Analytic sets and Borel isomorphisms

by

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Abstract. We show that if all analytic (i.e. Σ_1^1) games are determined, then any two analytic non-Borel sets of reals are carried one to the other by a Borel permutation of the reals. A similar theorem concerning homeomorphisms of the Cantor space is proved.

0. Introduction. We shall show that if all analytic (i.e. Σ_1^1) games are determined, then any two analytic non-Borel sets of reals are Borel isomorphic (carried one to the other by a Borel permutation of the reals). Thus there are, under this determinateness hypothesis, exactly two analytic Borel spaces ([1], p. 46).

The question whether all analytic games are determined has been the subject of much recent investigation. In our opinion, the existing evidence supports an affirmative answer. At any rate, it is shown in [3] that one cannot weaken this determinateness hypothesis here, for it is provable in ZFC that if all analytic non-Borel sets are Borel isomorphic, then all analytic games are determined.

The Borel isomorphisms we produce will actually be class (1, 1) homeomorphisms (cf. [4], p. 374). In § 3 we describe some cases in which full homeomorphisms can be obtained.

1. Preliminaries. Let Sq be the set of all finite sequences of zeros and ones, i.e. $Sq = \bigcup_{n\in\omega} {}^n2$. The Cantor space is the set ${}^\omega 2$ together with the topology generated by basic open sets of the form

$$[s] = \{x \in {}^\omega 2 \mid s \subseteq x\}$$

where $s \in Sq$. In fact, $[s]$ is clopen in this topology for $s \in Sq$. We work with the Cantor space, but the results of § 2 carry over to the real line since there is a Borel isomorphism of low complexity between the Cantor space and the real line.

In general, we use i, j, k, l, m, n to range over ω, p, q, r, s, t, u to range over Sq, v, w, x, y, z to range over ${}^\omega 2$, and A, B, C, D, E to range over subsets of ${}^\omega 2$.

Let $\sigma\colon Sq \to Sq$ be order preserving, i.e. $\forall s, t\big(s \subseteq t \Rightarrow \sigma(s) \subseteq \sigma(t)\big)$. Then

* This work formed a part of my Ph. D. thesis [6]. My warm thanks to Professor John Addison, my adviser, for helpful suggestions on this and other problems.

define f_σ by: $f_\sigma(x) = \bigcup_n \sigma(x \restriction n)$. The map f_σ is continuous on it's domain, which is a G_δ set; conversely, if $\operatorname{dom}(f)$ is a G_δ subset of ${}^\omega 2$ and $f\colon \operatorname{dom}(f) \to {}^\omega 2$ is continuous on $\operatorname{dom}(f)$, then $f = f_\sigma$ for some order preserving $\sigma\colon Sq \to Sq$.

Our proofs descend from an idea of William Wadge. For $A, B \subseteq {}^\omega 2$ let $A \leqslant_w B$ iff $\exists f\colon {}^\omega 2 \to {}^\omega 2$ (f is continuous $\wedge f^{-1}(B) = A$). If Γ is a class of subsets of ${}^\omega 2$, Γ is a pointclass iff $\forall A, B((B \in \Gamma \wedge A \leqslant_w B) \Rightarrow A \in \Gamma)$. For Γ a pointclass, let

$$\check{\Gamma} = \{{}^\omega 2 - A \mid A \in \Gamma\},$$

$$b\Gamma = \{A \mid A \text{ is a finite boolean combination of sets in } \Gamma\}.$$

Let "$\Gamma - AD$" abbreviate the statement "all games with payoff set in Γ are determined" (cf. [5]).

LEMMA 1 (Wadge; cf. [7]). *Let Γ be a pointclass so that $b\Gamma - AD$ holds. Then* $\forall A, B \in \Gamma(A \leqslant_w B \vee B \leqslant_w {}^\omega 2 - A)$, *and thus* $\forall A, B \in \Gamma(B \notin \check{\Gamma} \Rightarrow A \leqslant_w B)$.

The relation $\leqslant_w$ has been extensively studied; cf. [7] for a survey. The analogous relation for subsets of the real line has not been studied. One obstacle to such a study is that Lemma 1 fails on the real line; let Γ be the class of F_σ sets, B the set of rationals, and A the closed unit interval for a counterexample.

2. Borel isomorphisms. For A, $B \subseteq {}^\omega 2$ let $A \leqslant_1 B$ iff $\exists f\colon {}^\omega 2 \to {}^\omega 2$ (f is continuous $\wedge$ f is one-one $\wedge A = f^{-1}(B)$).

LEMMA 2. $\forall A, B \subseteq {}^\omega 2[(A \leqslant_1 B \wedge B \leqslant_1 A) \Rightarrow \exists h\colon {}^\omega 2 \to {}^\omega 2$ *(h is a class* $(1, 1)$ *homeomorphism* $\wedge h^{-1}(B) = A)]$.

Proof. One of the usual proofs of the Schroeder–Bernstein Theorem yields this refinement. ■

In view of Lemma 2, we would like to replace "$\leqslant_w$" by "$\leqslant_1$" in Lemma 1. This can be done if Γ is reasonably closed, by which we mean that $\forall A(A \in \Gamma \Rightarrow A^* \in \Gamma)$, where A^* is defined as follows. Fix $x \in {}^\omega 2$. If $\exists n \forall m > n(x(m) = 0)$, then $x \in A^*$; if $\exists n \forall m > n(x(m) = 1)$, then $x \notin A^*$. Otherwise x breaks up into infinitely many blocks of zeros separated by blocks of ones. Let $y(i) = 0$ if the ith block of zeros in x has even length, and $y(i) = 1$ otherwise. Then $x \in A^*$ iff $y \in A$.

All pointclasses occurring in classical descriptive set theory which properly include the F_σ sets are reasonably closed. (A precise description of the ordinals of such classes in the Wadge ordering $\leqslant_w$ can be abstracted from [6].)

LEMMA 3. *Let Γ be reasonably closed and suppose $b\Gamma - AD$ holds. Then* $\forall A, B((A \in \Gamma \wedge B \in \Gamma - \check{\Gamma}) \Rightarrow A \leqslant_1 B)$.

Proof. Let Γ, A, and B be given as above. Then $A^* \in \Gamma$, so by Lemma 1 we have a continuous $f\colon {}^\omega 2 \to {}^\omega 2$ so that $A^* = f^{-1}(B)$. Let $f = f_\sigma$ where $\sigma\colon Sq \to Sq$. We shall define $\tau\colon Sq \to Sq$ so that f_τ is a witness that $A \leqslant_w A^*$ and $f_\sigma \circ f_\tau$ is a witness that $A \leqslant_1 B$.

$\tau(s)$ is defined by induction on $\operatorname{dom}(s)$. Let $\tau(\emptyset) = \langle 1 \rangle$. Suppose $\tau(s)$ is defined, where $\operatorname{dom}(s) = \{n \mid n < i\}$, in such a way that

(i) the last entry in $\tau(s)$ is 1;

(ii) there are exactly i blocks of zeros in $\tau(s)$, and for $0 \leqslant n < i$, $s(n) = 0$ iff the nth block of zeros in $\tau(s)$ has even length.

Let $x(n) = y(n) = \tau(s)(n)$ if $n \in \operatorname{dom}(\tau(s))$; let $x(n) = 0$ and $y(n) = 1$ if $n \notin \operatorname{dom}(\tau(s))$. Then $f_\sigma(x) \in B$ and $f_\sigma(y) \notin B$, so for some k, $\sigma(x \restriction k)$ and $\sigma(y \restriction k)$ are incompatible. Let $\tau(s^\frown 0)$ and $\tau(s^\frown 1)$ be extensions of $x \restriction k$ and $y \restriction k$ respectively so that (i) and (ii) remain true; these are easily found. This completes the definition of τ.

By (ii), $\forall x(x \in A \Leftrightarrow f_\tau(x) \in A^*)$. Thus f_τ is a witness that $A \leqslant_w A^*$ and $f_\sigma \circ f_\tau$ a witness that $A \leqslant_w B$. But $f_\sigma \circ f_\tau$ is one-one by the construction, hence $A \leqslant_1 B$. ■

THEOREM 1. *Assume* $\Sigma_1^1 - AD$. *Then* $\forall A, B(A, B \in \Sigma_1^1 - \Pi_1^1 \Rightarrow \exists h\colon {}^\omega 2 \to {}^\omega 2$ *(h is a class* $(1, 1)$ *homeomorphism* $\wedge h^{-1}(B) = A))$.

Proof. $\Sigma_1^1 - AD$ implies $b\Sigma_1^1 - AD$ by [3] and unpublished work of D. Martin. The theorem now follows at once from Lemmas 1, 2 and 3. ■

Remark. Let $2 < \nu < \aleph_1$; then Π_ν^0 is a reasonably closed pointclass. By [5], $b\Pi_\nu^0 - AD$ holds. Thus any two properly Π_ν^0 sets are class $(1, 1)$ homeomorphic. It was known in classical times that any two uncountable Borel sets are Borel isomorphic, however, we conjecture that the refinement just mentioned is not provable by classical methods. That is, we conjecture that $\Delta_1^1 - AD$ is provable in second-order arithmetic together with the statement "$\forall \nu$ (if $2 < \nu < \aleph_1$, then any two properly Π_ν^0 sets are class $(1, 1)$ homeomorphic)" (cf. [2]).

Lemma 3 is due to L. Harrington. It provides a proof of Theorem 1 much simpler and of more general applicability than our original one.

3. Homeomorphisms. For $A, B \subseteq {}^\omega 2$ we say A is *homeomorphic* to B iff $\exists h\colon {}^\omega 2 \to {}^\omega 2$ (h is a homeomorphism $\wedge h(A) = B$). We believe that it would be of some interest to obtain a list of invariants characterizing the homeomorphism types of, say, Borel sets. Theorem 2 is a first step in that direction.

If $\forall s \in Sq(A \cap [s] \in \Gamma - \check{\Gamma})$, then we say that A is *everywhere properly* Γ.

THEOREM 2. *Let Γ be a reasonably closed pointclass so that $b\Gamma - AD$ holds, and suppose A and B are each everywhere properly Γ and meager. Then A is homeomorphic to B.*

Proof. Let A, B, and Γ be as in the hypotheses. Because A and B are meager there are sequences $\langle C_i \mid i < \omega \rangle$ and $\langle D_i \mid i < \omega \rangle$ of closed nowhere dense sets so that

(1) $\quad C_i \subseteq C_j \quad$ and $\quad D_i \subseteq D_j \quad$ for $\quad i < j$;

(2) $\quad A \subseteq \bigcup_i C_i \quad$ and $\quad B \subseteq \bigcup_i D_i$.

For $s \in Sq$ let $\bar{s} = s \restriction \operatorname{dom}(s) - 1$. The fact that A and B are everywhere properly Γ enables us to choose these sequences so that for each i

(3) $\quad$ (a) $B \cap D_0 \in \Gamma - \check{\Gamma}$;

(b) $\forall s \in Sq[([s] \cap C_i = \varnothing \wedge [\bar{s}] \cap C_i \neq \varnothing) \Rightarrow A \cap [s] \cap C_{i+1} \in \Gamma - \check{\Gamma}]$;

(c) $\forall s \in Sq[([s] \cap D_i = \varnothing \wedge [\bar{s}] \cap D_i \neq \varnothing) \Rightarrow B \cap [s] \cap D_{i+1} \in \Gamma - \check{\Gamma}]$.

The following observation allows one to arrange that (3) holds: if $X \in \Gamma - \check{\Gamma}$, then there is a closed nowhere dense set C such that $X \cap C \in \Gamma - \check{\Gamma}$. For let Y be any nowhere dense set in $\Gamma - \check{\Gamma}$, and let $Y \leqslant_1 X$ via the map f. Let $C = \{f(x) \mid x \in \text{closure of } Y\}$. Then the compactness of ${}^{\omega}2$ implies that C is closed nowhere dense, and $X \cap C \in \Gamma$ since Γ is reasonably closed. But $Y \leqslant_1 X \cap C$ via f, so $X \cap C \notin \check{\Gamma}$.

Our plan now is this: the desired homeomorphism will be the limit of a sequence $\langle h_n \mid n<\omega\rangle$ of partial homeomorphisms constructed by a back and forth argument using Lemma 3. The h_n's will satisfy:

(a) $h_{n+1} \restriction \operatorname{dom}(h_n) = h_n$;

(b) h_n is a homeomorphism between its domain and range;

(c) $\operatorname{dom}(h_{2n}) = C_{2n}$,
$\operatorname{ran}(h_{2n}) \subseteq D_{2n}$,
$\operatorname{dom}(h_{2n+1}) \subseteq C_{2n+1}$,
$\operatorname{ran}(h_{2n+1}) = D_{2n+1}$;

(d) $\forall x \in \operatorname{dom}(h_n)(x \in A \Leftrightarrow h_n(x) \in B)$.

(From (b) and (c) it follows that $\operatorname{dom}(h_n)$ and $\operatorname{ran}(h_n)$ are closed sets.) We define h_n by induction on n.

$n = 0$: Since $B \cap D_0 \in \Gamma - \check{\Gamma}$, Lemma 3 provides an h which witnesses that $A \leqslant_1 B \cap D_0$. Since A is dense, $\operatorname{ran}(h) \subseteq D_0$. Let $h_0 = h \restriction C_0$.

n odd: We are given h_{n-1} satisfying (a)–(d). Thus $\operatorname{dom}(h_{n-1}) = C_{n-1}$. Let

$$M = M_n = \{s \in Sq \mid [\bar{s}] \cap \operatorname{ran}(h_{n-1}) \neq \varnothing \wedge [s] \cap \operatorname{ran}(h_{n-1}) = \varnothing\}.$$

For $s \in M$, choose t_s so that $[t_s] \cap C_{n-1} \neq \varnothing$ and $h_{n-1}([t_s]) \subseteq [\bar{s}]$; such a t_s exists by (b). Pick $u_s \supseteq t_s$ so that $[u_s] \cap C_{n-1} = \varnothing$ and $A \cap [u_s] \cap C_n \in \Gamma - \check{\Gamma}$; this can be done by (3). It is not hard to see that we may choose u_s so that if $s \neq s'$ then $[u_s] \cap [u_{s'}] = \varnothing$. (This makes use of the reasonable closure of Γ.)

By Lemma 3 we have for each $s \in M$ a map h'_s so that $[s] \cap B \leqslant_1 [u_s] \cap A \cap C_n$ via h'_s. Let $h_s = h'_s \restriction D_n$. Finally, let

$$h_n = h_{n-1} \cup \bigcup_{s \in M} h_s^{-1}.$$

It is easy to verify all the properties (a)-(d) of h_n, except perhaps (b). For that, since h_n is clearly one-one, it is enough to see that h_n^{-1} is continuous on it's domain, D_n. Let $h_n^{-1} = h$. Suppose $\forall k(y_k \in D_n)$ and $\lim_{k\to\infty} y_k = y$. Since ${}^{\omega}2$ is compact, it is enough to see there is no subsequence $\langle y'_k \mid k<\omega\rangle$ of $\langle y_k \mid k<\omega\rangle$ so that $\lim_{k\to\infty} h(y'_k)$ exists and $\lim_{k\to\infty} h(y'_k) \neq h(y)$. Suppose $\langle y'_k \mid k<\omega\rangle$ were such a subsequence. Then for k large enough, $y'_k \notin \operatorname{ran}(h_{n-1})$ by property (b) of h_{n-1} and the definition of h_n. Further, there is no fixed $s \in M$ so that $y'_k \in [s]$ for k sufficiently large, by the choice of h_s and the definition of h_n. Thus by passing to a subsequence of $\langle y'_k \mid k<\omega\rangle$ we may assume that $\forall k(y'_k \in [s_k])$, where $s_k \in M$ and $i \neq k \Rightarrow s_i \neq s_k$.

Choose z_k in $\operatorname{ran}(h_{n-1}) \cap [\bar{s}_k]$. Now $\forall i \exists n \forall m>n(i \in \operatorname{dom}(\bar{s}_m))$, and $z_k \restriction \operatorname{dom}(\bar{s}_k) = y_k \restriction \operatorname{dom}(\bar{s}_k)$, so $\lim_{k\to\infty} z_k = y$. Thus $y \in \operatorname{ran}(h_{n-1})$ and $h(y) = h_{n-1}^{-1}(y)$.

Fix k. By (b) for h_{n-1} find l so that if $h_{n-1}(z) \in [y \restriction l]$ then $z \in [h(y) \restriction k]$. Now $y \restriction l \subseteq \bar{s}_i$ for i large enough, and so $h(y) \restriction k \subseteq t_{s_i}$ for i large enough. But $t_{s_i} \subseteq h(y'_i)$ by the definition of h_n. Since k was arbitrary, $\lim_{i\to\infty} h(y'_i) = h(y)$, a contradiction.

n even: We are given h_{n-1} satisfying (a)-(d). Thus $\operatorname{ran}(h_{n-1}) = D_{n-1}$. Let

$$M = M_n = \{s \in Sq \mid \operatorname{dom}(h_{n-1}) \cap [\bar{s}] \neq \varnothing \wedge \operatorname{dom}(h_{n-1}) \cap [s] = \varnothing\}.$$

For $s \in M$, choose t_s so that $h_{n-1}^{-1}([t_s]) \subseteq [\bar{s}]$ and $[t_s] \cap D_{n-1} \neq \varnothing$. Pick $u_s \supseteq t_s$ so that $[u_s] \cap D_{n-1} = \varnothing$ and $B \cap [u_s] \cap D_n \in \Gamma - \check{\Gamma}$. Let the u_s be chosen so that $s \neq s' \Rightarrow [u_s] \cap [u_{s'}] = \varnothing$.

By Lemma 3, let $[s] \cap A \leqslant_1 B \cap [u_s] \cap D_n$ via h'_s. Let $h_s = h'_s \restriction [s] \cap C_n$, and let

$$h_n = h_{n-1} \cup \bigcup_{s \in M} h_s.$$

Properties (a)-(d) are verified as in the case where n is odd.

Now let $h' = \bigcup_n h_n$. Since A is dense, for any $x \in {}^{\omega}2$ we can find $\langle x_k \mid k<\omega\rangle$ so that $\forall k(x_k \in \operatorname{dom}(h'))$ and $\lim_{k\to\infty} x_k = x$. Define $h(x)$ by:

$$h(x) = \lim_{k\to\infty} h'(x_k).$$

We shall show this limit exists and depends only on x.

Claim. *Let $s, t \in Sq$ and $h_n([s]) \subseteq [t]$, where $[s] \cap \operatorname{dom}(h_n) \neq \varnothing$. Suppose that $\forall t' \subseteq t$ $\forall i \in \{0, 1\}$ $(\operatorname{ran}(h_n) \cap [t'^\frown i] \neq \varnothing)$. Then $\forall m>n(h_m([s]) \subseteq [t])$.*

Proof. By induction on $m>n$. First, let m be even, and let

$$x \in [s] \cap (\operatorname{dom}(h_m) - \operatorname{dom}(h_{m-1})).$$

Then $x \in [r]$ for some $r \in M_m$. Since $[s] \cap \operatorname{dom}(h_{m-1}) \neq \varnothing$, $s \subseteq \bar{r}$. But then $t \subseteq t_r$. For if not, then $([t_r] - [t]) \cap \operatorname{ran}(h_{m-1}) \neq \varnothing$ by the claim hypothesis, and so $\exists y(y \notin [s] \wedge h_{m-1}(y) \in [t_r])$ by the induction hypothesis. Since $s \subseteq \bar{r}$, this contradicts the choice of t_r. Since $t \subseteq t_r$, $h_m(x) \in [t]$.

Now let m be odd, and let $x \in [s] \cap (\operatorname{dom}(h_m) - \operatorname{dom}(h_{m-1}))$. Then $x \in [t_r]$ for some $r \in M_m$. Now $h_{m-1}([t_r]) \subseteq [\bar{r}]$ and $h_{m-1}([s]) \subseteq [t]$ and $[t_r] \cap [s] \cap \operatorname{dom}(h_{m-1}) \neq \varnothing$. Thus $[\bar{r}] \cap [t] \neq \varnothing$. But $\bar{r} \nsubseteq t$ by the claim hypothesis and the definition of M_m, hence $t \subseteq \bar{r}$. Thus $h_m(x) \in [t]$. The claim is now proven.

Now fix x; we want to see that $h(x)$ is well-defined. By the compactness of ${}^{\omega}2$, it is enough to see that whenever $\lim_{k\to\infty} y_k = x$ and $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} h'(y_k) = z$ and $\lim_{k\to\infty} h'(x_k) = w$, then $w = z$. So suppose $\langle x_k \mid k<\omega\rangle$, $\langle y_k \mid k<\omega\rangle$, w, and z are a counterexample.

First, suppose $\exists n(x \in \operatorname{dom}(h_n))$. Let $t \subseteq h_n(x)$, $t \in Sq$. Let k be so large that

$x \in \operatorname{dom}(h_k)$ and if $t' \subseteq t$ and $i \in \{0, 1\}$ then $[t^\frown i] \cap \operatorname{ran}(h_k) = \emptyset$. Since h_k is a homeomorphism we have $s \in Sq$ so that $x \in [s]$ and $h_k([s]) \subseteq [t]$. By the claim, $h_j([s]) \subseteq [t]$ for all $j > k$. Thus $t \subseteq w$ and $t \subseteq z$. But t was arbitrary, so $w = z$.

Now suppose $\forall k(x \notin \operatorname{dom}(h_n))$ and $w \upharpoonright n = z \upharpoonright n$ but $w(n) \neq z(n)$. Let k be odd and so large that if $t \in Sq$ and $\operatorname{dom}(t) \leqslant n+2$ then $\operatorname{ran}(h_k) \cap [t] \neq \emptyset$. Let l be least so that $x \upharpoonright l \notin \operatorname{dom}(h_k)$. Let $s = x \upharpoonright l$; then $s \in M_{k+1}$. Let u_s be as in the definition of h_{k+1}. Then $[u_s] \cap D_k = \emptyset$, so $\operatorname{dom}(u_s) > n+2$ and $u_s \upharpoonright (n+1) \neq w \upharpoonright (n+1)$ or $u_s \upharpoonright (n+1) \neq z \upharpoonright (n+1)$. Say that $u_s \upharpoonright (n+1) \neq w \upharpoonright (n+1)$. Now $h_{k+1}([s]) \subseteq [u_s]$ by definition, and $h_j([s]) \subseteq [u_s \upharpoonright n+1]$ for all $j \geqslant k+1$ by the claim. This contradicts our assumption that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} h'(x_k) = w$.

Thus h is well-defined. Clearly h is continuous. A completely symmetric argument shows that h^{-1} is well-defined (i.e. h is one-one and onto) and continuous. Thus h is a homeomorphism of ${}^{\omega}2$. Clearly $h \upharpoonright \operatorname{dom}(h') = h'$, and thus $h(A) = B$. ∎

Both of the hypotheses that A and B are everywhere properly Γ and that A and B are meager are necessary in Theorem 2. One can, however, replace "meager" by "comeager" by passing to complements. In the case that $\Gamma = \Sigma_1^1$, the hypothesis of $b\Gamma - AD$ can also be shown necessary; this follows from [3] and the fact that any properly Σ_1^1 set is Borel isomorphic to a meager, everywhere properly Σ_1^1 set.

We conjecture that Theorem 2 holds for subsets of the real line. Of course, one must formulate the notion of reasonable closure properly in order to prove this.

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On fine shape theory II

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Abstract. Let $\mathfrak{C}_p^B$ be the proper shape category defined by Ball in terms of proper mutation. It is proved that the fine shape category $\mathfrak{C}_f$ is isomorphic to the full subcategory of $\mathfrak{C}_p^B$ whose objects are locally compact metric spaces of the form $X \times R_+$, where X is any compactum and R_+ is the space of non negative reals. The proper movability is defined and a characterization of pointed FANR in terms of proper movability is obtained.

1. Introduction. The notion of proper shape was introduced originally by Ball and Sher [3]. Their presentation paralleled Borsuk's one [1], using a notion of proper fundamental net in place of Borsuk's fundamental sequence. Ball [1] has established proper shape theory modeled on the ANR-systems of Mardešić–Segal [14], on the mutations of Fox [7] or on the shapings of Mardešić [13]. We mean by $\mathscr{C}_p$ the proper shape category in the sense of Ball and Sher [3] and by $\mathscr{C}_p^B$ the proper shape category in the sense of Ball [1]. As presented by Ball [1], whether $\mathscr{C}_p$ and $\mathscr{C}_p^B$ are isomorphic is an open question.

Recently the authors [12] have introduced the fine shape category $\mathscr{C}_f$ consisting of all compacta and proved that $\mathscr{C}_f$ is isomorphic to the full subcategory $\tilde{\mathscr{C}}_p$ of $\mathscr{C}_p$ whose objects consist of space of the form $X \times R_+$, where X is any compactum and R_+ is the space of non-negative reals. In this paper we first prove that $\mathscr{C}_f$ is isomorphic to the full subcategory $\tilde{\mathscr{C}}_p^B$ of $\mathscr{C}_p^B$ consisting of spaces of the form $X \times R_+$, X a compactum. This gives a partial answer to Ball's question mentioned above. In the second part of the paper we shall investigate a characteristic property of a pointed FANR in connection with the categories $\mathscr{C}_f$, $\mathscr{C}_p$ and $\mathscr{C}_p^B$. We use [12] as general reference for notions and notations. Throughout the paper all spaces are metrizable and maps are continuous. If X is a subset of a space M, then we denote by $U(X, M)$ the set of all neighborhoods of X in M.

2. $\mathscr{C}_f$ and $\mathscr{C}_p^B$. Ball has defined the proper shape categories $\mathscr{S}_p^1$, $\mathscr{S}_p^2$ and $\mathscr{S}_p^3$ whose objects consist of locally compact spaces and proved that these three categories are isomorphic to each other. (Cf. [1, §§ 2, 3 and 5, Theorems 4.6 and 5.3].) We shall identify the categories $\mathscr{S}_p^i$, $i = 1, 2, 3$, under Ball's isomorphism and denote it by $\mathscr{C}_p^B$.