On the existence of arcs in rational curves

by

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Abstract. It is an open question whether curves of finite rim-type contain arcs. A. Lelek and L. Mohler gave a positive answer for the case in which the curve is hereditarily unicoherent. In this paper the question is answered in the affirmative.

1. Preliminaries. Rational curves which contain no arcs have interested topologists for several years. First, Z. Janiszewski in 1912 (see [4]) constructed an arc-like rational curve of rim-type ω which contains no arcs. A. Lelek in [6] Problem 729 asked if it is true that every curve which contains no arcs has infinite rim-type. A. Lelek and L. Mohler in [7] Theorem 1.2 proved that if X is a hereditarily unicoherent curve which contains no arcs, then X has infinite rim-type. B. B. Epps, Jr. in [2] constructed an arc-like curve of rim-type $n>1$ all of whose subcurves are of rim-type either 1 or n. J. Grispolakis and E. D. Tymchatyn gave another example in [3] of an arc-like curve of rim-type 3 all of whose subcurves are of rim-type either 1 or 3.

In this note we answer Lelek's question in the affirmative by proving that every curve of finite rim-type contains arcs. Thus, Lelek's question is settled completely.

2. Rim-type of rational curves. A *continuum* is a connected, compact, metric space. A *curve* is a 1-dimensional continuum. If A is a subset of a topological space X, let A' denote the derived set of A. Let $A^{(0)} = A$ and by transfinite induction define $A^{(\alpha)}$ for each ordinal α, by $A^{(\alpha+1)} = (A^{(\alpha)})'$ and $A^{(\lambda)} = \bigcap \{A^{(\alpha)} |\ \alpha<\lambda\}$ for a limit ordinal λ. Let $\mathrm{Cl}\,A$ and $\mathrm{Bd}\,A$ denote the closure and the boundary, respectively, in X of a subset A of X. By $\mathrm{Int}\,A$ we denote the interior of A in X. Let N denote the set of natural numbers. If C is a compact, countable subset of a metric space, then there exists a countable ordinal α such that $C^{(\alpha)} = \emptyset$. We call the smallest such ordinal α the *topological type* of C. A curve X is said to be *rational* if X admits a basis of open sets with countable boundaries. Define the *rim-type* of X to be the smallest ordinal α such that X has a neighbourhood basis of open sets $(U_i)_{i\in N}$ such that the topological type of $\mathrm{Bd}\,U_i$ is $\leqslant\alpha$ for each $i \in N$. It is well-known (see [5], p. 290) that the rim-type of a rational continuum is an ordinal number strictly smaller

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than the first uncountable ordinal Ω. By a *mapping* we shall always mean a continuous function.

The following theorem is a well-known result (see [5], p. 216) and it is stated here in the form which we shall use.

2.1. THEOREM. *If X is an irreducible, hereditarily decomposable continuum, then there exists a finest monotone mapping of X onto the unit interval* $[0, 1]$. *The point-inverses under this mapping are nowhere dense subcontinua of X and are called the tranches of X.*

2.2. THEOREM. *Every curve of finite rim-type contains an arc.*

Proof. The proof is by induction on the rim-type of the curve X. If rim-type of X is 1, then X has a basis of open sets with finite boundaries. It is well-known that in this case X is locally connected, and hence, X contains an arc. Suppose that each curve of rim-type $\leqslant n-1$ contains an arc.

Just suppose that there exists a curve X of rim-type n such that X contains no arc. Then every non-degenerate subcontinuum of X has rim-type n. We may suppose, without loss of generality, that X is an irreducible continuum. Let $\pi\colon X \to [0, 1]$ be a finest monotone mapping of X onto $[0, 1]$. We may suppose since X is not an arc that $\pi^{-1}(1)$ is non-degenerate. Let $T_0 = \pi^{-1}(1)$ and let $Y_0 = X$.

Let $\mathcal{U}$ be a countable basis of open sets for X whose boundaries are pairwise disjoint and have topological type $\leqslant n$. This is possible since X is compact, $\mathcal{U}$ is countable, and the boundaries of the members of $\mathcal{U}$ are zero-dimensional. We may also suppose that if $U \in \mathcal{U}$ and $x \in \mathrm{Bd}\, U$, then $x \in \mathrm{Cl}(X \setminus \mathrm{Cl}\, U)$. Let $\{U_1, U_2, \ldots\}$ be the members of $\mathcal{U}$ which meet T_0.

For each $m = 1, 2, \ldots$ let $D_m = (T_0 \cap \mathrm{Bd}\, U_m) \setminus (\mathrm{Bd}\, U_m)'$. Then D_m is a discrete set for each $m = 1, 2, \ldots$ Let $D_m = \{x_{m1}, x_{m2}, \ldots\}$ for each $m = 1, 2, \ldots$ Let Y_1 be the compactification of $Y_0 \setminus D_1$ which is larger than Y_0 and such that if $g_1\colon Y_1 \to Y_0$ is the extension over Y_1 of the inclusion of $Y_0 \setminus D_1 \subset Y_1$ into Y_0, then $g_1^{-1}(x_{1i}) = \{y_{1i}, z_{1i}\}$. A basic open neighbourhood of y_{1i} (respectively, z_{1i}) is given by $g_1^{-1}(U \cap U_1) \cup \{y_{1i}\}$ (respectively, $g_1^{-1}(U \setminus \mathrm{Cl}\, U_1) \cup \{z_{1i}\}$), where U is a neighbourhood of x_{1i} in Y_0. Then Y_1 is an irreducible continuum of rim-type n and $g^{-1}(T_0) = T_1$ is a tranche of Y_1. Notice that the topological type of $T_1 \cap \mathrm{Bd}\big[\mathrm{Cl}\big(g_1^{-1}(U_1)\big)\big]$ is less than or equal to $n-1$ since g_1 maps $T_1 \cap \mathrm{Bd}\big[\mathrm{Cl}\big(g_1^{-1}(U_1)\big)\big]$ homeomorphically onto a subset of $(\mathrm{Bd}\, U_1)'$. Also g_1 maps $\mathrm{Bd}\big[\mathrm{Cl}\big(g_1^{-1}(U_1)\big)\big]$ homeomorphically onto $\mathrm{Bd}\, U_1 \setminus D_1$. We identify points and subsets of $Y_0 \setminus D_1$ with their preimages in Y_1.

Suppose that m is a positive integer and $Y_1, \ldots, Y_{m-1}$ are irreducible continua of rim-type n, $g_i\colon Y_i \to Y_{i-1}$ is a mapping of Y_i onto Y_{i-1}, g_i maps $Y_i \setminus g_i^{-1} \circ \ldots \circ g_1^{-1}(D_i)$ homeomorphically onto $Y_{i-1} \setminus g_{i-1}^{-1} \circ \ldots \circ g_1^{-1}(D_i)$, g_i is two-to-one at the points of $g_{i-1}^{-1} \circ \ldots \circ g_1^{-1}(D_i)$ and the topological type of $T_i \cap \mathrm{Bd}\big[\mathrm{Cl}\big(g_i^{-1} \circ \ldots \circ g_1^{-1}(U_i)\big)\big]$ is at most $n-1$, where $T_i = g_i^{-1}(T_{i-1})$ for each $i = 1, \ldots, m-1$. We identify points and subsets in $Y_{i-1} \setminus g_{i-1}^{-1} \circ \ldots \circ g_1^{-1}(D_i)$ with their preimages under g_i for each $i = 1, \ldots, m-1$.

Let Y_m be the compactification of $Y_{m-1} \setminus D_m$ that is larger than Y_{m-1} and such that if $g_m\colon Y_m \to Y_{m-1}$ is the extension over Y_m of the inclusion of $Y_{m-1} \setminus D_m \subset Y_m$ into Y_{m-1}, then $g_m^{-1}(x_{mi}) = \{y_{mi}, z_{mi}\}$. A basic open neighbourhood of y_{mi} (respectively, z_{mi}) is given by

$$g_m^{-1}\big(U \cap g_{m-1}^{-1} \circ \ldots \circ g_1^{-1}(U_m)\big) \cup \{y_{mi}\}$$

(respectively, $g_m^{-1}\big[U \setminus \mathrm{Cl}\big(g_{m-1}^{-1} \circ \ldots \circ g_1^{-1}(U_m)\big)\big] \cup \{z_{mi}\}$), where U is a neighbourhood of x_{mi} in Y_{m-1}. We identify points and subsets of $Y_{m-1} \setminus D_m$ with their preimages in Y_m under g_m. Then Y_m is an irreducible continuum of rim-type n and $g_m^{-1}(T_{m-1}) = T_m$ is a tranche of Y_m. Notice that the topological type of $T_m \cap \mathrm{Bd}\big[\mathrm{Cl}\big(g_m^{-1} \circ \ldots \circ g_1^{-1}(U_m)\big)\big]$ is less than or equal to $n-1$ since it is mapped by $g_m \circ \ldots \circ g_1$ homeomorphically onto a subset of $(\mathrm{Bd}\, U_m)'$. Also $g_1 \circ \ldots \circ g_m$ maps $\mathrm{Bd}\big[\mathrm{Cl}\big(g_m^{-1} \circ \ldots \circ g_1^{-1}(U_m)\big)\big]$ homeomorphically onto $\mathrm{Bd}\, U_m \setminus D_m$. By induction, Y_m and g_m are defined for each $m = 1, 2, \ldots$

Consider the inverse system $\{Y_m, g_m, N\}$ and let $Y_\infty = \{Y_m, g_m, N\}$ and let $g\colon Y_\infty \to Y_0 = X$ be the mapping induced by the inverse limit. Then g is one-to-one except at the points x_{mi}, for each $m, i = 1, 2, \ldots$, where g is two-to-one, and g maps $g^{-1} \circ \pi^{-1}([0, 1))$ homeomorphically onto $\pi^{-1}([0, 1))$. We also have that $g^{-1} \circ \pi^{-1}([0, 1))$ is dense in Y_∞, Y_∞ is irreducible, and $T_\infty = \varprojlim \{T_m, g_m | T_m, N\}$ is a tranche of Y_∞.

If $x \in Y_\infty \setminus T_\infty$, then a basic neighbourhood of x in Y_∞ (see [1], p. 218) is given by $g^{-1}(U)$ for some $U \in \mathcal{U}$ such that $\mathrm{Cl}\, U \cap T_0 = \emptyset$. Then $\mathrm{Bd}\big(g^{-1}(U)\big)$ is homeomorphic to $\mathrm{Bd}\, U$. If $x \in T_\infty$ but $x \neq x_{mi}$ for each $m, i = 1, 2, \ldots$, then a basic neighbourhood of x in Y_∞ is of the form $\mathrm{Int}\big[\mathrm{Cl}\big(g^{-1}(U_i)\big)\big]$ for some $U_i \in \mathcal{U}$. Then

$$\mathrm{Bd}\big[\mathrm{Int}\big(\mathrm{Cl}\big(g^{-1}(U_i)\big)\big)\big] = \mathrm{Bd}\big[\mathrm{Cl}\big(g^{-1}(U_i)\big)\big]$$

is homeomorphic to $\mathrm{Bd}\big[\mathrm{Cl}\big(g_i^{-1} \circ \ldots \circ g_1^{-1}(U_i)\big)\big]$ and the latter set has topological type less than or equal to n. Also, $T_\infty \cap \mathrm{Bd}\big[\mathrm{Cl}\big(g^{-1}(U_i)\big)\big]$ is homeomorphic to $\mathrm{Bd}\big[\mathrm{Cl}\big(g_i^{-1} \circ \ldots \circ g_1^{-1}(U_i)\big)\big] \cap T_\infty$. If $x = y_{mi}$ (respectively, $x = z_{mi}$), then a basic neighbourhood G of x is of the form $\mathrm{Int}\big[\mathrm{Cl}\big(g^{-1}(U_m \cap U_j)\big)\big] \cup \{y_{mi}\}$ (respectively, $\mathrm{Int}\big[\mathrm{Cl}\big(g^{-1}(U_j \setminus \mathrm{Cl}\, U_m)\big)\big] \cup \{z_{mi}\}$), where U_j is a basic neighbourhood of $g(x)$ in Y_0. The boundary of this neighbourhood in Y_∞ is contained in

$$\mathrm{Bd}\big(g^{-1}(U_m \cap U_j)\big) \subset \mathrm{Bd}\big(g^{-1}(U_m)\big) \cup \mathrm{Bd}\big(g^{-1}(U_j)\big)$$

(respectively, $\mathrm{Bd}\big(g^{-1}(U_j \setminus \mathrm{Cl}\, U_m)\big) \subset \mathrm{Bd}\big(g^{-1}(U_j)\big) \cup \mathrm{Bd}\big(g^{-1}(U_m)\big)$). This shows that the boundary of the neighbourhood G has topological type $\leqslant n$. Also $T_\infty \cap \mathrm{Bd}\, G$ is mapped by g homeomorphically onto a subset of $(\mathrm{Bd}\, U_m)' \cup (\mathrm{Bd}\, U_j)'$. Hence, the rim-type of Y_∞ is less than or equal to n and the rim-type of T_∞ is less than or equal to $n-1$. Therefore, T_∞ contains an arc. Since g is at most two-to-one $T_0 = g(T_\infty)$ also contains an arc. This contradicts the assumption that X does not contain any arc. The proof of the theorem is complete.

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On compact spaces which are locally Cantor bundles

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Abstract. The paper deals with what we call the local bundles over X, i.e. with compact Hausdorff spaces such that each point has a neighbourhood homeomorphic to the product $X \times J$, where X is a given totally disconnected compact Hausdorff space and J is an open interval. It is proved that each local bundle over X can be obtained from the disjoint union of some copies of the bundle $X \times [0, 1]$ by identifying points $\langle x, i\rangle$ with $h\langle x, i\rangle$, where h is a continuous involution without fixed points on some copies of $X \times \{0, 1\}$.

1. Preliminaries. If X is Hausdorff, I is the unit interval $\{t \in R\colon 0 \leqslant t \leqslant 1\}$, and h is a continuous involution on $X \times \{0, 1\}$, then we denote by $X \times I/h$ the quotient of the product $X \times I$, where points $\langle x, i\rangle$ and $h\langle x, i\rangle$ are identified.

If h has no fixed point, then each point of the space $X \times I/h$ has a neighbourhood homeomorphic to the product $X \times J$, where J is an open interval $\{t \in R\colon 0 < t < 1\}$. If the involution h is determined by a homeomorphism $f\colon X \to X$ in such a way that $h\langle x, 0\rangle = \langle f(x), 1\rangle$ and $h\langle x, 1\rangle = \langle f^{-1}(x), 0\rangle$, then we write $X \times I/f$ rather than $X \times I/h$.

Lemma. *Let X be a compact totally disconnected Hausdorff space, and let D be a closed-open subset of X. Let Y be a compact Hausdorff space each point of which has a neighbourhood homeomorphic to $X \times J$, where J is an open unit interval. Let Z be a closed subset of Y homeomorphic to $D \times I$ under a homeomorphism f and such that* $\mathrm{Int}_Y Z = f^{-1}(D \times J)$. *Then the quotient space $Y/D \times I$, which is obtained from Y by collapsing each arc in Z to a point, is homeomorphic to Y.*

Proof. For each point y of $f^{-1}(D \times \{0\})$ take a neighbourhood V_y homeomorphic to $X \times J$ under a homeomorphism g_y. Consider $g_y(Z \cap V_y)$ and $g_y(\mathrm{Int}_Y Z \cap V_y)$. Since Y is compact and Hausdorff and V_y is an open subset of Y, there exist a closed-open subset D_y of X and points $a_y < b_y$ of J such that $g_y^{-1}(D_y \times [a_y, b_y])$ contains y in its interior and has no point in common with $f^{-1}(X \times [\frac{1}{2}, 1])$ and the intersection of each arc of $D_y \times [a_y, b_y]$ with $g_y(Z \cap V_y)$ is a proper non-degenerate subinterval of that arc. Denote the set $g_y^{-1}(D_y \times (a_y, b_y))$