Functorial uniform reducibility

by

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Abstract. It is shown that if T is a Horn theory (possibly infinitary) such that every homomorphism of relativised reducts (say to a predicate M and a signature Σ) of models of T extends to a homomorphism of the whole models, then atomic sentences in the language of T are equivalent to positive existential sentences in the language of Σ with all quantifiers relativised to M. This extends to Horn theories a result proved for varieties by Isbell (Israel J. Math. 15 (1973), pp. 185–188). Some other results in infinitary Horn model theory are given, including a compactness theorem.

In [4], Isbell proves: let T be a varietal theory (not necessarily finitary), Σ a class of its operations, and σ an operation which is preserved by Σ-homomorphisms of T-algebras; then $y = \sigma(\bar{x})$ is equivalent in T to a formula of form $\exists\bar{z}\Phi$ where Φ is a conjunction of equations.

This looks like a theorem of logic, but the logical literature does not contain any theorem which it looks like. In fact it is notorious that Beth-type theorems are commonly not true in infinitary logics. Isbell's proof is entirely algebraic, and makes key use of a free T-algebra.

We shall prove a model-theoretic result which yields Isbell's in the case of varieties with rank. The appropriate setting, it seems, is infinitary Horn logic. A surprising amount of elementary model theory goes through in this setting; we give a few examples.

1. Horn formulae. Let $\varkappa$ be a regular cardinal, possibly ω. Similarity types Ω, Σ etc. will consist of sets of operation or relation symbols, each of arity $<\varkappa$. The *$\varkappa$-Horn* and *positive $\varkappa$-Horn* formulae (over Ω) are defined by:

(1) Every atomic formula (with operation and relation symbols all from Ω) is $\varkappa$-Horn and positive $\varkappa$-Horn; this shall include the *absurdity* formula $\perp$ (the empty disjunction).

(2) If φ is positive $\varkappa$-Horn and quantifier-free, and ψ is $\varkappa$-Horn, then $[\varphi \to \psi]$ is $\varkappa$-Horn.

(3) Any conjunction of $<\varkappa$ $\varkappa$-Horn formulae (resp. positive $\varkappa$-Horn formulae) is $\varkappa$-Horn (resp. positive $\varkappa$-Horn).

(4) If φ is $\varkappa$-Horn (resp. positive $\varkappa$-Horn), then so is $\overline{Qx}\varphi$, where $\overline{Qx}$ is a well-ordered string of $<\varkappa$ (resp. $<\omega$) quantifiers $\forall x_i$, $\exists x_j$.

$\neg\varphi$ is $[\varphi \to \bot]$, so that the negation of a positive quantifier-free $\varkappa$-Horn formula is $\varkappa$-Horn. Quantifiers in (4) are interpreted by Skolem functions or games as usual. The $\varkappa$-Horn language over Ω will be written $L(\Omega)$.

In algebraic contexts it is tiresome to have to keep excluding the empty structure. We shall say that a formula is $\exists$*-bounded* if it begins with an existential quantifier. An $\exists$-bounded formula is automatically false in the empty structure (in any sensible semantics); if Ω has individual constants, then every formula is logically equivalent to an $\exists$-bounded one and we can ignore the restriction.

A filter is said to be $\varkappa$*-complete* if it contains infs of all its subsets of cardinality $<\varkappa$. We shall use reduced products over proper $\varkappa$-complete filters; we call such products $\varkappa$*-reduced*. The basic facts about reduced products are in Chang and Keisler [1].

LEMMA 1. *Let I be a non-empty set, D a $\varkappa$-complete proper filter on I, and for each $i \in I$ let A_i be an Ω-structure. Let B be the reduced product of the A_i over D. Then*

(a) *for every $\varkappa$-Horn formula φ and sequence $\bar f/D$ of elements of B,*

$$\{i \in I\colon A_i \models \varphi[\bar f(i)]\} \in D \Rightarrow B \models \varphi[\bar f/D];$$

(b) *for every $\exists$-bounded positive $\varkappa$-Horn formula φ, the same holds with $\Leftarrow$ in place of $\Rightarrow$.*

The proof of Lemma 1 is entirely straightforward. The lemma at once implies:

LEMMA 2 (Compactness). *Let T be a set of $\varkappa$-Horn sentences of $L(\Omega)$, let I be the set of all subsets of T with $<\varkappa$ elements, and suppose every set in I has a model. Then T has a model.*

Proof. Every $S \in I$ has a model A_S. For each $S \in I$, let X_S be $\{S' \in I\colon S \subseteq S'\}$, and put

$$D = \{X \subseteq I\colon X_S \subseteq X \text{ for some } S \in I\}.$$

Then D is a $\varkappa$-complete filter on I; since all X_S are non-empty, D is proper. For each $\varphi \in T$,

$$\langle\{S \in I\colon A_S \models \varphi\} \supseteq X_{\{\varphi\}} \in D.$$

Hence by Lemma 1, the reduced product of the A_S over D is a model of T. This proves the lemma.

A *homomorphism* of Ω-structures, $f\colon A \to B$, is a map which preserves all atomic formulae (and hence all positive existential formulae) from A to B.

LEMMA 3. *Let K be a class of Ω-structures, and A an Ω-structure. Then the following are equivalent:*

(a) *A can be mapped homomorphically into a $\varkappa$-reduced product of structures in K.*

(b) *$A \models \neg\chi$ whenever χ is a sentence of $L(\Omega)$ of form $\exists\bar v \bigwedge S$ (S a set of atomic formulae) and $\neg\chi$ is true throughout K.*

Proof. (a)$\Rightarrow$(b) is by Lemma 1(a). (b)$\Rightarrow$(a): expand Ω by adding constants for all elements of A. By Lemma 2 it now suffices to find, for each set of $<\varkappa$ atomic sentences true in the expansion of A, a model of S which is an expansion of a structure in K. Condition (b) guarantees this can be done, concluding the proof.

Arbitrary Horn theories do not have free models. But they come somewhere near it, as the next lemma shows.

LEMMA 4. *Let T be a theory in $L(\Omega)$. Then T has a model B such that for every E-bounded positive sentence φ of $L(\Omega)$, $B \models \varphi$ iff $T \vdash \varphi$.*

Proof. For every $\exists$-bounded positive φ not entailed by T, let A_φ be a model of T in which φ is false, and let B be the product of these A_φ. Apply Lemma 1(b) to deduce the lemma.

Structures B with the property of Lemma 3 will be called *loose* models of T. Anand Pillay noticed that they can be used to prove the following:

LEMMA 5. *Let T be a theory in $L(\Omega)$, and let S be a non-empty set of $\exists$-bounded positive sentences of $L(\Omega)$ such that $T \vdash \bigvee S$. Then $T \vdash \varphi$ for some $\varphi \in S$.*

Proof. By hypothesis some sentence in S is true in a loose model of T, hence in all models of T.

2. Uniform reduction. Uniform reduction theorems have the following setting:

(+) Ω, Σ and Σ' are similarity types, $\Sigma \subseteq \Sigma' \subseteq \Omega$, M is a 1-ary relation symbol of Ω, and T is a theory in $L(\Omega)$ such that for any model A of T, M_A is closed under the functions of Σ.

Since M_A is closed under the functions of Σ, the structure $(A \restriction \Sigma) \cap M_A$ is well-defined; we abbreviate it to $A_M \restriction \Sigma$. The general problem is: given (+) and further conditions on T, how do the properties of $A \restriction \Sigma'$ depend on those of $A_M \restriction \Sigma$? (See Feferman [2].)

We present three related uniform reduction theorems. Like Isbell's theorem they rely on some degree of functoriality.

THEOREM 1. *Let T be a theory in $L(\Omega)$ such that (+) holds. Suppose that for every pair A, B of models of T, each homomorphism $h\colon A_M \restriction \Sigma \to B_M \restriction \Sigma$ extends to a homomorphism $h^*\colon A \restriction \Sigma' \to B \restriction \Sigma'$. Then each sentence φ of $L(\Sigma')$ of form $\exists\bar w \bigwedge T$ (T a set of atomic formulae of $L(\Sigma')$) is equivalent in T to a sentence of form $\exists\bar v[\bigwedge M\bar v \wedge \bigwedge S]$ where S is a set of atomic formulae of $L(\Sigma)$. ($\bigwedge M\bar v$ is an abbreviation for $\bigwedge\{Mv_\gamma\colon \gamma < \alpha\}$.)*

Proof. Let φ be a sentence of the stated form. We may assume $T \cup \{\varphi\}$ is consistent, for otherwise φ is equivalent to $\bot$ in T. Let Φ be the set of sentences χ of form $\exists\bar v[\bigwedge M\bar v \wedge \bigwedge S]$, where S is a set of atomic formulae of $L(\Sigma)$, such that T, $\varphi \vdash \chi$. It suffices to show that $T, \Phi \vdash \varphi$ and invoke compactness.

Let C be an Ω-structure which is a model of T, Φ; let the Ω-structure A be a loose model of $T \cup \{\varphi\}$. Suppose $A_M \restriction \Sigma \models \exists \bar{v} \bigwedge S$, where S is a set of atomic formulae of $L(\Sigma)$. Then $A \models \exists \bar{v} [\bigwedge M\bar{v} \wedge \bigwedge S]$, and so T, $\varphi \vdash \exists \bar{v} [\bigwedge M\bar{v} \wedge \bigwedge S]$ since A is loose. Hence $C \models \exists \bar{v} [\bigwedge M\bar{v} \wedge \bigwedge S]$ and so $C_M \restriction \Sigma \models \exists \bar{v} \bigwedge S$. By Lemma 3, there is therefore a $\varkappa$-reduced power B of C and a homomorphism $h\colon A_M \restriction \Sigma \to B_M \restriction \Sigma$. Since T is $\varkappa$-Horn, we have $B \models T$ by Lemma 1. Hence by assumption, h extends to a homomorphism $h^*\colon A \restriction \Sigma' \to B \restriction \Sigma'$. Now $A \models \varphi$ and φ is a positive existential $\varkappa$-Horn sentence of $L(\Sigma')$; so $B \models \varphi$ and hence $C \models \varphi$ by Lemma 1 again. Hence T, $\Phi \vdash \varphi$ as required. The theorem is proved.

Isbell's theorem in [4], when the similarity type is restricted to be a set, is essentially the special case of Theorem 1 where T is a set of identities together with $\forall v Mv$.

THEOREM 2. *Let T be a theory in $L(\Omega)$ such that* (+) *holds. Suppose that*

(a) *the class of structures $A_M \restriction \Sigma$ (A a model of T) is closed under substructures,*

(b) *for every pair A, B of models of T, each embedding $e\colon A_M \restriction \Sigma \to B_M \restriction \Sigma$ extends to an embedding $e^*\colon A \restriction \Sigma' \to B \restriction \Sigma'$.*

Then each atomic sentence of $L(\Sigma')$ is equivalent in T to a quantifier-free sentence of $L(\Sigma)$.

Proof. Let φ be an atomic sentence of $L(\Sigma')$. Let Φ be the set of all quantifier-free formulae χ of $L(\Sigma)$ such that $T, \varphi \vdash \chi$. It suffices to show that $T, \Phi \vdash \varphi$ and appea to compactness.

Let C be a model of $T \cup \Phi$. Let S be the set of all atomic or negated atomic sentences ψ of $L(\Sigma)$ such that $C \models \psi$. We claim that the set $T \cup S \cup \{\varphi\}$ is consistent. For otherwise T, φ, $S_1 \vdash \bigvee S_2$, where S_1 is the set of atomic sentences of $L(\Sigma)$ true in C, and S_2 is the set of atomic sentences of $L(\Sigma)$ false in C. We may apply Lemma 5 and compactness to find $\chi \in S_2$ and a set S' of $<\varkappa$ sentences of S_1 such that T, $\varphi \vdash [\bigwedge S' \to \chi]$. Then $[\bigwedge S' \to \chi] \in \Phi$, which contradicts the choice of C.

Thus $T \cup S \cup \{\varphi\}$ has a model A. By (a) there is an Ω-structure D which is a model of T such that $D_M \restriction \Sigma$ is the minimal substructure of $A_M \restriction \Sigma$, whose elements are all named by terms of $L(\Sigma)$. The inclusion of $D_M \restriction \Sigma$ in $A_M \restriction \Sigma$ extends to an embedding of $D \restriction \Sigma'$ in $A \restriction \Sigma'$ by (b), so that $D \models \varphi$. Since every element of $D_M \restriction \Sigma$ is named by a closed term of $L(\Sigma)$, and $D_M \restriction \Sigma \models S$, there is an embedding $e\colon D_M \restriction \Sigma \to C_M \restriction \Sigma$. By (b) again, e extends to an embedding $e^*\colon D \restriction \Sigma' \to C \restriction \Sigma'$. Hence $C \models \varphi$ as required. The theorem is proved.

A suitable combination of the proofs of these two theorems gives the following:

THEOREM 3. *Under the hypotheses of Theorem* 2, *suppose further that*

(c) *for every pair A, B of models of T, each homomorphism $h\colon A_M \restriction \Sigma \to B_M \restriction \Sigma$ extends to a homomorphism $h^*\colon A \restriction \Sigma' \to B \restriction \Sigma'$.*

Then each atomic sentence of $L(\Sigma')$ is equivalent in T to a conjunction of atomic sentences of $L(\Sigma)$.

3. Remarks. Using $\varkappa$-reduced products, one can carry a surprising amount of first-order model theory up into infinitary Horn logic. For example the following can both be proved by adapting Lemma 3 above:

FACT 1. *Let K be a class of Ω-structures. Then the following are equivalent:*

(a) *K is closed under isomorphism, substructures and $\varkappa$-reduced products.*

(b) *K is the class of models of a set of universal sentences of $L(\Omega)$.*

FACT 2. *Let T be a theory in $L(\Omega)$, and φ a formula of $L(\Omega)$ whose negation is equivalent in T to a disjunction of $<\varkappa$ formulae of $L(\Omega)$. Then φ is preserved in substructures for models of T iff φ is equivalent in T to a universal formula of $L(\Omega)$.*

It is known that some restriction on φ is necessary for the conclusion of Fact 2: in $L_{\omega_1\omega_1}$ Tarski gives the example of the sentence which expresses "There are at most countably many things".

With a similar restriction on φ one can also prove Lyndon's theorem: φ is preserved in homomorphic images iff φ is equivalent to a positive formula. But the proof of this result is much harder and will appear elsewhere.

Kueker [3] has independently noted that $\varkappa$-reduced products can be used to get compactness results for infinitary Horn sentences. Gonzalo Reyes tells me that Theorem 1 says the same thing about Horn logic as Theorem 1.1 of his paper [5] with Makkai says (in categorical language) about "logical categories".

References

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