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Uniform quotients of metric spaces

by

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Abstract. Uniform quotients of (pseudo-) metric spaces are investigated and several conditions implying the pseudometrizability of such quotients are given. Most important, however, it is shown that pseudometrizability is not preserved under uniform quotient maps, thus answering in the negative a question of Himmelberg.

1. Introduction. The question of when a given class of maps preserves metrizability of a topological space has long been of interest in general topology. Recently, questions relating to the preservation of (pseudo-) metrizability have arisen in connection with various types of maps between proximity and uniform spaces (see, for example, [1], [4] and [8]). In [4] Himmelberg investigates uniform quotient maps and provides necessary and sufficient conditions for such maps to preserve pseudometrizability. Left open in [4], however, is the question of whether uniform quotient maps, in general, preserve this property.

In Section 3 of this paper we construct a metric space having a non-metrizable (T_2) uniform quotient and, in doing so, answer Himmelberg's question in the negative. In section 4 we provide several conditions sufficient for a uniform quotient map to preserve pseudometrizability.

2. Preliminaries. Let d be a pseudometric on a set X. For a positive real ε set $B(d,\varepsilon) = \{(x,y)\colon d(x,y)<\varepsilon\}$. Let $\mathcal{U}(d)$ denote the uniformity determined d, i.e. the uniformity with base $\{B(d,\varepsilon)\colon \varepsilon>0\}$.

For a uniformity $\mathcal{U}$ on X, $\mathcal{G}(\mathcal{U})$ will denote the gage of $\mathcal{U}$ [5, p. 189].

Let $[X, \mathcal{U}]$ be a uniform space, Y a set and $\varrho\colon X\to Y$ a surjection. The *quotient uniformity*, determined by $[X, \mathcal{U}]$ and ϱ, is defined to be the largest uniformity $\mathcal{V}$ on Y such that $\varrho\colon [X,\mathcal{U}]\to[Y,\mathcal{V}]$ is uniformly continuous. This uniformity will be indicated by $\mathcal{U}_\varrho$ and, when Y is endowed with $\mathcal{U}_\varrho$, ϱ will be called a *uniform quotient map*.

The formulation in [6] of the quotient uniformity is essential to the definitions of the spaces in § 3. This formulation, with minor modifications, is outlined below.

Let $\mathscr{D}$ be the set of diadic rationals in [0, 1] and let $\mathscr{A}$ denote the family of all maps $\alpha\colon \mathscr{D} \to [0, 1]$ satisfying:

(ri) $\alpha(r) = 0$ if and only if $r = 0$,

(rii) $q \leqslant r$ implies $\alpha(q) \leqslant \alpha(r)$, and

(riii) $r+s \leqslant 1$ implies $\alpha(r)+\alpha(s) \leqslant \alpha(r+s)$.

Given a pseudometric d on X, $n \in N$ and $\alpha \in \mathscr{A}$, let $d[n, \alpha]$ be the set of all $(u, v) \in Y \times Y$ such that for some $m \in N$ there exist $r_1, \ldots, r_m \in \mathscr{D}$ and $a_1, \ldots, a_{m+1} \in X$ satisfying the following conditions:

(Q1) $\Sigma_1^m r_i \leqslant 2^{1-n}$,

(Q2) $\varrho(a_1) = u$, $\varrho(a_{m+1}) = v$, and

(Q3) $d([a_i], [a_{i+1}]) < \alpha(r_i)$, $i = 1, \ldots, m$, where $[x] = \varrho^{-1}(\varrho(x))$, $x \in X$.

If $r_1, \ldots, r_m \in \mathscr{D}$ and $a_1, \ldots, a_{m+1} \in X$ satisfy (Q1)–(Q3), we say that $(r_1, \ldots, r_m)$ and $([a_1], \ldots, [a_{m+1}])$ are *$d[n, \alpha]$-related to* (u, v).

2.1. THEOREM [7, 2.10]. *For each base $\mathscr{E}$ for $\mathscr{G}(\mathscr{U})$ the collection*

$$\mathscr{B}_1(\mathscr{E}) = \{d[n, \alpha]\colon d \in \mathscr{E}, n \in N, \alpha \in \mathscr{A}\}$$

is a base for $\mathscr{U}_\varrho$.

For the purpose of simplifying the arguments of § 3, an alternate base for $\mathscr{U}_\varrho$ will now be associated with each base for $\mathscr{G}(\mathscr{U})$ (2.3).

2.2. *Let $\alpha\colon \mathscr{D} \to [0, 1]$ be monotone increasing with $\alpha(r) = 0$ if and only if $r = 0$. Then $\beta\colon r \to r\alpha(r)$, $r \in \mathscr{D}$, is a member of $\mathscr{A}$.*

2.3. THEOREM. *Let $\mathscr{A}' = \{\alpha \in \mathscr{A}\colon \alpha(r) \leqslant r^2, r \in \mathscr{D}\}$. For each base $\mathscr{E}$ for $\mathscr{G}(\mathscr{U})$, the collection $\mathscr{B}_2(\mathscr{E}) = \{d[1, \alpha]\colon d \in \mathscr{E}, \alpha \in \mathscr{A}'\}$ is a base for $\mathscr{U}_\varrho$.*

Proof. Let $d[n, \alpha] \in \mathscr{B}_1(\mathscr{E})$ and define $\beta\colon \mathscr{D} \to [0, 1]$ by $r \to r^2\alpha(r2^{1-n})$. According to 2.2, $\beta \in \mathscr{A}'$, hence $d[1, \beta] \in \mathscr{B}_2(\mathscr{E})$. Furthermore, if $(r_1, \ldots, r_m)$ and $([a_1], \ldots, [a_{m+1}])$ are $d[1, \beta]$-related to (u, v) then $(r_1 2^{1-n}, \ldots, r_m 2^{1-n})$ and $([a_1], \ldots, [a_{m+1}])$ are $d[n, \alpha]$-related to (u, v).

We conclude this section with several properties of the collection $\mathscr{A}'$ defined in 2.3.

2.4. *Let $\{\beta_n\}$ be a sequence in $\mathscr{A}'$ such that $\beta_{n+1}(r) \leqslant \beta_n(r)$ for all $r \in \mathscr{D}$ and all $n \in N$. Then there exists $\gamma \in \mathscr{A}'$ such that*

(i) *$\gamma(r) < 2^{-1}\beta_n(2^{-n}) \leqslant 2^{-2n-1}$ for all $r \leqslant 2^{-n}$, and*

(ii) *$\gamma(r) < 2^{-1}r$ for $r \neq 0$.*

Proof. The right-hand inequality in (i) holds since $\beta_n \in \mathscr{A}'$. Set $I_1 = [2^{-1}, 1]$ and, for $n \geqslant 2$, set $I_n = [2^{-n}, 2^{-n+1})$. Define $\beta\colon \mathscr{D} \to [0, 1]$ as follows:

$$\beta(r) = \begin{cases} 0 & \text{if } r = 0, \\ 2^{-2}\beta_n(2^{-n}) & \text{if } r \in I_n. \end{cases}$$

Since β is a decreasing function, it follows from 2.2 that $\gamma\colon r \to r^2\beta(r)$ is in $\mathscr{A}'$. Furthermore, if $r \leqslant 2^{-n}$,

$$\gamma(r) \leqslant \gamma(2^{-n}) = 2^{-2n}\beta(2^{-n}) < 2^{-1}\beta_n(2^{-n}).$$

Thus γ satisfies (i). Finally, $r < 2^{-1}$ implies $\gamma(r) < 2^{-1}r$ and $r \in I_1$ implies $\gamma(r) \leqslant \beta(1) = 2^{-2}\beta_1(2^{-1}) \leqslant 2^{-3} < 2^{-1}r$.

2.5. *If $\alpha, \beta \in \mathscr{A}'$ then $\alpha \wedge \beta \in \mathscr{A}'$.*

3. Main example. Provided in this section is an example of a metric space which has a non-pseudometrizable uniform quotient. Although the existence of such a space is not unexpected, the difficulties associated with its construction might be. Some indication of the inherent nontrivial nature of such an example is given in Section 4; in particular, see 4.8 and 4.9.

For each $\beta \in \mathscr{A}'$ and $n \in N$ let $X(\beta, n)$ denote the unit interval [0, 1] and let $P(\beta, n)$ be a fixed set of points $a_1 = 0 < a_2 < \ldots < a_{2^n+1} = 1$ satisfying

(ci) $|a_{i+1} - a_i| = 2^{-1}\beta(2^{-n})$ for each odd integer i, $1 \leqslant i \leqslant 2^{n+1}$, and

(cii) $|a_{i+1} - a_i| > 2^{-n}$ for each even integer i, $1 \leqslant i \leqslant 2^{n+1}$.

The existence of such a set $P(\beta, n)$ follows from the assumption that $\beta(r) \leqslant r^2$ for all $r \in \mathscr{D}$. Let $B_1 = \{0\}$, $B_{2^n+1} = \{1\}$ and for $k = 2, 3, \ldots, 2^n$, let

$$B_k = [a_{2k-2}, a_{2k-1}].$$

Define the equivalence relation $R(\beta, n)$ on $X(\beta, n)$ according to $(x, y) \in R(\beta, n)$ if and only if $x = y$ or $\{x, y\} \in B_k$, $1 \leqslant k \leqslant 2^n+1$, and let $Y(\beta, n)$ denote the resulting quotient set.

Now let X denote the disjoint union of the collection $\{X(\beta, n)\colon \beta \in \mathscr{A}', n \in N\}$ and let d be the metric on X defined by $d(x, y) = |x-y|$ if x and y are in the same $X(\beta, n)$, $d(x, y) = 1$ otherwise. Finally, let Y denote the disjoint union of the quotient sets $Y(\beta, n)$ and let ϱ be the natural map of X onto Y.

It will soon be proved that the (Hausdorff) uniform space $[Y, \mathscr{U}(d)\varrho]$ is not metrizable. First, however, we wish to note several facts concerning each of the sets $X(\beta, n)$.

3.1. *For the subsets $B_1, \ldots, B_{2^n+1}$ of $X(\beta, n)$ defined above, the following hold:*

(a) $d(B_k, B_{k+1}) = 2^{-1}\beta(2^{-n}) \leqslant 2^{-2n-1}$,

(b) $\sum_{k=1}^{2^n} d(B_k, B_{k+1}) \leqslant 2^{-n}$,

(c) *for $k = 2, \ldots, 2^n$, $\operatorname{diam}\{B_k\} > 2^{-n}$, and*

(d) $d(B_k, B_{k+j}) > (j-1)2^{-n}$.

3.2. *For each $\beta \in \mathscr{A}'$ and $n \in N$, $(\varrho(0), \varrho(1)) \in d[1, \beta]$ where 0 and 1 are the endpoints of $X(\beta, n)$.*

Proof. If $r_i = 2^{-n}$ for $i = 1, \ldots, 2^n$, then $(r_1, \ldots, r_{2^n})$ and $(B_1, \ldots, B_{2^n+1})$ are $d[1, \beta]$-related to $(\varrho(0), \varrho(1))$ (3.1 (a)).

3.3. *For* $0, 1 \in X(\beta, n)$ *and* $\gamma \in \mathscr{A}'$, $(\varrho(0), \varrho(1)) \in d[1, \gamma]$ *if and only if there exist* $r_1, \ldots, r_s \in \mathscr{D}$ *and a subset* $\{i_1 = 1 < i_2 < \ldots < i_{s+1} = 2^n+1\}$ of $\{1, 2, \ldots, 2^n+1\}$ *such that* $(r_1, \ldots, r_s)$ *and* $(B_{i_1}, \ldots, B_{i_{s+1}})$ *are* $d[1, \gamma]$*-related to* $(\varrho(0), \varrho(1))$.

Proof. Necessity. Let $(q_1, \ldots, q_k)$ and $(C_1, \ldots, C_{k+1})$ be $d(1, \gamma]$-related to $(\varrho(0), \varrho(1))$, where each C_i is of the form $[c_i]$. A procedure will now be given by which $(q_1, \ldots, q_k)$ and $(C_1, \ldots, C_{k+1})$ can be "reduced" to sets having the required properties.

(1) If $C_i = C_j$, $j > i$, then $q_i, \ldots, q_{j-1}$ and $C_{i+1}, \ldots, C_j$ can be dropped from $(q_1, \ldots, q_k)$ and $(C_1, \ldots, C_{k+1})$, respectively, and the resulting sets will be $d[1, \gamma]$-related to $(\varrho(0), \varrho(1))$.

(2) If C_j is a singleton for some $j \in \{2, 3, \ldots, k\}$, then

$$d(C_{j-1}, C_{j+1}) \leqslant d(C_{j-1}, C_j) + d(C_j, C_{j+1}) \leqslant \gamma(q_{j-1} + q_j).$$

Thus if q_{j-1} is replaced by $q_{j-1}+q_j$ and q_j and C_j are deleted from $(q_1, \ldots, q_k)$ and $(C_1, \ldots, C_{k+1})$, respectively, the newly defined sets will again be $d[1, \gamma]$-related to $(\varrho(0), \varrho(1))$.

Using the procedures described in (1) and (2) above, we can select a subset $(q_{i_1}, \ldots, q_{i_s})$ of $(q_1, \ldots, q_k)$ and a pairwise disjoint subcollection $(C_{i_1}, \ldots, C_{i_s})$ of $(C_1, \ldots, C_{k+1})$ where the newly chosen sets are $d[1, \gamma]$-related to $(\varrho(0), \varrho(1))$ and $C_{i_j} \in \{B_1, \ldots, B_{2^n+1}\}$ for $j = 1, \ldots, s$.

3.4. *Let* $\gamma, \beta \in \mathscr{A}'$ *satisfy* $\gamma(r) < 2^{-1} r$ *for all* $r \neq 0$ *and* $\gamma(r) < 2^{-1}\beta(2^{-n})$ *for* $r \leqslant 2^{-n}$. *Then* $(\varrho(0), \varrho(1)) \notin d[1, \gamma]$ *where* 0 *and* 1 *are the endpoints of* $X(\beta, n)$.

Proof. Assuming the contrary, select $(r_1, \ldots, r_s)$ and $(B_{i_1}, \ldots, B_{i_{s+1}})$ as in 3.3. For $j = 1, \ldots, s$ set $k_j = (i_{j+1} - i_j) - 1$ and let $m = \sum_{j=1}^{s} k_j$. Let $K = \{k_1, \ldots, k_s\}$, $K_1 = \{k_j \in K\colon k_j \neq 0\}$ and $K_2 = K \setminus K_1$.

A. $m \leqslant 2^{n-1} - 1$: By 3.1 (d),

$$k_j \cdot 2^{-n} < d(B_{i_j}, B_{i_{j+1}}) < \gamma(r_j) < 2^{-1} r_j,$$

hence $r_j > k_j \cdot 2^{-n+1}$ for each j. Therefore,

$$1 \geqslant \sum_{j=1}^{s} r_j > \sum_{j=1}^{s} k_j \cdot 2^{-n+1} = m \cdot 2^{-n+1}.$$

B. $2^n - 2m \leqslant |K_2| < 2^n$: Let $H = \{1, \ldots, 2^n\}$ and for each $k_j \geqslant 1$ let

$$H_{k_j} = \{k\colon i_j \leqslant k \leqslant i_j + k_j\}.$$

Now observe that $|H_{k_j}| < 2k_j$ and $K_2 = H \setminus \bigcup H_{k_j}$.

C. $\sum \{r_j\colon k_j \in K_2\} \leqslant 1 - m \cdot 2^{-n+1}$: Suppose $k_j \in K_1$. Then $d(B_{i_j}, B_{i_{j+1}}) \geqslant k_j \cdot 2^{-n}$ (3.1 (d)) and, since $\gamma(r) < 2^{-1} r$, it follows that $r_j \geqslant k_j \cdot 2^{-n+1}$. Thus

$$\sum \{r_j\colon k_j \in K_1\} \geqslant \sum \{k_j \cdot 2^{-n+1}\colon k_j \in K_1\} = m \cdot 2^{-n+1}.$$

D. For some j, $1 \leqslant j \leqslant s$, $k_j = 0$ and $r_j \leqslant 2^{-n}$: According to B above, $|K_2| \geqslant 2^n - 2m$. If $r_j > 2^{-n}$ for all j such that $k_j \in K_2$, then

$$\sum \{r_j\colon k_j \in K_2\} > 2^{-n}(2^n - 2m) = 1 - m \cdot 2^{-n+1},$$

contradicting C.

Selecting $j \in \{1, \ldots, s\}$ such that $k_j = 0$ and $r_j \leqslant 2^{-n}$, we note that

$$\begin{aligned} d(B_{i_j}, B_{i_{j+1}}) &= 2^{-1}\beta(2^{-n}) \quad (3.1\,(a)) \\ &> \gamma(2^{-n}) \\ &\geqslant \gamma(r_j), \end{aligned}$$

whence $(r_1, \ldots, r_s)$ and $(B_{i_1}, \ldots, B_{i_{s+1}})$ are not $d[1, \gamma]$-related to $(\varrho(0), \varrho(1))$. This completes the proof of 3.4.

3.5. *The uniform quotient* $[Y, \mathscr{U}(d)_\varrho]$ *is not pseudometrizable.*

Proof. Let $\mathscr{E} = \{d\}$ and let $\mathscr{S} = \{d[1, \beta_n]\colon n \in N\}$ be a countable subcollection of $B_2(\mathscr{E})$ (2.3) where $\beta_{n+1} \leqslant \beta_n$ for all n. If $\gamma \in \mathscr{A}'$ satisfies (i) and (ii) of 2.4 then, by 3.2 and 3.4, $d[1, \gamma]$ is not a member of the filter generated by $\mathscr{S}$.

4. Preservation of pseudometrizability. In § 2 the quotient uniformity was described in terms of a fundamental system of entourages. For the purpose of investigating the pseudometrizability of this uniformity, a formulation in terms of pseudometrics on the quotient set will prove most useful.

Throughout this section, $[X, \mathscr{U}]$ will represent an arbitrary uniform space, R an equivalence relation on X and ϱ the natural map of X onto X/R.

For each $d \in \mathscr{G}(\mathscr{U})$ define a pseudometric d_R on X/R by setting

$$d_R(\varrho(a), \varrho(b)) = \inf\{\sum_{i=1}^{n} d([a_i], [a_{i+1}])\colon a = a_1, \ldots, a_{n+1} = b\},$$

$a, b \in X$ (recall that $[x] = \varrho^{-1}\varrho(x)$), and let $\mathscr{B}_R = \{d_R\colon d \in \mathscr{G}(\mathscr{U})\}$.

4.1. THEOREM. *The collection* $\mathscr{B}_R$ *is a base for the gage of* $\mathscr{U}_\varrho$.

Proof. Since $\varrho\colon [X, \mathscr{U}] \to [X/R, d_R]$ is uniformly continuous for each $d \in G(\mathscr{U})$, $\mathscr{B}_R$ is a subcollection of $\mathscr{G}(\mathscr{U}_\varrho)$. Now let e be any member of $\mathscr{G}(\mathscr{U}_\varrho)$ and select $d \in \mathscr{G}(\mathscr{U})$ such that $\varrho\colon [X, d] \to [X/R, e]$ is uniformly continuous. Defining the pseudometric p on X according to

$$p(x, y) = d(x, y) + e(\varrho(x), \varrho(y)),$$

it can easily be shown that p and d are uniformly equivalent, hence $p \in \mathscr{G}(\mathscr{U})$ and $p_R \in \mathscr{B}_R$. Furthermore, $p_R(\varrho(a), \varrho(b)) \geqslant e(\varrho(a), \varrho(b))$ for all $a, b \in X$; for if ε is any positive real,

$$p_R(\varrho(a), \varrho(b))+\varepsilon > \sum_{i=1}^{n} p([a_i], [a_{i+1}]) \geqslant \sum_{i=1}^{n} e(\varrho(a_i), \varrho(a_{i+1})) \geqslant e(\varrho(a), \varrho(b))$$

for some $a_1 = a, a_2, \ldots, a_{n+1} = b$ in X.

As a corollary to the above we have the equivalence of (i) and (iv) of 4.2. The equivalence of (i), (ii) and (iii) is proved in [4, Thm. 7].

4.2. THEOREM. *For a pseudometrizable uniformity $\mathcal{U}$ on X the following are equivalent*:

(i) *$\mathcal{U}_\varrho$ is pseudometrizable.*

(ii) *There exist a pseudometric p on X and a collection Q of pseudometrics on X/R such that $\mathcal{U} = \mathcal{U}(p)$, Q generates $\mathcal{U}_\varrho$, and ϱ is distance decreasing relative to p and e for all $e \in Q$.*

(iii) *There exist a pseudometric p on X and a collection Q of pseudometrics on X/R such that $\mathcal{U} = \mathcal{U}(p)$, Q generates $\mathcal{U}_\varrho$, and for each $e \in Q$ and $\varepsilon>0$ there exists $\delta>0$ such that*

$$\sum_{i=1}^{n} p([a_i], [a_{i+1}])<\delta \quad \textit{implies} \quad \sum_{i=1}^{n} e(\varrho(a_i), \varrho(a_{i+1}))<\varepsilon$$

for all $a_1, \ldots, a_{n+1}$ in X.

(iv) *There exists a pseudometric p on X such that $\mathcal{U} = \mathcal{U}(p)$ and $\mathcal{U}_\varrho = \mathcal{U}(p_R)$.*

4.3. Remark. Using (iii′) below, the equivalence of (i) and (iii) is easily established without constructing the space $[Z, \mathcal{W}]$ in [4]. Observe that (iii)→(iii′)→(iv).

(iii′) There exists a pseudometric p on X and a collection Q of pseudometrics on X/R such that $\mathcal{U} = \mathcal{U}(p)$, Q generates $\mathcal{U}_\varrho$, and for each $e \in Q$ and $\varepsilon>0$ there exists $\delta>0$ such that $B(p_R, \delta) \subseteq B(e, \varepsilon)$.

Suppose now that both $\mathcal{U}$ and $\mathcal{U}_\varrho$ are pseudometrizable, in which case $\mathcal{U}_\varrho = \mathcal{U}(p_R)$ for *some* p that generates $\mathcal{U}$ (4.2). It is natural to ask if $\mathcal{U}_\varrho = \mathcal{U}(p_R)$ for *each* p that generates $\mathcal{U}$. The answer to this (no) is provided in 4.5. Here we make use of a pseudometrizability condition given in [3].

A function f of a uniform space $[X, \mathcal{U}]$ onto a set Y is said to *preserve* the uniformity if the image filter $\{(f\times f)[U]: U \in \mathcal{U}\}$ is a uniformity on Y. Clearly, if a surjection preserves $\mathcal{U}$, the image filter coincides with the quotient uniformity and each base for $\mathcal{U}$ is carried to a base for the quotient. The next result now follows from Theorem 1 of [3].

4.4. THEOREM. *Let $\mathcal{U}$ be a pseudometrizable uniformity on X. Then $\mathcal{U}_\varrho$ is pseudometrizable if either of the following equivalent conditions holds*:

(a) *ϱ: $X \to X/R$ preserves $\mathcal{U}$.*

(b) *For each pseudometric d on X satisfying $\mathcal{U} = \mathcal{U}(d)$ and each $\varepsilon>0$, there exists $\delta>0$ such that $B(d, \delta) \circ R \circ B(d, \delta) \subseteq R \circ B(d, \varepsilon) \circ R$.*

4.5. EXAMPLE. For $n \in N$ let $X_n = \{0\} \cup [1/(n+1)^2, 1/n^2]$. Let X denote the disjoint union of $\{X_n: n \in N\}$ and define metrics d and e on X according to

$$d(x, y) = d(y, x) = \begin{cases} |x-y|, & x, y \in X_n, \\ \displaystyle\sum_{j=1}^{i+1} \frac{1}{n+j}, & x \in X_n,\ y \in X_{n+i} \end{cases}$$

and

$$e(x, y) = \begin{cases} |x^{1/2}-y^{1/2}|, & x, y \in X_n, \\ d(x, y), & x \in X_n,\ y \in X_{n+i}. \end{cases}$$

Observe that d and e are uniformly equivalent metrics on X, i.e. $\mathcal{U}(d) = \mathcal{U}(e)$.

Let 0_n denote the point 0 in X_n. Set $A_1 = X_1 \setminus \{0_1\}$ and for $n>1$ set $A_n = \{0_{n-1}\} \cup X_n \setminus \{0_n\}$. Now let R be the equivalence relation defined by the decomposition $\{A_n: n \in N\}$ and let ϱ denote the natural map of X onto X/R. It will be shown that ϱ preserves $\mathcal{U}(d)$, hence that $\mathcal{U}_\varrho$ is pseudometrizable, and that d_R does not generate $\mathcal{U}_\varrho$.

Given $\varepsilon \in (0, 1)$ select $m \in N$ such that

$$\frac{1}{m}+\frac{1}{m+1}<\frac{1}{3}\varepsilon$$

and set

$$\delta = \frac{1}{(m+1)^2}.$$

Now suppose $(a, b) \in B(d, \delta) \circ R \circ B(d, \delta)$. Then for some $(x, y) \in R$, $d(a, x)<\delta$ and $d(y, b)<\delta$. If $(a, x) \in R$ then

$$(a, b) \in R \circ B(d, \delta) \subseteq R \circ B(d, \varepsilon) \circ R.$$

On the other hand, if $(a, x) \notin R$ and $x, y \in A_n$, then

$$\frac{1}{(n+1)^2} = d(A_n, X \setminus A_n)<\delta = \frac{1}{(m+1)^2},$$

hence $n>m$. It follows that

$$d(x, y)<\operatorname{diam}\{A_n\}<\tfrac{1}{3}\varepsilon.$$

Since $\delta<\frac{1}{3}\varepsilon$, $d(a, b)\leqslant d(a, x)+d(x, y)+d(y, b)<\varepsilon$, i.e. $(a, b) \in B(d, \varepsilon)$. From 4.4 we conclude that ϱ preserves $\mathcal{U}(d)$.

Finally, it will be proved that d_R and e_R are not uniformly equivalent. It is not difficult to show that for $n, q \in N$,

$$d_R(\varrho(0_n), \varrho(0_{n+q})) = \sum_{i=2}^{q+1} \frac{1}{(n+i)^2}$$

and

$$e_R(\varrho(0_n), \varrho(0_{n+q})) = \sum_{i=2}^{q+1} \frac{1}{n+i}.$$

Thus, given $0<\varepsilon<1$ and $\delta>0$ there exist $n, q \in N$ such that

$$(\varrho(0_n), \varrho(0_{n+q})) \in B(d_R, \delta)\setminus B(e_R, \varepsilon).$$

For the remainder of this section we assume that $\mathcal{U}$ is generated by a pseudo-metric d. Provided now are several conditions sufficient for d_R to generated $\mathcal{U}_\varrho$.

4.6. THEOREM. *The uniformity $\mathcal{U}_\varrho$ is generated by d_R if the following property* $(*)$ *is satisfied*:

$(*)$ *There exists $\delta>0$ and $j \in N$ such that for every $\varepsilon<\delta$,*

$$d_R(\varrho(a), \varrho(b))<\varepsilon \quad \textit{implies} \quad \sum_{i=1}^{k} d([a_i], [a_{i+1}])<\varepsilon$$

for some $a = a_1, \ldots, a_{k+1} = b$ where $k \leqslant j$.

Proof. It must be shown that for each $\alpha \in \mathcal{A}$ and $n \in N$ there exists $\varepsilon>0$ such that $d_R(\varrho(a), \varrho(b))<\varepsilon$ implies $(\varrho(a), \varrho(b)) \in d[n, \alpha]$. Select $m \in N$ such that $2^m \geqslant j$ and $\alpha(2^{1-n-m})<\delta$. Set $\varepsilon = \alpha(2^{1-n-m})$ and let $(\varrho(a), \varrho(b)) \in B(d_R, \varepsilon)$. According to $(*)$, $\sum_{i=1}^{n} d([a_i], [a_{i+1}])<\varepsilon$ for some choice of $a = a_1, \ldots, a_{k+1} = b$, $k \leqslant j$. Setting $r_i = 2^{1-n-m}$, $i = 1, \ldots, k$, we see that $d([a_i], [a_{i+1}])<\alpha(r_i)$ and

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} 2^{-m}(2^{1-n}) \leqslant 2^{1-n}.$$

It follows that $(\varrho(a), \varrho(b)) \in d[n, \alpha]$.

4.7. *For distinct $\varrho(a)$ and $\varrho(b)$ in X/R, $d_R(\varrho(a), \varrho(b))<\varepsilon$ if and only if there exist $a_1 = a$, $a_2, \ldots, a_{m+1} = b$ in X such that*

(si) $[a_i] \neq [a_j]$ *for* $i \neq j$,

(sii) $|[a_i]| \geqslant 2$ *for* $1<i<m+1$, *and*

(siii) $\sum_{i=1}^{n} d([a_i], [a_{i+1}])<\varepsilon$.

Proof. Use the procedures (1) and (2) in the proof 3.3.

Recall that a family $\{T_\lambda\colon \lambda \in \Lambda\}$ of subsets of $[X, d]$ is said to be d-discrete if for some $\delta>0$, $d(T_{\lambda_1}, T_{\lambda_2})>\delta$ whenever $\lambda_1 \neq \lambda_2$ [2, 15.15].

4.8. THEOREM. *If $\mathcal{T} = \{[a]\colon |[a]| \geqslant 2\}$ is d-discrete then d_R generates $\mathcal{U}_\varrho$.*

Proof. Select $0<\delta<1$ such that the d-distance between distinct members of $\mathcal{T}$ is at least δ. If $\varepsilon<\delta$ and $a_1, \ldots, a_{m+1}$ satisfy (si)–(siii) of 4.7, then $m \leqslant 2$. We conclude that $(*)$ (4.6) holds for $j = 2$.

4.9. THEOREM. *If $\mathcal{T} = \{[a]\colon |[a]| \geqslant 2\}$ is finite then d_R generates $\mathcal{U}_\varrho$.*

Proof. If $a_1, \ldots, a_{m+1}$ satisfy (si) and (sii) of 4.7, then $m \leqslant |\mathcal{T}|+1$ hence $(*)$ holds for $j = |\mathcal{T}|+1$.

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