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Continua whose hyperspace is a product

by

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Abstract. Let X be a nondegenerate metric continuum. By the *hyperspace of* X is meant $C(X) = \{A\colon A$ is a nonempty subcontinuum of $X\}$ with the Hausdorff metric. An investigation is made of when $C(X)$ is homeomorphic to a cartesian product of nondegenerate continua. Some examples are given using techniques in infinite-dimensional topology and some unsolved problems are stated.

1. Introduction. In [21] are some results concerning the structure of all finite-dimensional continua whose hyperspace and cone are homeomorphic. Among other results, I showed that there are *exactly eight* such hereditarily decomposable continua [21, (1.1)] and that for such continua which are indecomposable, each proper subcontinuum is an arc. In [25] we showed that any finite-dimensional continuum whose hyperspace and suspension are homeomorphic must be an arc. Using [15, 5.4], 9.7 of [10] may be restated as follows:

(1.1) THEOREM [10]. *Let X be a locally connected continuum. If $C(X)$ is a finite-dimensional cartesian product of* (*nondegenerate*) *continua, then X is an arc or a circle* (*and conversely*).

The above-mentioned results provide the principal motivation for the following question:

(Q) For what continua X is $C(X)$ homeomorphic to a cartesian product (of nondegenerate continua)?

In this paper I give some answers to (Q). The next section is devoted to giving some general results, and some complete answers to (Q) in some special cases. In Section 3, I consider the situation when X is locally connected and $C(X)$ is an infinite-dimensional cartesian product. The section contains several examples and Theorem (3.15) which hopefully [see (3.19)] will lead to a characterization completely answering (Q).

I adopt the following notation. The term *nondegenerate* means consisting of more than one point. The letters X, Y, and Z always denote continua (a *continuum* is a nondegenerate compact connected metric space). I refer the reader to [15] for preliminary information about the space $C(X)$. Whenever I say $C(X)$ *is a cartesian product*, I mean that $C(X)$ is homeomorphic to the cartesian product of (nondegenerate) continua. A *simple triod* is a continuum homeomorphic to a figure "T". For each $n = 1, 2, \ldots$, let R^n denote Euclidean n-space,

$$B^n = \{(x_1, x_2, \ldots, x_n) \in R^n:\ \sum_{i=1}^{n} x_i^2 \leqslant 1\},$$

and

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in R^n:\ \sum_{i=1}^{n} x_i^2 = 1\}.$$

Let l_2 denote the Hilbert space of all square-summable sequences of real numbers and let $I_\infty = \prod_{j=1}^{\infty} [0, 2^{-j}]$, where $\prod$ denotes cartesian product. I consider I_∞ as contained in l_2 by inclusion. The symbol "$\times$" denotes cartesian product and $\pi_Y\colon Y \times Z \to Y$ denotes projection. The slash "$\setminus$" denotes complementation for sets. The symbol "$\cong$" means "is homeomorphic to", and cl denotes closure. By a *free arc* in X I mean an arc $A \subset X$ such that $A \setminus \{\text{end points of } A\}$ is an open subset of X.

A *dendroid* is an arcwise connected hereditarily unicoherent continuum. In [6] Borsuk observes that a one-dimensional acyclic continuum is hereditarily unicoherent. Hence, we have the following lemma which we will use several times.

(1.2) Lemma [6]. *If X is one-dimensional, arcwise connected, and acyclic, then X is a dendroid.*

2. General results and the finite-dimensional case. The main result of this section is (2.13) where it is shown that if $C(X)$ is a finite-dimensional cartesian product and X is a-triodic, then X is an arc or a circle. I had hoped to give an affirmative answer to the following question which, by (2.13), is answered affirmatively for the class of a-triodic continua.

(2.0) Question. If $C(X)$ is a finite-dimensional cartesian product, then must X be an arc or a circle?

Since the cone and suspension of X are formed from $X \times [0, 1]$, perhaps the question most closely related to results in [21] and [25] is: For what finite-dimensional continua X is $C(X) \cong X \times [0, 1]$? The answer to this question is in (2.8). Many of the results in this section bear specifically on (2.0), though some give general information [for example, (2.4) and (2.12)]. We begin with the following lemma.

(2.1) Lemma. *If $C(X) \cong Y \times Z$, then Y and Z are each arcwise connected and acyclic.*

Proof. Since $C(X)$ is arcwise connected ([7] or [15]) and acyclic [30, 1.2], $Y \times Z$ is arcwise connected and acyclic. Thus, using projections, we see that Y and Z are each arcwise connected and acyclic.

(2.2) Theorem. *If $C(X) \cong Y \times Z$ where X is chainable or circle-like, then X is an arc or a circle.*

Proof. By [11, Corollary 1], $C(X)$ and, hence, $Y \times Z$ is 2-dimensional. Thus, [13, pp. 33–34], $\dim[Y] = 1 = \dim[Z]$. Therefore, by (2.1) and (1.2), Y and Z are dendroids. Now, since X is an inverse limit of arcs or circles, $C(X)$ [30, 1.1] and, hence, $Y \times Z$ is an inverse limit of 2-cells. Therefore, using the projections from $Y \times Z$ into the 2-cells and using basic properties of inverse limits (notably, [8, 2.8(ii)]), it follows that:

(#) For each $\varepsilon > 0$, there is an ε-map of any subcontinuum of $Y \times Z$ into R^2 (recall that $f\colon A \to B$ is an ε-map if and only if $\operatorname{diam}[f^{-1}(f(a))] < \varepsilon$ for all $a \in A$).

We show that Y and Z must each be arcs. Suppose Y is not an arc. Then, since Y is a dendroid, Y contains a simple triod T [23, Lemma 2]. Let A be an arc in Z. There is a continuum M in $T \times A$ such that M is homeomorphic to

$$\{(x, y, 0) \in R^3:\ x^2 + y^2 \leqslant 1\} \cup \{(0, 0, z) \in R^3:\ 0 \leqslant z \leqslant 1\}.$$

By (#) there is an ε-map of M into R^2 for each $\varepsilon > 0$, but this contradicts [3, Theorem 3]. Hence, Y is an arc. A similar argument shows that Z is an arc. Thus, $C(X)$ is a 2-cell and, hence [15, 4.4 and 5.3], X is an arc or a circle (depending on whether X is chainable or circle-like).

Next I show that if $C(X)$ is a cartesian product then X is decomposable and, if $C(X)$ is a finite-dimensional product, X is hereditarily decomposable. First, the following lemma.

(2.3) Lemma. *If $C(X) \cong Y \times Z$, then $C(X) \setminus \{A\}$ is arcwise connected for any $A \in C(X)$.*

Proof. By (2.1), Y and Z are each arcwise connected. Thus it follows that for any $(y, z) \in [Y \times Z]$, $[Y \times Z] \setminus \{(y, z)\}$ is arcwise connected. The lemma now follows.

(2.4) Theorem. *If $C(X) \cong Y \times Z$, then X is decomposable.*

Proof. The result is a simple consequence of taking $A = X$ in (2.3) and applying 8.2 of [15].

(2.5) Theorem. *If $C(X) \cong Y \times Z$ where Y and Z are each finite-dimensional, then X is hereditarily decomposable and, hence, one-dimensional.*

Proof. Since Y and Z are each finite-dimensional, $Y \times Z$ [13, p. 33] and, hence, $C(X)$ is finite-dimensional. Therefore, (2.3) above and Proposition 5 of [28] may be applied to show that X is hereditarily decomposable. Now, by a result of Bing (4, Theorem 5], X is one-dimensional.

Kelley [15, 5.4] showed that for locally connected continua X, $C(X)$ is finite-dimensional if and only if X is a finite graph. The next lemma is an analogue of [15, 5.4] for dendroids.

(2.6) Lemma. *Let D be a dendroid. Then: $C(D)$ is finite-dimensional if and only if D is a finite graph.*

Proof. By the above-mentioned result [15, 5.4], we need only show that if $C(D)$ is finite-dimensional, then D is locally connected. To do this assume D is not locally connected. If D has only a finite number of ramification points, then one of them, say p, is of infinite order. It then follows that $C(D)$ contains a Hilbert cube (take Y in [22, Theorem 6] to be $\{p\}$ and $\{Y_i\}_{i=1}^{\infty}$ to be a null sequence of arcs in D emanating from p and disjoint except for p). Now, assume that D has an infinite number of ramification points. Let n be a natural number and let $p_1, p_2, \ldots, p_n$ be n distinct ramification points of D. For each $i = 2, 3, \ldots, n$, let A_i be an arc in D with end-points p_1 and p_i. Let $F = \bigcup_{i=2}^{n} A_i$. We extend F to obtain a continuum F^* as follows: For each p_i such that $\mathrm{ord}_{p_i}[F] = j(i) \in \{1, 2\}$, let M_i be the union of $[\{1, 2\} \setminus \{j(i)\}]$ arcs which intersect F and each other (if $j(i) = 1$) only in the point p_i. Let

$$F^* = F \cup [\bigcup \{M_i\colon\ \mathrm{ord}_{p_i}[F] \leqslant 2\}].$$

Evidently, F^* is a finite connected graph and each p_i is a ramification point of F^* such that $\mathrm{ord}_{p_i}[F^*] \geqslant 3$. Hence, by [15, 5.5] (see [10, p. 278]),

$$\dim[C(F^*)] \geqslant 2 + \sum_{i=1}^{n} (\mathrm{ord}_{p_i}[F^*] - 2) \geqslant 2 + n.$$

Thus, $\dim[C(D)] \geqslant 2+n$. Therefore, since n was arbitrary, we have that $C(D)$ is infinite-dimensional. We have now proved the implication in the first sentence of the proof, and the lemma follows.

(2.7) Theorem. *If $C(X) \cong X \times Z$ where X and Z are each finite-dimensional, then X is an arc.*

Proof. By (2.1), X is arcwise connected and acyclic. Therefore, since X is one-dimensional by (2.5), X is a dendroid by (1.2). Now, since $X \times Z$ [13, p. 33] and, hence, $C(X)$ is finite-dimensional, we have by (2.6) that X is a finite graph. Thus, by (1.1), X is an arc or a circle. Therefore, since X is acyclic, X must be an arc.

We mention the following corollary because of its natural relationship to results in [21] and [25].

(2.8) Corollary. *If $C(X) \cong X \times [0, 1]$ where X is finite-dimensional, then X is an arc.*

As a corollary to the proof of (2.7) we have the following result.

(2.9) Corollary. *If $C(X) \cong Y \times Z$ where X is arcwise connected and acyclic and Y and Z are each finite-dimensional, then X is an arc.*

Our final results, (2.13) and (2.14), are extensions of (2.2) (see (2.15)). The next two lemmas will be used to prove (2.12).

(2.10) Lemma. *Let X be an a-triodic continuum such that given any nondegenerate proper subcontinuum E of X, there exists a subcontinuum Y_0 of X such that $Y_0 \cap E \neq \emptyset \neq Y_0 \cap [X \setminus E]$ and $Y_0 \not\supset E$. Then, each proper subcontinuum of X is unicoherent.*

Proof. Since X is a-triodic we have, by [14, Lemma 1], that

(2.10.1) the intersection of any two subcontinua of X can have at most two components.

Now, suppose P is a proper subcontinuum of X such that P is not unicoherent. Let P_1 and P_2 be subcontinua of P such that $P = P_1 \cup P_2$ and $P_1 \cap P_2$ is not connected. By (2.10.1), $P_1 \cap P_2$ has exactly two components Q and R. Let E_1 be a subcontinuum of P_1 such that E_1 is irreducible (as a continuum) with respect to intersecting Q and R. Let $Q_1 = E_1 \cap Q$ and let $R_1 = E_1 \cap R$. Let E_2 be a subcontinuum of P_2 such that E_2 is irreducible with respect to intersecting Q_1 and R_1. Let $M = E_2 \cap Q_1$ and let $N = E_2 \cap R_1$. We note the following facts, each of which is easy to verify:

(2.10.2) $E_1 \cap E_2 = M \cup N$;

(2.10.3) $M \cap N = \emptyset$ (since $M \subset Q$ and $N \subset R$);

(2.10.4) M and N are the two components of $E_1 \cap E_2$ (this follows from (2.10.1) through (2.10.3));

(2.10.5) for each $i \in \{1, 2\}$, E_i is irreducible with respect to being a subcontinuum of P_i which intersects M and N.

Now, let $E = E_1 \cup E_2$. Then, since E is a nondegenerate proper subcontinuum of X, there exists a subcontinuum Y_0 of X such that $Y_0 \cap E \neq \emptyset \neq Y_0 \cap [X \setminus E]$ and $Y_0 \not\supset E$. Let K denote a component of $Y_0 \cap E$. Recall that, by (2.10.1), $Y_0 \cap E$ has atmost two components; hence, using [16, p. 172], we see that there is a subcontinuum Z of Y_0 such that $Z \cap E = K$ and $Z \cap [Y_0 \setminus E] \neq \emptyset$. Now we verify (a) through (c) below.

(a) $K \not\subset E_i$ for any $i \in \{1, 2\}$.

Proof of (a). Suppose $K \subset E_i$ for some i, say $i = 1$. Using [16, p. 172], there are disjoint subcontinua A_1 and A_2 of E_2 such that $A_1 \cap M \neq \emptyset \neq A_1 \cap [E_2 \setminus E_1]$ and $A_2 \cap N \neq \emptyset \neq A_2 \cap [E_2 \setminus E_1]$. Then, since $[Z \cup E_1 \cup A_1 \cup A_2] \setminus E_1$ is the union of three nonempty mutually separated sets, $Z \cup E_1 \cup A_1 \cup A_2$ is a triod in the sense of [5, p. 653]. This contradicts the a-triodicness of X.

(b) $K \not\supset E_i$ for any $i \in \{1, 2\}$.

Proof of (b). Suppose $K \supset E_i$ for some i, say $i = 1$. Then note that $K \supset [M \cup N]$. Suppose $K \cap E_2$ were connected. Then, by (2.10.5), $K \cap E_2 = E_2$ which implies $K \supset E$. Thus, since $Y_0 \supset K$, $Y_0 \supset E$. This is a contradiction and, hence, $K \cap E_2$ is not connected. By (2.10.1), $K \cap E_2$ has exactly two components B_1 and B_2. Now, using [16, p. 172], there are disjoint subcontinua C_1 and C_2 of E_2 such that $C_i \cap B_i \neq \emptyset \neq C_i \cap [E_2 \setminus K]$ for each $i \in \{1, 2\}$. It is easy to see that $[Z \cup C_1 \cup C_2] \setminus K$ is the union of three nonempty mutually separated sets. Hence, $Z \cup C_1 \cup C_2$ is a triod [5, p. 653], a contradiction.

(c) $K \cap M = \emptyset$ or $K \cap N = \emptyset$.

Proof of (c). Suppose $K \cap M \neq \emptyset \neq K \cap N$. If $K \cap E_i$ were connected for some $i \in \{1, 2\}$ then, by (2.10.5), $K \cap E_i = E_i$. Hence, $K \supset E_i$ which contradicts (b). Therefore, $K \cap E_i$ is not connected for any $i \in \{1, 2\}$. Thus, by (2.10.1), $K \cap E_i$ has exactly two components S_i and T_i for each $i \in \{1, 2\}$. Now suppose $S_i \cap M \neq \emptyset \neq S_i \cap N$ for some fixed $i \in \{1, 2\}$. Then, by (2.10.5), $S_i = E_i$ and, thus, $K \supset E_i$. This contradicts (b). Therefore, $S_i \cap M = \emptyset$ or $S_i \cap N = \emptyset$ for each $i \in \{1, 2\}$; similarly, $T_i \cap M = \emptyset$ or $T_i \cap N = \emptyset$ for each $i \in \{1, 2\}$. Thus, since $K \cap M \neq \emptyset \neq K \cap N$, we assume by renaming if necessary that $S_i \cap M \neq \emptyset \neq S_i \cap N$ and $T_i \cap M \neq \emptyset \neq T_i \cap N$, $i \in \{1, 2\}$. Then, $K = [S_1 \cup S_2] \cup [T_1 \cup T_2]$ and $[S_1 \cup S_2] \cap [T_1 \cup T_2] = \emptyset$. This contradicts the connectedness of K. Therefore, (c) is proved.

Now, by (c), we assume without loss of generality that $K \cap M = \emptyset$. By (a) there exists a point $x_i \in [K \setminus E_i]$ for each $i \in \{1, 2\}$. Therefore, since K is connected and $K \cap M = \emptyset$, $K \cap N \neq \emptyset$. Let U be an open subset of E such that $U \supset M$ and $\mathrm{cl}[U] \cap [K \cup N] = \emptyset$. For each $i \in \{1, 2\}$, let D_i be the component of $E_i \setminus U$ containing x_i. By [16, p. 172], $D_i \cap \mathrm{cl}[E_i \cap U] \neq \emptyset$ for each $i \in \{1, 2\}$. Hence, $D_i \setminus [K \cup N] \neq \emptyset$ for each $i \in \{1, 2\}$. It now follows easily that

$$[Z \cup N \cup D_1 \cup D_2] \setminus [K \cup N]$$

is the union of three nonempty mutually separated sets. Therefore, since $K \cup N$ is a subcontinuum of $Z \cup N \cup D_1 \cup D_2$, we have that $Z \cup N \cup D_1 \cup D_2$ is a triod [5, p. 653]. This contradicts the a-triodicness of X, and the lemma is proved.

The following lemma is part of (4.4) of [18].

(2.11) LEMMA [18]. *Let E be a nondegenerate proper subcontinuum of X. If $C(X) \setminus \{E\}$ is arcwise connected, then there exists a subcontinuum Y_0 of X such that $Y_0 \cap E \neq \emptyset \neq Y_0 \cap [X \setminus E]$ and $Y_0 \not\supset E$.*

The next theorem states nothing about products. It gives some facts about a-triodic continua X such that $C(X)$ is not arcwise disconnected by (removing) any of its points ($\neq X$). In [18, Section 4] an extensive study is made of points of $C(X)$ which do, or do not, arcwise disconnect $C(X)$. We include (2.12) because it seems to be of interest especially in relation to results in [18].

(2.12) THEOREM. *Let X be an a-triodic continuum such that $C(X) \setminus \{E\}$ is arcwise connected for each proper subcontinuum E of X. Then:*

(2.12.1) *Each proper subcontinuum of X is unicoherent.*

(2.12.2) *If each proper subcontinuum of X is decomposable, then each proper subcontinuum of X is chainable and, hence, X is either chainable, circle-like, or indecomposable.*

(2.12.3) *If X is hereditarily decomposable, then X is chainable or circle-like.*

Proof. From (2.10) and (2.11) we have (2.12.1). The first part of (2.12.2) follows from (2.12.1) and [5, Theorem 11]. The second part of (2.12.2) follows from the first part of (2.12.2) and [14, Theorem 4]. Finally, (2.12.2) implies (2.12.3).

The next result answers (2.0) for the case of a-triodic continua.

(2.13) THEOREM. *If $C(X) \cong Y \times Z$ where Y and Z are each finite-dimensional and X is a-triodic, then X is an arc or a circle.*

Proof. By (2.3) and (2.5), all the hypotheses of (2.12.3) are satisfied and so X is chainable or circle-like. Hence, by (2.2), X is an arc or a circle.

(2.14) THEOREM. *If $C(X) \cong Y \times Z$ where $\dim[C(X)] = 2$, then X is an arc or a circle.*

Proof. By [29, Corollary 1], X is a-triodic. Hence, by (2.13), X is an arc or a circle.

I mention that $C(X)$ can be two-dimensional without X being chainable or circle-like. For example, such is the case for any compactification of a half line with a circle as the remainder.

(2.15) Remark. In [11, Corollary 1] we showed that if X is chainable or circle-like, then $\dim[C(X)] = 2$. Hence, (2.14) is an extension of (2.2), as is (2.13).

(2.16) Remark. I mention an application of (2.12) outside the realm of hyperspaces. Assume X is an arcwise connected a-triodic continuum. Then it is easy to show that all the hypotheses of (2.12.3) are satisfied and, hence, it follows that X is an arc or an arcwise connected circle-like continuum. This is a different proof, than that given in [27], for an important special case of Theorem 2 of [27].

3. The infinite-dimensional case. In this Section I consider the problem of determining when $C(X)$ is an infinite-dimensional cartesian product. I will restrict my attention to the case when X is locally connected.

First observe that if X is a (nondegenerate) locally connected continuum (i.e., a *Peano continuum*) such that X contains no free arc, then $C(X) \cong I_\infty$ [9, Theorem 2]. Now, I give an example of a Peano continuum X containing a free arc L such that $C(X) \cong Y \times [0, 1]$ where Y is infinite-dimensional. For use in the example, I adopt the following notation. Let $\tau_j: I_\infty \to [0, 2^{-j}]$ denote projection, let $\theta = (0, 0, \ldots, 0, \ldots) \in l_2$, and let $J_\infty = \prod_{j=1}^{\infty} [a_j, b_j] \subset l_2$ where $[a_1, b_1] = [-2^{-1}, 0]$ and $[a_j, b_j] = [0, 2^{-j}]$ for each $j > 1$. We say that a function f is a *homeomorphism of $(A_1, A_2, \ldots, A_n)$ onto $(B_1, B_2, \ldots, B_n)$*, written $f: (A_1, A_2, \ldots, A_n) \xrightarrow{\text{onto}} (B_1, B_2, \ldots, B_n)$, if and only if f maps A_i onto B_i homeomorphically for each $i = 1, 2, \ldots, n$. I refer the reader to [1] for definition of and some facts about property Z.

(3.1) EXAMPLE. Let $X = B^2 \cup L$ where $L = \{(x, 0) \in R^2 \colon 1 \leqslant x \leqslant 2\}$. Let $\Omega = \{A \in C(L) \colon (1, 0) \in A\}$, $\Gamma = C(B^2)$, $C(X; A) = \{K \in C(X) \colon K \cap L = A\}$ for each $A \in \Omega$, and $\Lambda = \bigcup \{C(X); A\} \colon A \in \Omega\}$. Note that $\Gamma \cap \Lambda = C(X; \{(1, 0)\})$. We first prove (i) through (iii) below.

(i) $\Gamma \cap \Lambda$ has property Z in Γ (see [1, p. 366]).

Proof of (i). Let U be a homotopically trivial, nonempty open subset of Γ, note that $U \setminus [\Gamma \cap \Lambda] \neq \emptyset$ (because $\Gamma \cap \Lambda$ is nowhere dense in Γ), and let

$f\colon S^k \to U\setminus[\Gamma \cap \Lambda]$ be continuous. Since U is homotopically trivial, there is an extension $F\colon B^{k+1} \to U$ of f. For each ε, $0<\varepsilon<1$, let

$$r_\varepsilon\colon B^2 \xrightarrow{\text{onto}} B^2\setminus\{(x,y)\in B^2\colon [(x-1)^2+y^2]^{1/2}<\varepsilon\}$$

be a retraction; for each r_ε, let $\hat{r}_\varepsilon\colon \Gamma \to \Gamma$ be the "induced" map given by $\hat{r}_\varepsilon(M) = \{r_\varepsilon(m)\colon m\in M\}$ $(= r_\varepsilon[M])$ for each $M\in\Gamma$. Now, since $F(B^{k+1})$ is a compact subset of U and $f(S^k)$ is a compact subset of $U\setminus[\Gamma\cap\Lambda]$, it is easy to see that there exists ε_0 such that $\hat{r}_{\varepsilon_0}[F(B^{k+1})]\subset U$ and $\hat{r}_{\varepsilon_0}(M) = M$ for each $M\in f(S^k)$. Hence, $\hat{r}_{\varepsilon_0}\circ F$ is an extension of f and $\hat{r}_{\varepsilon_0}\circ F$ maps B^{k+1} into $U\setminus[\Gamma\cap\Lambda]$. This proves (i) [1, p. 366].

(ii) There is a homeomorphism

$$h_1\colon (\Gamma, \Gamma\cap\Lambda, \{(1,0)\}) \xrightarrow{\text{onto}} (I_\infty, \tau_1^{-1}(0), \theta).$$

Proof of (ii). By [9, Theorem 2], there is a homeomorphism $k_1\colon \Gamma \xrightarrow{\text{onto}} I_\infty$. By (i), $k_1[\Gamma\cap\Lambda]$ has property Z in I_∞. Therefore, since $k_1[\Gamma\cap\Lambda]\cong\tau_1^{-1}(0)$ [9, Theorem 3] and $\tau_1^{-1}(0)$ has property Z in I_∞ [1, Theorem 8.2 with $K=\emptyset$], there is a homeomorphism $k_2\colon (I_\infty, k_1[\Gamma\cap\Lambda]) \xrightarrow{\text{onto}} (I_\infty, \tau_1^{-1}(0))$ [1, Corollary 10.3]. Now, by the homogeneity of the cube $\tau_1^{-1}(0)$, there is a homeomorphism

$$k_3\colon \big(\tau_1^{-1}(0), k_2\circ k_1(\{(1,0)\})\big) \xrightarrow{\text{onto}} (\tau_1^{-1}(0), \theta).$$

Let $\bar{k}_3\colon I_\infty \xrightarrow{\text{onto}} I_\infty$ be the extension of k_3 given by:

$$\bar{k}_3((x_1, x_2, \ldots, x_n, \ldots)) = (x_1, y_2, y_3, \ldots)$$

where

$$(0, y_2, y_3, \ldots, y_n, \ldots) = k_3((0, x_2, x_3, \ldots, x_n, \ldots)).$$

Let $h_1 = \bar{k}_3\circ k_2\circ k_1$.

(iii) Let $h = h_1|[\Gamma\cap\Lambda]$ (the restriction of h_1 to $\Gamma\cap\Lambda$) and let

$$\alpha = \{(t,0,0,\ldots)\in J_\infty\colon -2^{-1}\leqslant t\leqslant 0\}.$$

Then h can be extended to a homeomorphism $h_2\colon (\Lambda, \Omega) \xrightarrow{\text{onto}} (J_\infty, \alpha)$.

Proof of (iii). It is easy to see that Ω is an arc and that $\{(1,0)\}$ is an end point of Ω; hence, there is a homeomorphism $g_1\colon (\Omega, \{(1,0)\}) \xrightarrow{\text{onto}} (\alpha, \theta)$. Now note that for each $K\in\Lambda$, $[K\cap B^2]\in[\Gamma\cap\Lambda]$ and the function $g_2\colon \Lambda \xrightarrow{\text{onto}} \Gamma\cap\Lambda$, given by $g_2(K) = K\cap B^2$ for each $K\in\Lambda$, is a continuous retraction. Also, it is easy to see that the function $g_3\colon \Lambda \xrightarrow{\text{onto}} \alpha$, given by $g_3(K) = g_1(K\cap L)$ for each $K\in\Lambda$, is continuous. Define $h_2\colon \Lambda\to J_\infty$ by $h_2(K) = g_3(K)+h\circ g_2(K)$ for each $K\in\Lambda$, where $+$ means vector addition in l_2. Since g_2 is a retraction of Λ onto $\Gamma\cap\Lambda$ and $g_3(K) = \theta$ whenever $K\in[\Gamma\cap\Lambda]$, h_2 is an extension of h. It is also easy to verify that h_2 is a homeomorphism of (Λ, Ω) onto (J_∞, α). This proves (iii).

Now, let

$$G_1(K) = \begin{cases} h_1(K), & K\in\Gamma, \\ h_2(K), & K\in\Lambda. \end{cases}$$

Then, we have from (ii) and (iii) that $G_1\colon (\Gamma\cup\Lambda, \Omega) \xrightarrow{\text{onto}} (I_\infty\cup J_\infty, \alpha)$ is a homeomorphism. Let

$$\beta = \{(t,0,0,\ldots)\in[I_\infty\cup J_\infty]\colon -2^{-1}\leqslant t\leqslant 2^{-1}\}.$$

Since α and β each have property Z in the cube $I_\infty\cup J_\infty$ (use [1, Theorem 8.2 with $K=\emptyset$]), there is [1, Corollary 10.3] a homeomorphism

$$G_2\colon (I_\infty\cup J_\infty, \alpha) \xrightarrow{\text{onto}} (I_\infty\cup J_\infty, \beta).$$

Let $G = G_2\circ G_1$. Now note that $C(X) = \Gamma\cup\Lambda\cup C(L)$, $C(L)$ is a 2-cell such that $C(L)\cap[\Gamma\cup\Lambda] = \Omega$, and Ω is an arc in the manifold boundary of $C(L)$. Let $\Delta = \{(t_1, t_2, 0, 0, \ldots)\in l_2\colon -2^{-1}\leqslant t_1\leqslant 2^{-1}$ and $-1\leqslant t_2\leqslant 0\}$. Let g denote the restriction of G to Ω. We note the following simple consequence of the Schönflies theorem [16, p. 535]: If A is a 2-cell in R^2, B is any 2-cell, λ is an arc in the manifold boundary of B, and f is a homeomorphism of λ onto an arc in the manifold boundary of A, then f can be extended to a homeomorphism of all of B onto A. Hence, since Δ may be considered as in R^2, g can be extended to a homeomorphism $G^*\colon C(L) \xrightarrow{\text{onto}} \Delta$. Now, define $e\colon C(X) \xrightarrow{\text{onto}} [I_\infty\cup J_\infty\cup\Delta]$ by

$$e(K) = \begin{cases} G(K), & K\in[\Gamma\cup\Lambda], \\ G^*(K), & K\in C(L). \end{cases}$$

It is easy to verify that e is a homeomorphism of $C(X)$ onto $I_\infty\cup J_\infty\cup\Delta$. Therefore, letting $\gamma = \{(2^{-1}, t_2, 0, 0, \ldots)\in\Delta\}$, we see that $C(X)\cong[\tau_1^{-1}(2^{-1})\cup\gamma]\times\text{arc}$.

The techniques employed in (3.1) can be used to verify that $C(X)$ is a cartesian product for some other Peano continua X containing free arcs. Such is the case, for example, for the continua X in (3.2) and (3.3) below.

(3.2) Example. Let $X = B^2\cup L_1\cup L_2$ where $L_1 = \{(x,0)\in R^2\colon 1\leqslant x\leqslant 2\}$ and $L_2 = \{(0,y)\in R^2\colon 1\leqslant y\leqslant 2\}$.

(3.3) Example. Let $X = B^2\cup[\bigcup_{i=1}^{3} L_i]$ where L_1 and L_2 are as in (3.2) and $L_3 = \{(x, -x+2)\in R^2\colon 0\leqslant x\leqslant 2\}$.

However, as we will see in (3.16) and (3.17), there are simple Peano continua, closely related to those above, such that their hyperspace is not a cartesian product.

A necessary condition on Peano continua X in order that $C(X)$ be a cartesian product is given in (3.15). In order to prove (3.15), a number of preliminary lemmas will be proved.

(3.4) Lemma. *If A is a (nondegenerate) Peano continuum such that there is an open subset U of $C(A)$ such that $A\in U$ and $\dim[U]\leqslant 2$, then A is an arc or a circle.*

Proof. Let F be a finite connected graph in A. Using the arcwise connectedness of A, it follows easily that there is a finite connected graph $F^* \supset F$ such that $F^* \in U$. Let $U^* = C(F^*) \cap U$. Then, U^* is an open subset of $C(F^*)$ such that $F^* \in U^*$ and, since $U^* \subset U$, $\dim[U^*] \leqslant 2$. Since F^* is a finite connected graph, these properties of $C(F^*)$ and U^* easily imply that F^* is an arc or a circle. Thus, since $F^* \supset F$, F is an arc or a circle. Since F was an arbitrary finite connected graph in A, we have proved that any finite connected graph in A is an arc or a circle. Hence, A is a Peano continuum which contains no simple triod. Therefore, A is an arc or a circle.

(3.5) LEMMA. *Let X be a Peano continuum. If $A \in C(X)$ such that there is an open subset U of $C(X)$ such that $A \in U \cong R^2$, then A is an arc or a circle; furthermore, A is a circle if and only if $A = X$.*

Proof. We first show that

(3.5.1) A is nondegenerate.

Proof of (3.5.1). Suppose $A = \{a\}$. Since X is locally connected and U is an open subset of $C(X)$ such that $\{a\} \in U$, there is a connected open subset V of X such that $a \in V$ and, letting $A_1 = \mathrm{cl}[V]$, $C(A_1) \subset U$. Hence, since $U \cong R^2$, $C(A_1)$ is embeddable in R^2 and so, by [24, Theorem 2.3], A_1 is an arc or a circle. Thus, $C(A_1) \cong B^2$ and $\{a\}$ is in the manifold boundary of $C(A_1)$. Let $W = \{K \in C(X)\colon K \subset V\}$. Since V is an open subset of X, W is an open subset of $C(X)$ and, since $W \subset C(A_1) \subset U$, $W \subset U$. Hence, W is an open subset of $U \cong R^2$. But, since $\{a\} \in W \subset C(A_1) \cong B^2$ and $\{a\}$ is in the manifold boundary of $C(A_1)$, we have a contradiction to basic facts about open subsets of R^2. Therefore, we have proved (3.5.1).

Next we show that

(3.5.2) A is locally connected.

Proof of (3.5.2). Suppose there are three distinct points a_1, a_2, and a_3 of A such that a_i is arcwise accessible from $X \setminus A$ for each $i \in \{1, 2, 3\}$. Then there are three mutually disjoint arcs J_1, J_2, and J_3 in X such that for each $i \in \{1, 2, 3\}$, a_i is an endpoint of J_i and $J_i \cap A = \{a_i\}$. Moreover, by choosing appropriate subarcs of J_i if necessary, we assume without loss of generality that $\Lambda \subset U$ where

$$\Lambda = \{B \in C(X)\colon A \subset B \subset A \cup [\bigcup_{i=1}^{3} J_i]\}.$$

It follows easily that Λ is a 3-cell (see the proof in [15, 5.3]). Since $\Lambda \subset U \cong R^2$, we have a contradiction. Hence, at most two points of A are arcwise accessible from $X \setminus A$. From this, and the local arcwise connectivity of X [16, p. 254], it follows easily that each point of A which is not arcwise accessible from $X \setminus A$ has a neighborhood base in A consisting entirely of connected open subsets of X. Thus, A is locally connected at all but at most two points. Therefore [31, 12.3, p. 19], A is locally connected. This proves (3.5.2).

Now we complete the proof of (3.5). By (3.5.1), (3.5.2), and properties of U in (3.5), A and $U_0 = U \cap C(A)$ satisfy the hypotheses of (3.4). Hence, by (3.4), A is an arc or a circle. This proves the first part of (3.5). We prove the second part. Assume A is a circle. Suppose $A \neq X$. Then, since X is arcwise connected, there is an arc α in X from a point of $X \setminus A$ to a point of A. Without loss of generality, assume $[A \cup \alpha] \in U$. Then, by (3.4) applied to the Peano continuum $A \cup \alpha$ and $U \cap C(A \cup \alpha)$, we have a contradiction (since $A \cup \alpha$ is not an arc or a circle). Therefore, $A = X$. Conversely, assume A is not a circle; then, by the first part of (3.5), A is an arc. Suppose $A = X$. Then $C(A) = C(X)$ is a 2-cell with A in its manifold boundary. Hence, no U as in (3.5) exists. Therefore, $A \neq X$. This completes the proof of (3.5).

NOTATION. In what follows it will be convenient to let $\mathring{J}$ denote $J \setminus \{p, q\}$ whenever J is an arc with end points p and q.

(3.6) LEMMA. *Let X be a Peano continuum such that X is not a circle and let $A \in C(X)$. Then, there is an open subset U of $C(X)$ such that $A \in U \cong R^2$ if and only if there exists a free arc J in X such that $A \subset \mathring{J}$ and A is nondegenerate.*

Proof. Assume U is an open subset of $C(X)$ such that $A \in U \cong R^2$. Therefore, since X is not a circle, we have by (3.5) that A is an arc. It follows using [16, p. 254] that A is free in X. Let p and q denote the end points of A. By (3.5), $A \neq X$. Thus, since A is free in X, p or q must be a limit point of $X \setminus A$. Therefore, using [16, p. 254], we see that there is an arc B_1 from a point b_1 of $X \setminus A$ to a point $x \in A$ such that $B_1 \cap A = \{x\}$ and such that $[A \cup B_1] \in U$. Since A is free in X, $x = p$ or $x = q$, say $x = p$. Again using (3.5) and [16, p. 254], we see that $A \cup B_1$ is a free arc in X. Since $C(A \cup B_1)$ is a 2-cell with A in its manifold boundary and since $A \in U \cong R^2$, it follows that $C(A \cup B_1) \cap U$ can not be a neighborhood in $C(X)$ of A. Hence, since U is an open subset of $C(X)$, it follows that $[A \cup B_1] \setminus \{b_1\}$ is not an open subset of X. Therefore, since $A \cup B_1$ is a free arc in X, it now follows that q is a limit point of $X \setminus A$. So, using [16, p. 254], we obtain an arc B_2 from a point of $X \setminus A$ to q such that $B_2 \cap A = \{q\}$ and such that $[A \cup B_1 \cup B_2] \in U$. Again, $A \cup B_1 \cup B_2$ is a free arc in X. Letting J denote $A \cup B_1 \cup B_2$, we see that $A \subset \mathring{J}$. This proves half of (3.6); the other half is clear from examining the structure of the hyperspace of any arc.

(3.7) Remark. Let

$$X_1 = \mathrm{cl}(\{(x, \sin[1/x])\colon 0 < x \leqslant 1\}),$$
$$X_2 = \mathrm{cl}(\{(x, \sin[1/x])\colon 0 < x \leqslant 2^{-1}\}),$$
$$X_3 = \{(0, y) \in R^2\colon -2 \leqslant y \leqslant -1\},$$
$$X_4 = \{(0, y) \in R^2\colon -\tfrac{3}{2} \leqslant y \leqslant 0\}.$$

Then: By taking $A = X_1 \cup X_3$, we see the necessity for the requirement in (3.4) that A be a Peano continuum. By taking $X = X_1 \cup X_3$ and $A = X_2 \cup X_4$, we see the necessity for the requirement in (3.5) that X be a Peano continuum. By taking $X = X_1 \cup X_3$ and $A = X_4$, we see the necessity for the requirement in (3.6) that X be a Peano continuum.

(3.8) LEMMA. *Let Y and Z be continua (not necessarily locally connected) and let $(y, z) \in [Y\times Z]$. Then, there is an open subset U of $Y\times Z$ such that $(y, z) \in U \cong R^2$ if and only if there are free arcs $J(y)$ and $J(z)$ in Y and Z respectively such that $y \in \mathring{J}(y)$ and $z \in \mathring{J}(z)$.*

Proof. Assume U is an open subset of $Y\times Z$ and $(y, z) \in U \cong R^2$. Then, it is easy to see that there is a connected open subset V of $Y\times Z$ such that $(y, z) \in V$, $\mathrm{cl}(V) \cong B^2$, and $[K\times L] \subset U$ where $K = \pi_Y[\mathrm{cl}(V)]$ and $L = \pi_Z[\mathrm{cl}(V)]$. Thus, K and L are nondegenerate locally connected continua such that $K\times L$ is embeddable in R^2. Hence neither K nor L can contain a simple triod, and so K and L are both arcs, or one of them is an arc and the other is a circle (comp. [22, Lemma 2]). Therefore, by choosing a smaller V if necessary, we assume without loss of generality that K and L are both arcs. Now observe that:

(3.8.1) $\pi_Y[V]$ is an open subset of Y and $\pi_Z[V]$ is an open subset of Z;

(3.8.2) $\pi_Y[V]\times\pi_Z[V]$ is an open subset of $U \cong R^2$ and $K\times L$ is a 2-cell in U such that $(\pi_Y[V]\times\pi_Z[V]) \subset (K\times L)$.

It follows from (3.8.2) and elementary properties of the topology of R^2 that

(3.8.3) $\pi_Y[V] \subset \mathring{K}$ and $\pi_Z[V] \subset \mathring{L}$.

Also, using (3.8.1), we see that

(3.8.4) $\pi_Y[V]$ (resp., $\pi_Z[V]$) is a dense connected open subset of K (resp., L).

Hence, by (3.8.3) and (3.8.4), $\pi_Y[V] = \mathring{K}$ and $\pi_Z[V] = \mathring{L}$. It now follows from (3.8.1) that, by taking $J(y) = K$ and $J(z) = L$, half the lemma is proved. The other half is clear.

(3.9) For any continuum M let

$$F(M) = \{A \in C(M)\colon A \subset \mathring{J} \text{ for some free arc } J \text{ in } M \text{ and } A \text{ is nondegenerate}\}.$$

Observe that

(3.9.1) $\bigcup F(M) = \{p \in M\colon p \in \mathring{J} \text{ for some free arc } J \text{ in } M\}$.

(3.10) LEMMA. *If X is a Peano continuum such that X is not a circle and if $h\colon C(X) \xrightarrow{\text{onto}} Y\times Z$ is a homeomorphism, then $h[F(X)] = [\bigcup F(Y)]\times[\bigcup F(Z)]$.*

Proof. The equality is a direct consequence of (3.6), (3.8), and (3.9.1).

(3.11) LEMMA. *Let M be a Peano continuum and let K be a nonempty connected subset of $\bigcup\{\mathring{J}_\alpha\colon \alpha \in I\}$, where J_α is a free arc in M for each α in the index set I. Then, one of the following holds:*

(i) *$K \subset \mathring{J}$ for some free arc J in M;*

(ii) *M is a circle;*

(iii) *$K = C\setminus\{c\}$ where C is a circle and $C \cap \mathrm{cl}(M\setminus C) = \{c\}$.*

Proof. Let $p \in K$ and let $N = \{x \in K\colon$ there is a free arc A in M such that $p, x \in \mathring{A}\}$. We first show:

(3.11.1) $N = K$.

Proof of (3.11.1). Since $p \in N$, $N \neq \emptyset$. Now, let $x_0 \in N$. Then, there is a free arc A_0 in M such that $p, x_0 \in \mathring{A}_0$. Since $\mathring{A}_0 \cap K$ is an open subset of K and since $x_0 \in [\mathring{A}_0 \cap K] \subset N$, N is an open subset of K. Next we prove N is a closed subset of K. To do this let $\{x_i\}_{i=1}^{\infty}$ be a sequence in N such that $x_i \to x_\infty \in K$. Since $x_\infty \in K \subset \bigcup\{\mathring{J}_\alpha\colon \alpha \in I\}$, there exists $\beta \in I$ such that $x_\infty \in \mathring{J}_\beta$. Since $\mathring{J}_\beta$ is an open subset of M and $x_i \to x_\infty \in \mathring{J}_\beta$, there exists k such that $x_k \in \mathring{J}_\beta$. Since $x_k \in N$, there exists a free arc A_1 in M such that $p, x_k \in \mathring{A}_1$. Consider the set $\mathring{A}_1 \cup \mathring{J}_\beta$. First assume $\mathring{A}_1 \cup \mathring{J}_\beta$ contains a circle S. Then, it is easy to see that $\mathring{A}_1 \cup \mathring{J}_\beta = S$. Hence, S is both open and closed in M. Hence, $S = M$ and it is easy to see that $K = N$. Next, assume $\mathring{A}_1 \cup \mathring{J}_\beta$ does not contain a circle. Then it follows easily that $\mathring{A}_1 \cup \mathring{J}_\beta$ contains a free arc A in M such that $p, x_\infty \in \mathring{A}$. Thus, $x_\infty \in N$. Therefore, N is a closed subset of K. We now conclude, from the connectedness of K, that $N = K$.

Now, let

$$G = \bigcup\{\mathring{A}\colon A \text{ is a free arc in } M \text{ and } p \in \mathring{A}\}.$$

Note the following properties of G:

(3.11.2) G is arcwise connected;

(3.11.3) G is locally homeomorphic to R^1;

(3.11.4) $G \supset K$ (this follows from (3.11.1)).

Hence, G is an arcwise connected space which contains no simple triod by (3.11.3). So, by [19, 3.2], G is a one-to-one continuous image of a connected subset of R^1 which we again denote by L. By (3.11.3), L is not homeomorphic to $[0, 1]$. Also, since $K \neq \emptyset$ and $K \subset G$ (by (3.11.4)), L must be nondegenerate. Thus, there are two cases:

Case 1. $L \cong [0, +\infty)$. Suppose G is not compact. Then, since G is locally compact (by (3.11.3)), we have by [26, Theorem 7.1, p. 69] that $G \cong [0, +\infty)$. But, by (3.11.3), this is false. Hence, G is compact. Also, by (3.11.3), G is locally connected and contains no simple triod. Therefore, it follows easily from the Structure Theorem in [20, p. 128] that G must be a circle. However, G is also an open subset of M since, from the definition of G, G is a union of open subsets of M. Hence, $G = M$ and M is a circle.

Case 2. $L \cong R^1$. Then, by (3.11.3) and Theorem 1 of [17, p. 320], it follows easily that $G \cong R^1$. Thus, since M is locally arcwise connected [16, p. 254], it follows from the definition of G that $\mathrm{cl}(G)\setminus G$ consists of at most two points. First assume that $\mathrm{cl}(G)\setminus G$ consists of exactly two points q_1 and q_2. Then $J = \mathrm{cl}(G)$ is an arc with endpoints q_1 and q_2; furthermore, since G is an open subset of M, J is a free arc in M. Also, by (3.11.4), $K \subset \mathring{J}$. Thus, (i) of (3.11) holds. Next assume that $\mathrm{cl}(G)\setminus G$ consists of only one point c. Then $C = \mathrm{cl}(G)$ is a circle. Since G is an open subset of M and $G \cap [M\setminus C] = \emptyset$, $G \cap \mathrm{cl}[M\setminus C) = \emptyset$. It now follows that

$[C\cap \mathrm{cl}(M\setminus C)]\subset\{c\}$. If $M\setminus C=\emptyset$, then $M=C$ and (ii) of (3.11) holds. Assume $M\setminus C\neq\emptyset$. Then, $C\cap\mathrm{cl}(M\setminus C)\neq\emptyset$ (otherwise M would not be connected) and so $C\cap\mathrm{cl}(M\setminus C)=\{c\}$. It follows easily from (3.11.4) that (i) or (iii) of (3.11) holds.

(3.12) Lemma. *Assume $C(X)\cong Y\times Z$ where X is a Peano continuum such that X is not a circle. If K is a component of $\bigcup F(Y)\neq\emptyset$ (resp. $\bigcup F(Z)\neq\emptyset$), then there is a free arc J in Y (resp., Z) such that $K=\tilde{J}$.*

Proof. By (3.9.1), K satisfies the hypotheses of (3.11). Hence, since X is not a circle, (i) or (iii) of (3.11) must hold. Suppose (iii) holds. Then define $r\colon Y\xrightarrow{\text{onto}} C$ by

$$r(y)=\begin{cases} y, & y\in C,\\ c, & y\in\mathrm{cl}(M\setminus C).\end{cases}$$

Since r is a retraction of Y onto the circle C, we have contradicted (2.1) which states that Y is acyclic. Therefore, (iii) does not hold and, so, (i) holds. Hence, there is a free arc J in Y such that $K\subset\tilde{J}$. Therefore, since K is a component of $\bigcup F(Y)$ and $\tilde{J}\subset[\bigcup F(Y)]$, $K=\tilde{J}$.

(3.13) Lemma. *Assume $C(X)\cong Y\times Z$ where X is a Peano continuum such that X is not a circle. If K is a component of $\bigcup F(X)\neq\emptyset$, then there is a free arc J in X such that $K=\tilde{J}$.*

Proof. Since X is not a circle, we have by (3.9.1) and (3.11) that (i) or (iii) of (3.11) must hold. Suppose (iii) holds. Let $\Gamma=\{A\in F(X)\colon A\subset K\}$. Let $h\colon C(X)\xrightarrow{\text{onto}} Y\times Z$ be a homeomorphism. It is easy to verify that Γ is a component of $F(X)$. Hence, by (3.10), $h(\Gamma)$ is a component of $[\bigcup F(Y)]\times[\bigcup F(Z)]$ and so $\pi_Y(h[\Gamma])$ and $\pi_Z(h[\Gamma])$ are components of $\bigcup F(Y)$ and $\bigcup F(Z)$ respectively and $h[\Gamma]=\pi_Y(h[\Gamma])\times\pi_Z(h[\Gamma])$. Now, by (3.12), $\pi_Y(h[\Gamma])=\tilde{J}_Y$ and $\pi_Z(h[\Gamma])=\tilde{J}_Z$ for free arcs J_Y and J_Z in Y and Z respectively. Hence, $\mathrm{cl}(h[\Gamma]=\mathrm{cl}(\tilde{J}_Y\times\tilde{J}_Z)=J_Y\times J_Z$; in other words, $\mathrm{cl}(h[\Gamma])$ is a 2-cell. However, it is easy to see that

$$\mathrm{cl}(\Gamma)\cong B^2\setminus\{(x,y)\in R^2\colon\ [x-\tfrac12]^2+y^2<\tfrac14\}$$

and therefore, since $h[\mathrm{cl}(\Gamma)]=\mathrm{cl}(h[\Gamma])$, we have a contradiction. Thus, (i) of (3.11) must hold. Hence, there is a free arc J in X such that $K\subset\tilde{J}$. Therefore, since K is a component of $\bigcup F(X)$ and $\tilde{J}\subset[\bigcup F(X)]$ by (3.9.1), $K=\tilde{J}$.

(3.14) Lemma. *Assume $C(X)\cong Y\times Z$ where X is a Peano continuum. If K_1 and K_2 are distinct components of $\bigcup F(X)$, then $\mathrm{cl}(K_1)\cap\mathrm{cl}(K_2)=\emptyset$.*

Proof. Assume, for the purpose of proof, X is not a circle. By (3.13), there are free arcs J_i in X such that $K_i=\tilde{J}_i$, $i\in\{1,2\}$. For each $i\in\{1,2\}$, let

$$\Gamma_i=\{A\in F(X)\colon\ A\subset K\}.$$

It is easy to verify that Γ_1 and Γ_2 are distinct components of $F(X)$. Now suppose $\mathrm{cl}(K_1)\cap\mathrm{cl}(K_2)\neq\emptyset$. Then, noting that $\mathrm{cl}(K_1)\cap\mathrm{cl}(K_2)=J_1\cap J_2$ and that J_1 and J_2 are free arcs in X such that $\tilde{J}_1\cap\tilde{J}_2=\emptyset$, we see that $J_1\cap J_2=\{p\}$ or $J_1\cap J_2=\{p,q\}$ where p and q are end points of J_1 and J_2. Note that

$$(*)\qquad \mathrm{cl}(\Gamma_i)=C(J_i)\quad\text{for each } i\in\{1,2\}.$$

Therefore, since $J_1\cap J_2=\{p\}$ or $J_1\cap J_2=\{p,q\}$,

(**) $\mathrm{cl}(\Gamma_1)\cap\mathrm{cl}(\Gamma_2)$ consists of at least one point and at most two points.

Now, let $h\colon C(X)\xrightarrow{\text{onto}} Y\times Z$ be a homeomorphism. Since Γ_i is a component of $F(X)$ for each $i\in\{1,2\}$, it follows (see part of the proof of (3.13)) that $h[\mathrm{cl}(\Gamma_i)]=J_i(Y)\times J_i(Z)$ where: For each $i\in\{1,2\}$, $J_i(Y)$ and $J_i(Z)$ are free arcs in Y and Z respectively and $\tilde{J}_i(Y)=\pi_Y(h[\Gamma_i])$ and $\tilde{J}_i(Z)=\pi_Z(h[\Gamma_i])$ are components of $\bigcup F(Y)$ and $\bigcup F(Z)$ respectively.

For future use we prove

$$(\#)\qquad \tilde{J}_1(Z)\cap\tilde{J}_2(Z)=\emptyset.$$

Proof of (#). For each $i\in\{1,2\}$, $h[\mathrm{cl}(\Gamma_i)]=J_i(Y)\times J_i(Z)$. Hence

$$h[\mathrm{cl}(\Gamma_i)]\cap h[\mathrm{cl}(\Gamma_2)]=[J_1(Y)\cap J_2(Y)]\times[J_1(Z)\cap J_2(Z)].$$

Suppose $\tilde{J}_1(Z)\cap\tilde{J}_2(Z)\neq\emptyset$. Then, since $\tilde{J}_1(Z)$ and $\tilde{J}_2(Z)$ are components of $\bigcup F(Z)$, $\tilde{J}_1(Z)=\tilde{J}_2(Z)$. Hence, $J_1(Z)=J_2(Z)$ and so

$$h[\mathrm{cl}(\Gamma_1)]\cap h[\mathrm{cl}(\Gamma_2)]=[J_1(Y)\cap J_2(Y)]\times J_1(Z).$$

It now follows that $\mathrm{cl}(\Gamma_1)\cap\mathrm{cl}(\Gamma_2)$ is either empty or contains a copy of $J_1(Z)$, both possibilities being incompatible with (**). This proves (#).

Let $L=\tilde{J}_1(Y)\times\tilde{J}_2(Z)$. Since L is a component of $[\bigcup F(Y)]\times[\bigcup F(Z)]$, $h^{-1}(L)$ is a component of $F(X)$ (by (3.10)). Hence, $\bigcup h^{-1}(L)$ is a connected subset of $\bigcup F(X)$. Thus, by (3.13), there is a free arc J in X such that $[\bigcup h^{-1}(L)]\subset\tilde{J}$. Therefore, since $h^{-1}(L)$ is a component of $F(X)$, it follows easily that $\bigcup h^{-1}(L)=\tilde{J}$. Now, since $\{p\}\in C(J_i)=\mathrm{cl}(\Gamma_i)$ for each $i\in\{1,2\}$ (by (*)),

$$h(\{p\})\in h[\mathrm{cl}(\Gamma_i)]=J_i(Y)\times J_i(Z)\quad\text{for each } i\in\{1,2\}.$$

Hence, $h(\{p\})\in[J_1(Y)\times J_2(Z)]=\mathrm{cl}(L)$. Thus, $\{p\}\in\mathrm{cl}[h^{-1}(L)]$ which implies $p\in\mathrm{cl}[\bigcup h^{-1}(L)]=J$. Therefore, $J_1\cap J\neq\emptyset$. Since $L=\tilde{J}_1(Y)\times\tilde{J}_2(Z)$ and $h[\Gamma_1]\subset[\tilde{J}_1(Y)\times\tilde{J}_1(Z)]$ (actually equality holds) and $\tilde{J}_1(Z)\cap\tilde{J}_2(Z)=\emptyset$ (by (#)), we have that $h^{-1}(L)\cap\Gamma_1=\emptyset$. Hence, since $\Gamma_1=\{A\in F(X)\colon A\subset\tilde{J}_1\}$ and (as is easy to verify) $h^{-1}(L)=\{A\in F(X)\colon A\subset\tilde{J}\}$ and since J_1 and J are free arcs in X, it follows that $\tilde{J}_1\cap\tilde{J}=\emptyset$. Thus we conclude that $J_1\cap J$ is nonempty and consists only of the common endpoint(s) of J_1 and J. Therefore, since $\mathrm{cl}(\Gamma_1)=C(J_1)$ and $\mathrm{cl}[h^{-1}(L)]=C(J)$, it follows that

(***) $h[\mathrm{cl}(\Gamma_1)]\cap\mathrm{cl}(L)$ consists of at least one point and at most two points.

But, we also have

$$h[\mathrm{cl}(\Gamma_1)] \cap \mathrm{cl}(L) = [J_1(Y) \times J_1(Z)] \cap [J_1(Y) \times J_2(Z)]$$
$$= J_1(Y) \times [J_1(Z) \cap J_2(Z)]$$

which contradicts (***). Hence, $\mathrm{cl}(K_1) \cap \mathrm{cl}(K_2) = \emptyset$ and we have proved the lemma.

(3.15) THEOREM. *Let X be a Peano continuum. If $C(X) \cong Y \times Z$, then one of the following must hold*:

(3.15.1) *X is a circle*;

(3.15.2) *X contains no free arc*;

(3.15.3) *The closure of any component of $\bigcup F(X)$ is a free arc (in X) which is disjoint from any free arc (in X) not contained in it.*

Proof. Assume (3.15.1) and (3.15.2) do not hold. Then, letting K be a component of $\bigcup F(X)$, we see that (3.13) may be applied giving us that there is a free arc J in X such that $K = \mathring{J}$. Hence, $\mathrm{cl}(K) = J$ is a free arc in X which, by (3.14), is disjoint from any free arc in X not contained in it. Therefore, (3.15.3) holds.

The continuum X in the following example should be compared with the continuum in (3.2).

(3.16) EXAMPLE. Let $X = B^2 \cup L_1 \cup A$ where L_1 is as in (3.2) and $A = \{(x, x-1) \in R^2\colon 1 \leqslant x \leqslant 2\}$. Then, by (3.15), $C(X)$ is not a cartesian product.

The continuum X in the following example should be compared with the continuum in (3.3).

(3.17) EXAMPLE. Let $X = B^2 \cup L_1 \cup A \cup L$ where L_1 and A are as in (3.16) and $L = \{(2, y)\colon 0 \leqslant y \leqslant 1\}$. Then, by (3.15), $C(X)$ is not a cartesian product.

(3.18) Remark. Note that (3.15) yields 9.7 of [10] (and, hence, (1.1) above) as a corollary. Though the proof given for 9.7 in [10] provided inspiration for some ideas used above, the procedure used to prove (3.15) is different from that used to prove 9.7 in [10].

(3.19) QUESTION. Is the converse of (3.15) true? Since $C(X) \cong [0, 1] \times [0, 1]$ when X is a circle and since $C(X) \cong I_\infty$ when X is a Peano continuum containing no free arc [9, Theorem 2], the question reduces to whether, for Peano continua, (3.15.3) implies $C(X)$ is a cartesian product. It is clear from what was done in (3.1) that knowledge about the union of Hilbert cubes that intersect in a Hilbert cube is intimately connected with this question. Hence, more information about the conjecture in [2, p. 213] than is currently available would seem to be necessary to answer the question (see (3.20)). One further comment: The question is stated in the context of Peano continua; it may in fact be true that if $C(X)$ is a cartesian product, then X *is* a Peano continuum. If this is true and if the converse of (3.15) is true, then a complete characterization of those continua X such that $C(X)$ is a cartesian product would be obtained.

(3.20) Remark. Sher has recently shown the conjecture in [2, p. 213] is false (Proc. Amer. Math. Soc. 63 (1977), pp. 150–152). However, perhaps some special Hilbert cube sum theorems could be obtained to help answer the question in (3.19).

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Uniform quotients of metric spaces

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Abstract. Uniform quotients of (pseudo-) metric spaces are investigated and several conditions implying the pseudometrizability of such quotients are given. Most important, however, it is shown that pseudometrizability is not preserved under uniform quotient maps, thus answering in the negative a question of Himmelberg.

1. Introduction. The question of when a given class of maps preserves metrizability of a topological space has long been of interest in general topology. Recently, questions relating to the preservation of (pseudo-) metrizability have arisen in connection with various types of maps between proximity and uniform spaces (see, for example, [1], [4] and [8]). In [4] Himmelberg investigates uniform quotient maps and provides necessary and sufficient conditions for such maps to preserve pseudometrizability. Left open in [4], however, is the question of whether uniform quotient maps, in general, preserve this property.

In Section 3 of this paper we construct a metric space having a non-metrizable (T_2) uniform quotient and, in doing so, answer Himmelberg's question in the negative. In section 4 we provide several conditions sufficient for a uniform quotient map to preserve pseudometrizability.

2. Preliminaries. Let d be a pseudometric on a set X. For a positive real ε set $B(d,\varepsilon)=\{(x,y)\colon d(x,y)<\varepsilon\}$. Let $\mathscr{U}(d)$ denote the uniformity determined d, i.e. the uniformity with base $\{B(d,\varepsilon)\colon \varepsilon>0\}$.

For a uniformity $\mathscr{U}$ on X, $\mathscr{G}(\mathscr{U})$ will denote the gage of $\mathscr{U}$ [5, p. 189].

Let $[X,\mathscr{U}]$ be a uniform space, Y a set and $\varrho\colon X\to Y$ a surjection. The *quotient uniformity*, determined by $[X,\mathscr{U}]$ and ϱ, is defined to be the largest uniformity $\mathscr{V}$ on Y such that $\varrho\colon [X,\mathscr{U}]\to[Y,\mathscr{V}]$ is uniformly continuous. This uniformity will be indicated by $\mathscr{U}_\varrho$ and, when Y is endowed with $\mathscr{U}_\varrho$, ϱ will be called a *uniform quotient map*.

The formulation in [6] of the quotient uniformity is essential to the definitions of the spaces in § 3. This formulation, with minor modifications, is outlined below.