say $A$ is homeomorphic to $B$ iff $\exists h$: $h$ is a homeomorphism $\land h(A) = B)$. We believe that it would be of some interest to obtain a list of invariants characterizing the homeomorphism types of, say, Borel sets. Theorem 2 is a first step in that direction.

If $\forall x \in \text{Sq}(a \in \mathcal{G} \Gamma)$, then we say that $A$ is everywhere properly $\mathcal{G}$.

**Theorem 2.** Let $\Gamma$ be a reasonably closed pointclass so that $b \Gamma - \mathcal{AD}$ holds, and suppose $A$ and $B$ are each everywhere properly $\mathcal{G}$ and meager. Then $A$ is homeomorphic to $B$.

**Proof.** Let $A, B$, and $\Gamma$ be given as in the hypotheses. Because $A$ and $B$ are meager there are sequences $\{C_i : i < \omega\}$ and $\{D_j : j < \omega\}$ of closed nowhere dense sets so that

(1) $C_i \subseteq C_j$ and $D_j \subseteq D_i$ for $i < j$;
(2) $A = \bigcup \{C_i : i < \omega\}$ and $B = \bigcup \{D_j : j < \omega\}$.

For $a \in \text{Sq}$ let $i(a)$ be $\text{dom}(a)-1$. The fact that $A$ and $B$ are everywhere properly $\mathcal{G}$ enables us to choose these sequences so that for each $i$
(b) \(\forall s \in S\) \((\{[t] \cap C_t = \emptyset \cap (C_t \cap \emptyset) \cap \emptyset = A \cap \{[s] \cap C_{t+1} \cap \emptyset = G - \Gamma^*\})\); 

(c) \(\forall s \in S\) \((\{[t] \cap D_t = \emptyset \cap \{[s] \cap D_{t+1} \cap \emptyset = B \cap \{[s] \cap D_{t+1} \cap \emptyset = G - \Gamma^*\})\); 

The following observation allows one to arrange that (3) holds: if \(X \in \Gamma - \Gamma^*\), then there is a closed nowhere dense set \(C\) such that \(X \cap C \in \Gamma - \Gamma^*\). For let \(Y\) be any non-
here dense set in \(\Gamma - \Gamma^*\), and let \(Y^* \equiv Y\) via the map \(f\). Let \(C = \{f(x) \cap \emptyset = \emptyset\}\). Then the compactness of \(\mathcal{K}\) implies that \(C\) is closed nowhere dense, and \(X \cap C \in \Gamma - \Gamma^*\).

Our plan now is this: the desired homeomorphism will be the limit of a sequence \(\langle h_n \rangle_{n < \omega}\) of partial homeomorphisms constructed by a back and forth argument using Lemma 3. The \(h_n\)'s will satisfy:

(a) \(h_{n+1} \cap \text{dom}(h_n) = h_n\); 

(b) \(h_n\) is a homeomorphism between its domain and range; 

(c) \(\text{dom}(h_{n+1}) = C_{n+1}, \text{ran}(h_{n+1}) = D_{n+1}, \text{ran}(h_{n+1}) = D_{n+1}, \text{ran}(h_{n+1}) = D_{n+1}\); 

(d) \(\forall x \in \text{dom}(h_n)(x \in A \iff h_n(x) \in B)\). 

From (b) and (c) it follows that \(\text{dom}(h_n)\) and \(\text{ran}(h_n)\) are closed sets. We define \(h_n\) by induction on \(n\).

\(n = 0:\) Since \(B \cap D_0 \in \Gamma - \Gamma^*\), Lemma 3 provides an \(h\) which witnesses that \(A \preceq B \cap D_0\). Since \(A\) is dense, \(\text{ran}(h) = \emptyset\). Let \(h_n = h^* \cap C_0\).

\(n < \omega:\) We are given \(h_{n-1}\) satisfying (a)-(d). Then \(\text{dom}(h_{n-1}) = C_{n-1}\). Let 

\(M = M_0 = \{s \in S\} \cap \text{ran}(h_{n-1}) \cap \emptyset = \emptyset \cap \text{ran}(h_{n-1}) = \emptyset\); 

For \(s \in M\), choose \(t_s\) so that \([t_s] \cap C_{s+1} \cap \emptyset \neq \emptyset\) and \([t_s] \cap C_{s+1} \cap \emptyset \neq \emptyset\); such a \(t_s\) exists by (b). Pick \(u_{(s,t)}\) so that \([u_{(s,t)}] \cap C_{s+1} = \emptyset\) and \(A \cap \{[u_{(s,t)}] \cap C_{s+1} \cap \emptyset = \emptyset\); this can be done by (3). It is not hard to see that we may choose \(u_{(s,t)}\) so that \(s \neq s'\) then \([u_{(s,t)}] \neq [u_{(s,t')}] \neq \emptyset\). (This makes use of the reasonable closure of \(\Gamma\).

By Lemma 3 we have for each \(s \in M\) a map \(h^*_s\) so that \([s] \cap B \preceq [u_{(s,t)}] \cap A \cap C\) via \(h^*_s\). Let \(h_n = h_n^{(s)} \cap D_0\). Finally, let 

\(h_n = h_n^{(s)} \cup \bigcup h_n^{(s)}\). 

It is easy to verify all the properties (a)-(d) of \(h_n\), except perhaps (b). For that, since \(h_n\) is clearly one-one, it is enough to see that \(h_n^{-1}\) is continuous on \(h_n\)’s domain, \(D_0\). Let \(h_n^{-1} = h\). Suppose \(\text{dom}(h_n) = D_0\) and \(\lim y_n = y\). Since \(\mathcal{K}\) is compact, it is enough to see that there is no subsequence \(\langle y_{k_m} \rangle_{k_m < \omega}\) of \(\langle y_n \rangle_{k < \omega}\) so that \(\lim h(y_{k_m}) \neq h(y)\). Hence \(\langle y_{k_m} \rangle_{k_m < \omega}\) was such a subsequence. Then for \(k\) large enough, \(y_{k_m} \neq \text{ran}(h_{n-1})\) by property (b) of \(h_{n-1}\); and the definition of \(h_n\). Further, there is no \(x \in M\) so that \(y_n \in [s] \cap [t_{s+1}] \cap \emptyset = \emptyset\), large, by the choice of \(h_n\) and the definition of \(h_n\). Thus by passing to a subsequence of \(\langle y_{k_m} \rangle_{k_m < \omega}\) we may assume that \(\text{dom}(h_{n+1}) = D_n\), where \(s_n \in M\) and \(t_{s+1} = t_{s+1} \neq h_n\).
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$x \in \text{dom}(h)$ and if $t \subseteq t$ and $t \in \{0, 1\}$ then $[t]^{[t]} \cap \text{ran}(h) = \emptyset$. Since $h$ is a homeomorphism we have $s \in Sq$ so that $x \in [s]$ and $h(s) \in [t]$. By the claim, $h(s) \in [t]$ for all $j > k$. Thus $t \subseteq w$ and $t \subseteq z$. But $t$ was arbitrary, so $w = z$.

Now suppose $\forall k \in \{x \notin \text{dom}(h)\}$. Let $n = n + 1$ be odd and so large that if $t \in Sq$ and $\text{dom}(f) \subseteq n + 2$ then $\text{ran}(h) \cap [t] = \emptyset$. Let $l$ be least so that $n + 1 > \text{dom}(h)$. Let $x = x(1) l$, then $x = x(l \cdot l)$. Let $u_0$ be as in the definition of $h_{n+1}$. Then $x_0 \land x_0 = \emptyset$, so $\text{dom}(x_0) > n + 2$ and $u_0 \not\in [n + 1] \not\in [n + 1]$. Say that $u_0 \not\in [n + 1] \not\in [n + 1]$. Now $h_{n+1}(x) \subseteq [u_0]$ for definability, and $h_{n+1}(x) \subseteq [u_0]$. This contradicts our assumption that $\lim x = x$ and $\lim h(x) = w$.

Thus $h$ is well-defined. Clearly $h$ is continuous. A completely symmetric argument shows that $h^{-1}$ is well-defined (i.e., $h$ is one-to-one and onto) and continuous. Thus $h$ is a homeomorphism of $\mathbb{R}$. Clearly $h(\emptyset) = h(\emptyset)$, and thus $h(A) = B, B$

Both of the hypotheses that $A$ and $B$ are everywhere properly $\Gamma$ and that $A$ and $B$ are meager are necessary in Theorem 2. One can, however, replace "meager" by "comeager" by passing to complements. In the case that $\Gamma = X^*_1$, the hypothesis of $h_{\emptyset} - AD$ can also be shown necessary; this follows from [3] and the fact that any properly $X^*_1$ set is Borel isomorphic to a meager, everywhere properly $X^*_1$ set.

We conjecture that Theorem 2 holds for subsets of the real line. Of course, one must formulate the notion of reasonable closure properly in order to prove this.

References


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On fine shape theory II

Abstract. Let $C^p$ be the proper shape category defined by Ball in terms of proper mutation. It is proved that the fine shape category $C^p$ is isomorphic to the full subcategory of $C^p$ whose objects are locally compact metric spaces of the form $X \times R^+$, where $X$ is any compactum and $R^+$ is the space of non-negative reals. The proper movability is defined and a characterization of pointed $\text{FANR}$ in terms of proper movability is obtained.

1. Introduction. The notion of proper shape was introduced originally by Ball and Sher [3]. Their presentation paralleled Borsuk's one [1], using a notion of proper fundamental net in place of Borsuk's fundamental sequence. Ball [1] has established proper shape theory modeled on the ANR-systems of Mardellić-Segal [14], on the mutations of Fox [7] or on the shapings of Mardellić [13]. We mean by $C^p$ the proper shape category in the sense of Ball and Sher [3] and by $C^p$ the proper shape category in the sense of Ball [1]. As presented by Ball [1], whether $C^p$ and $C^p$ are isomorphic is an open question.

Recently the authors [12] have introduced the fine shape category $C^p$ consisting of all compacta and proved that $C^p$ is isomorphic to the full subcategory of $C^p$ whose objects consist of space of the form $X \times R^+$, where $X$ is any compactum and $R^+$ is the space of non-negative reals. In this paper we first prove that $C^p$ is isomorphic to the full subcategory $C^p$ of $C^p$ consisting of spaces of the form $X \times R^+$, $X$ a compactum. This gives a partial answer to Ball's question mentioned above. In the second part of the paper we shall investigate a characteristic property of a pointed FANR in connection with the categories $C^p$, $C^p$ and $C^p$.

We use [12] as general reference for notions and notations. Throughout the paper all spaces are metrizable and maps are continuous. If $X$ is a subset of a space $M$, then we denote by $U(X, M)$ the set of all neighborhoods of $X$ in $M$.

2. $C^p$ and $C^p$. Ball has defined the proper shape categories $C^p$, $C^p$ and $C^p$ whose objects consist of locally compact spaces and proved that these three categories are isomorphic to each other. (Cf. [1, §§ 2, 3 and 5, Theorems 4.6 and 5.3] We shall identify the categories $C^p$, $i = 1, 2, 3$, under Ball's isomorphism and denote it by $C^p$.}