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On compact spaces which are locally Cantor bundles

by

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Abstract. The paper deals with what we call the local bundles over X, i.e. with compact Hausdorff spaces such that each point has a neighbourhood homeomorphic to the product $X\times J$, where X is a given totally disconnected compact Hausdorff space and J is an open interval. It is proved that each local bundle over X can be obtained from the disjoint union of some copies of the bundle $X\times[0,1]$ by identifying points $\langle x,i\rangle$ with $h\langle x,i\rangle$, where h is a continuous involution without fixed points on some copies of $X\times\{0,1\}$.

1. Preliminaries. If X is Hausdorff, I is the unit interval $\{t\in R\colon 0\leqslant t\leqslant 1\}$, and h is a continuous involution on $X\times\{0,1\}$, then we denote by $X\times I/h$ the quotient of the product $X\times I$, where points $\langle x,i\rangle$ and $h\langle x,i\rangle$ are identified.

If h has no fixed point, then each point of the space $X\times I/h$ has a neighbourhood homeomorphic to the product $X\times J$, where J is an open interval $\{t\in R\colon 0<t<1\}$. If the involution h is determined by a homeomorphism $f\colon X\to X$ in such a way that $h\langle x,0\rangle=\langle f(x),1\rangle$ and $h\langle x,1\rangle=\langle f^{-1}(x),0\rangle$, then we write $X\times I/f$ rather than $X\times I/h$.

LEMMA. *Let X be a compact totally disconnected Hausdorff space, and let D be a closed-open subset of X. Let Y be a compact Hausdorff space each point of which has a neighbourhood homeomorphic to $X\times J$, where J is an open unit interval. Let Z be a closed subset of Y homeomorphic to $D\times I$ under a homeomorphism f and such that $\mathrm{Int}_Y Z=f^{-1}(D\times J)$. Then the quotient space $Y/D\times I$, which is obtained from Y by collapsing each arc in Z to a point, is homeomorphic to Y.*

Proof. For each point y of $f^{-1}(D\times\{0\})$ take a neighbourhood V_y homeomorphic to $X\times J$ under a homeomorphism g_y. Consider $g_y(Z\cap V_y)$ and $g_y(\mathrm{Int}_Y Z\cap V_y)$. Since Y is compact and Hausdorff and V_y is an open subset of Y, there exist a closed-open subset D_y of X and points $a_y<b_y$ of J such that $g_y^{-1}(D_y\times[a_y,b_y])$ contains y in its interior and has no point in common with $f^{-1}(X\times[\frac{1}{2},1])$ and the intersection of each arc of $D_y\times[a_y,b_y]$ with $g_y(Z\cap V_y)$ is a proper non-degenerate subinterval of that arc. Denote the set $g_y^{-1}(D_y\times(a_y,b_y))$

by W_y. Since $f^{-1}(D\times\{0\})$ is compact, there is a finite collection $\{W_1, \ldots, W_n\}$ of sets W_y covering $f^{-1}(D\times\{0\})$. We may construct a collection of open subsets $U_1, \ldots, U_m$ of the sets $W_1, \ldots, W_n$ such that the collection covers $f^{-1}(D\times\{0\})$, the sets $\mathrm{cl}_Y U_1, \ldots, \mathrm{cl}_Y U_m$ are pairwise disjoint, each $\mathrm{cl}_Y U_i$ is homeomorphic under $f_i = g_y|_{\mathrm{cl}_Y U_i}$ to $D_i\times I_i$, where D_i is a closed-open subset of D_y and $I_i = [a_i, b_i] \subset (a_y, b_y)$ for a certain W_y containing U_i, and $U_i = f_i^{-1}(D_i\times(a_i, b_i))$. We can assume that $f_i^{-1}(D_i\times\{a_i\})\subset Z$. Let $p_2(y)$ denote the second coordinate of $f_i(y)$ if $y\in \mathrm{cl}_Y U_i$. Let $q(y)$ be the second coordinate of $f_i(f^{-1}\langle x, 0\rangle)$, where $f^{-1}\langle x, 0\rangle$ and y lie on the same arc of U_i. The mappings p_2 and q are both continuous. Define a continuous mappings s from $A = \bigcup\{\mathrm{cl}_Y U_i\colon i = 1, \ldots, m\} - \mathrm{Int}_Y Z$ into the set of reals by setting $s(y) = q(y)-p_2(y)$. Let us observe that $A\cap Z = f^{-1}(D\times\{0\})$, $s|_{A\cap Z}\equiv 0$ and that $s(y)\leqslant 0$ for $y\in A$.

Using the same arguments, we can construct a finite collection of open subsets $U_1', \ldots, U_k'$ covering $f^{-1}(D\times\{1\})$ such that their closures are pairwise disjoint and disjoint with A and are homeomorphic under f_j' to the sets $D_j'\times I_j'$, where D_j' are closed-open subsets of X and $I_j' = [a_j', b_j']$ are subintervals of I, and

$$U_j' = f_j'^{-1}(D_j'\times(a_j', b_j'))\,.$$

We can assume that the common part of each arc of $\mathrm{cl}_Y U_j'$ and of Z is a proper non-degenerate subinterval of that arc, and that $f_j'^{-1}(D_j'\times\{a_j'\})$ are subsets of Z. Let $p_1'(y)$ be the second coordinate of $f_j'(y)$ if $y\in U_j'$, and let $q'(y)$ be the second coordinate of $f_j'(f^{-1}\langle x, 1\rangle)$, where $f^{-1}\langle x, 1\rangle$ and y lie on the same arc of U_j'. Define a continuous mapping s' from $B = \bigcup\{\mathrm{cl}_Y U_j'\colon j = 1, \ldots, k\} - \mathrm{Int}_Y Z$ into the set of reals by setting $s'(y) = 1+p_2'(y)-q'(y)$. The function p from the union of Z and of the sets $U_1, \ldots, U_m, U_1', \ldots, U_k'$ into the set of reals defined by $p(y) = s(y)$ for $s\in A$, $p(y) = s'(y)$ for $y\in B$ and $p(y)$ being the second coordinate of $f(y)$ for $y\in Z$, is continuous.

Denote by $r(y)$ the real $s(f_i^{-1}\langle x, b_i\rangle)$, where $y\in A\cup Z$ and both y and $f_i^{-1}\langle x, b_i\rangle$ lie on the same arc of $A\cup Z$. Denote by $r'(y)$ the real $s'(f_j'^{-1}\langle x, b_j'\rangle)$, where $y\in B\cup Z$ and both y and $f_j^{-1}\langle x, b_j'\rangle$ lie on the same arc of $B\cup Z$.

Let Q_Y be the quotient map from Y onto $Y/D\times I$.

The map F from Y onto $Y/D\times I$ defined by setting

$$F(f^{-1}\langle x, \tfrac{1}{2}\rangle) = Q_Y(f^{-1}\langle x, \tfrac{1}{2}\rangle)\,,$$

$$F(y) = f_i^{-1}\left\langle x, \frac{\frac{1}{2}-p(y)}{\frac{1}{2}-r(y)}\, r(y)\right\rangle \text{ if } y \text{ is a point of } A\cup Z \text{ which lies on the arc}$$

passing through $f_i^{-1}\langle x, b_i\rangle$ and if $p(y)<\frac{1}{2}$,

$$F(y) = f_j'^{-1}\left\langle x, 1+\frac{2p(y)-1}{2r'(y)-1}\, r'(y)\right\rangle \text{ if } y \text{ is such a point of } B\cup Z \text{ that}$$

$p(y)>\frac{1}{2}$ and both y and $f_j'^{-1}\langle x, b_j'\rangle$ lie on the same arc of $B\cup Z$,

$$F(y) = y \text{ if } y \text{ do not belong to } A\cup B\cup Z,$$

is the desired homeomorphism.

2. Basic theorem. We prove the following

THEOREM. *Let X be a compact totally disconnected Hausdorff space. If Y is a compact Hausdorff space each point of which has a neighbourhood homeomorphic to $X\times J$, where $J = \{t\in R\colon 0<t<1\}$, then there are a positive integer n and an involution h from $X'\times\{0, 1\}$ onto $X'\times\{0, 1\}$ with no fixed point such that Y is homeomorphic to $X'\times I/h$, where X' is the disjoint union of n copies of X.*

Proof. Each point of Y has a neighbourhood whose closure is homeomorphic to $X\times I$, and if f is a given homeomorphism then that neighbourhood is equal to $f^{-1}(X\times J)$. So there is a finite irreducible cover $V_1, \ldots, V_n$ of Y such that $\mathrm{cl}_Y V_i$ are homeomorphic to $X\times I$ and, if $f_i\colon \mathrm{cl}_Y V_i\to X\times I$ are given homeomorphisms, then $V_i = f_i^{-1}(X\times J)$. We shall construct a finite collection of open pairwise disjoint subsets $U_1, \ldots, U_n$ of Y such that each $\mathrm{cl}_Y U_i$ will be homeomorphic to $X\times I$ and the quotient space Y' obtained from Y by collapsing each arc outside the union $U_1\cup\ldots\cup U_n$ to a point will be homeomorphic to Y.

Let us observe that there are such continuous mappings s_i from X into V_i that $f_i(s_i(x))$ belongs to $\{x\}\times J$ and the sets $s_1(X), \ldots, s_n(X)$ are pairwise disjoint. Since $f_1\circ s_1$ is continuous, there exists an open subset U_1' of $X\times J$ containing $f_1(s_1(X))$ such that $\mathrm{cl}_{X\times J} U_1'$ is homeomorphic to $X\times I$ under a homeomorphism g_1', $g_1'(U_1') = X\times J$ and $f_1^{-1}(U_1')$ has no point in common with $s_2X, \ldots, s_nX$ in its closure, and each arc of $f_1^{-1}(U_1')$ contains a point $s_1(x)$. Let $U_1 = f_1^{-1}(U_1')$ and $g_1 = g_1'\circ f_1|\mathrm{cl}_Y U$. Clearly, g_1 is a homeomorphism from $\mathrm{cl}_Y U$ onto $X\times I$.

Suppose we have defined open subsets $U_1, \ldots, U_k$ of Y, $k<n$, the closures of which are pairwise disjoint and disjoint with $s_{k+1}(X), \ldots, s_n(X)$, such that $\mathrm{cl}_Y U_j$ are homeomorphic under g_j to $X\times I$ and $s_j(X)\subset U_j = g_j^{-1}(X\times J)\subset V_j$, $j = 1, \ldots, k$. Since $f_{k+1}\circ s_{k+1}$ is continuous, there is an open subset U_{k+1}' of $X\times J$ containing $f_{k+1}(s_{k+1}(X))$ such that $\mathrm{cl}_{X\times J} U_{k+1}'$ is homeomorphic under g_{k+1}' to $X\times I$, $U_{k+1}' = g_{k+1}'^{-1}(X\times J)$ and the closure of $U_{k+1} = f_{k+1}(U_{k+1}')$ is disjoint with closures of the sets $U_1, \ldots, U_k$ and with $s_{k+2}(X), \ldots, s_n(X)$. The map

$$g_{k+1} = g_{k+1}'\circ f_{k+1}|\mathrm{cl}_Y U_{k+1}$$

is a homeomorphism from $\mathrm{cl}_Y U_{k+1}$ onto $X\times I$.

Thus we have a collection of open subsets $U_1, \ldots, U_n$ of Y containing $s_1(X), \ldots, s_n(X)$ and such that their closures are pairwise disjoint and homeomorphic to $X\times I$ under homeomorphisms $g_1, \ldots, g_n$.

Denote by A the union of the sets $\mathrm{cl}_Y U_1, \ldots, \mathrm{cl}_Y U_n$. Clearly, A is homeomorphic under f_A to the disjoint union $X'\times I$ of n copies of $X\times I$.

Let $B = Y - \mathrm{Int}_Y A$. B is the complement of the union of the sets $U_1, \ldots, U_n$ and $A\cap B$ is the set of the endpoints of arcs of A; so $A\cap B$ is the union of images of $X\times\{0, 1\}$ under $g_1, \ldots, g_n$. For each point b of $A\cap B$ let us consider a subset V_b

of $A \cap B$ such that it is contained in an image of $X \times \{0\}$ or $X \times \{1\}$ under a homeomorphism g_i for a certain i, its inverse image under g_i is a closed-open subset of $X \times \{0, 1\}$ and there is no arc in B joining any points of V_b. Let W_b be the set of arcs in B arising from V_b. The set W_b is homeomorphic to the cartesian product $D_b \times I$, where D_b is a closed-open subset of X and $\mathrm{cl}_Y \mathrm{Int}_Y W_b = W_b$. Since B is a compact subset of Y and each W_b is closed-open in B, there is a finite collection $W_1, \ldots, W_k$ of these subsets covering the whole of B. The sets $W_1' = W_1$, $W_2' = W_2 - W_1, \ldots, W_k' = W_k - (W_1 \cup \ldots \cup W_{k-1})$ are pairwise disjoint and have the properties required in our lemma. Thus, using the lemma k times, we prove that Y is homeomorphic to the quotient Y' obtained from Y by collapsing to a point each arc outside $\mathrm{Int}_Y A$. But Y' is a quotient $X' \times I/h$ obtained from $X' \times I$, where X' is the disjoint union of n copies of X and where h is an involution without fixed points on the disjoint union of n copies of $X \times \{0, 1\}$ defined by $h\langle a, i\rangle = \langle b, j\rangle$ if and only if $g_k^{-1}\langle a, i\rangle$ and $g_m^{-1}\langle b, j\rangle$ are the endpoints of the same arc in B, $\langle a, i\rangle$ and $\langle b, j\rangle$ being the points of the kth and the mth copies of $X \times \{0, 1\}$, respectively.

This completes the proof.

3. Remarks. We cannot require n to be equal to one. To see that take as X a space consisting of one point and let Y be equal to the disjoint union of two circles.

Furthermore, we cannot require the involution h to be introduced by a homeomorphism from n copies of $X \times \{0\}$ onto n copies of $X \times \{1\}$. To see this, consider the following:

Let X be the space consisting of a sequence $x_1, x_2, \ldots, x_i \neq x_j$ for $i \neq j$, and of its limit x_0.

Let $h\colon X \times \{0, 1\} \to X \times \{0, 1\}$ be defined by setting

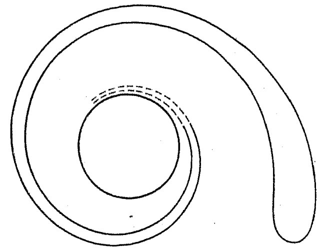
$$\begin{aligned} h\langle x_0, 0\rangle &= \langle x_0, 1\rangle, \\ h\langle x_0, 1\rangle &= \langle x_0, 0\rangle, \\ h\langle x_1, 0\rangle &= \langle x_2, 0\rangle, \\ h\langle x_2, 0\rangle &= \langle x_1, 0\rangle, \\ h\langle x_n, 0\rangle &= \langle x_{n-2}, 1\rangle, \quad n>2, \\ h\langle x_n, 1\rangle &= \langle x_{n+2}, 0\rangle, \quad n = 1, 2, \ldots \end{aligned}$$

The space $Y = X \times I/h$ is a planar continuum (see the figure).

This continuum has two arc-components, one being a circle and the other a line.

The idea of the proof is the following. If the space Y is $X \times I/f$ for some homeomorphism f on X, then $f(x_0) = x_0$ and $\{x_1, x_2, \ldots\}$ is an orbit of f. Thus this homeomorphism induces an order on the arc-component which is a line in the following way: if $\langle x, t\rangle$ and $\langle y, s\rangle$ are different points of this arc-component, then if $x = y$ then $\langle x, t\rangle$ is less than $\langle y, s\rangle$ if and only if $t < s$, and if $x \neq y$, then $\langle x, t\rangle$ is less than $\langle y, s\rangle$ if and only if $x = f^n(y)$ for some positive integer n, where f^n is a superposition of n maps each equal to f. This order coincides with the order of the set of reals, and induces an orientation of the arc component which is a circle.

This does not apply to the space $\mathscr{B} \times I/h$. If the arc-component which is a line has the order of reals, then this order cannot induce an orientation of the other arc-component.



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