Let $C = \bigcup_{j=0}^{\infty} (\{h(b_j)\} \cup \{h^{-1}(b_j)\})$ and $B' = \{b_0, b_1, \ldots\}$. Notice that B' and C do not intersect. For example, if $b_n = h(b_j)$, then $n \neq j$, else $b_n \in N$, and $n \not> j$, else $b_n \in \{h(b_j)\}$, and $n \not> j$, else $b_j \in \{h^{-1}(b_n)\}$. B' is dense in $S-C$ though, for it is dense in I, and $M \cap (S-C)^2$ is closed (relative to $(S-C)^2$) and still uncountable, but it is forced to miss the B'-grid in $(S-C)^2$.

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Dimension of free L-spaces

by

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Abstract. We introduce the class of free L-spaces which is countably productive and hereditary. The class is an intermediate class between that of L-spaces and that of M_1-spaces. The class has excellent features in dimension theory a part of which is clarified in this paper.

0. Introduction. In a previous paper [6] we introduced the notion of L-spaces which constitute an intermediate class between that of Lašnev spaces and that of M_1-spaces. As was noted there the class of L-spaces is not even finitely productive. In this paper we introduce the notion of free L-spaces in Section 1 which generalizes the notion of L-spaces. The class of free L-spaces is not only hereditary and countably productive but also has many excellent features in dimension theory. In Section 2 we show that even the dimension-raising theorem is valid for the class of free L-spaces. As trivial corollaries of this theorem there are the decomposition theorem and the coincidence theorem for two basic dimensions. A characterization theorem for a free L-space X with $\dim X = n$ is also presented. Our characterization assures the existence of equi-dimensional G_δ-envelopes as in Theorem 2.8 below. In Section 3 we show that the universal space for free L-spaces is the countable product of almost polyhedral spaces. As a special case we prove, in Theorem 3.8 below, that each space X is a free L-space with $\dim X \leqslant 0$ if and only if it is embedded in the countable product of almost discrete spaces. Thus a role played by Baire's 0-dimensional spaces in the theory of metric spaces is done by the countable products of almost discrete spaces in the theory of free L-spaces.

In this paper all spaces are assumed to be Hausdorff topological spaces, maps to be continuous onto, and images to be those under maps. The letter N denotes the positive integers. For undefined terminology refer to [2] and [6].

1. Definition of free L-spaces.

1.1. Definition. Let X be a space, F a closed set of X, and $\mathcal{U}$ an anti-cover of F. If S is a subset of X, $\mathcal{U}(S)$ denotes the star $\bigcup \{U \in \mathcal{U}: U \cap S \neq \emptyset\}$. $\mathcal{U}^i(S)$ is defined inductively by the formulae: $\mathcal{U}^1(S) = \mathcal{U}(S)$ and $\mathcal{U}^i(S) = \mathcal{U}(\mathcal{U}^{i-1}(S))$. A set V of X is said to be a *canonical neighborhood* of F (with respect to $\mathcal{U}$) if V is an open neighborhood of F such that, for each i, $\mathrm{Cl}\,\mathcal{U}^i(X-V)$ does not meet F.

1.2. DEFINITION. For a space X consider a pair $\mathscr{P} = (\mathscr{F}, \{\mathscr{U}_F: F \in \mathscr{F}\})$ of a σ-discrete closed collection $\mathscr{F}$ of X and a collection of anti-covers $\mathscr{U}_F$ of $F \in \mathscr{F}$. $\mathscr{P}$ is said to be a *free L-structure* if for each point $x \in X$ and each open neighborhood U of x there exist a finite subcollection $\{F_1, \ldots, F_k\}$ of $\mathscr{F}$ and a canonical neighborhood U_i of each F_i with $x \in \bigcap_{i=1}^{k} F_i \subset \bigcap_{i=1}^{k} U_i \subset U$. X is said to be a *free L-space* if X is a paracompact space admitting a free L-structure.

If $\mathscr{P}$ is a free L-structure, the collection of all finite intersections of elements of $\mathscr{F}$ is a σ-discrete network of X. Thus each free L-space is a σ-space. Each open neighborhood of F is canonical with respect to $\mathscr{U}_F$ if and only if $\mathscr{U}_F$ is approaching to F in the sense of Nagami [6].

1.3. THEOREM. *To be a free L-space is hereditary and countably productive property.*

Proof. Let X be a free L-space, $(\mathscr{F}, \{\mathscr{U}_F: F \in \mathscr{F}\})$ a free L-structure of X, and S a subset of X. Since X is a paracompact σ-space, S is paracompact. Since the restriction $(\mathscr{F}|S, \{\mathscr{U}_F|S: F \in \mathscr{F}\})$ is, as can easily be seen, a free L-structure of S, then S is a free L-space.

Let X_i, $i \in N$, be a sequence of free L-spaces, and $(\mathscr{F}_i, \{\mathscr{U}_F: F \in \mathscr{F}_i\})$, $i \in N$, a sequence of corresponding free L-structures. Set $X = \prod X_i$. Since each X_i is a paracompact σ-space, X is paracompact. Let π_i: $X \to X_i$, $i \in N$, be the projections. Set $\mathscr{H} = \bigcup \pi_i^{-1}(\mathscr{F}_i)$. Then $\mathscr{H}$ is a σ-discrete closed collection. Let us see that $(\mathscr{H}, \{\pi_i^{-1}(\mathscr{U}_F): F \in \mathscr{F}_i, i \in N\})$ is a free L-structure of X. If $F \in \mathscr{F}_i$, $\pi_i^{-1}(\mathscr{U}_F)$ is an anti-cover of $\pi_i^{-1}(F)$. Let $x = (x_i)$ be an arbitrary point of X, and U an arbitrary open neighborhood of x. Choose a finite subset M of N and open neighborhoods U_i of x_i, $i \in M$, such that $\bigcap \{\pi_i^{-1}(U_i): i \in M\} \subset U$. For each $i \in M$, choose a finite subcollection $\{F(i,j): j = 1, \ldots, n(i)\} \subset \mathscr{F}_i$ and canonical neighborhoods $U(i,j)$ of $F(i,j)$ with respect to $\mathscr{U}_{F(i,j)}$, $j = 1, \ldots, n(i)$, such that

$$x_i \in \bigcap \{F(i\ j): j = 1, \ldots, n(i)\} \subset \bigcap \{U(i,j): j = 1, \ldots, n(i)\} \subset U_i .$$

Then $\pi_i^{-1}(U(i,j))$ is a canonical neighborhood of $\pi_i^{-1}(F(i,j))$ with respect to $\pi_i^{-1}(\mathscr{U}_{F(i,j)})$ and

$$\begin{aligned} x &\in \bigcap \{\pi_i^{-1}(F(i,j)): j = 1, \ldots, n(i), i \in M\} \\ &\subset \bigcap \{\pi_i^{-1}(U(i,j)): j = 1, \ldots, n(i), i \in M\} \\ &\subset \bigcap \{\pi_i^{-1}(U_i): i \in M\} . \end{aligned}$$

That completes the proof.

2. Dimension for free L-spaces.

2.1. LEMMA. *Let X be a hereditarily paracompact space. Let F, H be a disjonit pair of closed sets of X. Let $\mathscr{V}$ be a σ-locally finite open cover of X with $V^5(F) \cap H = \emptyset$. Then there exists an open set D such that $F \subset D \subset \overline{D} \subset X - H$ and $\partial D \subset \bigcup \{\partial V: V \in \mathscr{V}\}$. If each binary open cover of X can be refined by a σ-locally finite open cover $\mathscr{V}$ such that* $\operatorname{Ind} \partial V \leqslant n-1$ *for each $V \in \mathscr{V}$, then* $\operatorname{Ind} X \leqslant n$.

Cf. Nagami [2], Theorem 11.12.

2.2. LEMMA (Nagami [5], Lemma 4). *Let X and Y be paracompact σ-spaces and f a closed map of X onto Y. If order $f = n$ and, for each point $y \in Y$, $f^{-1}(y)$ consists of exactly n points, then* $\dim Y \leqslant \dim X$.

2.3. THEOREM. *For a free L-space X the following four conditions are equivalent.*

(1) $\dim X \leqslant n$.

(2) *X is the image of a free L-space Z with $\dim Z \leqslant 0$ under a closed map f of order $\leqslant n+1$.*

(3) *X is the sum of $n+1$ subsets Z_i, $i = 1, \ldots, n+1$, with $\dim Z_i \leqslant 0$ for each i.*

(4) $\operatorname{Ind} X \leqslant n$.

Proof. The implication (2) → (3) is a direct consequence of Lemma 2.2. The implications (3) → (4) → (1) are already known (cf. Nagami [2], Theorem 12.6).

To prove that (1) implies (2) let $(\mathscr{F} = \bigcup \mathscr{F}_i, \{\mathscr{U}_F: F \in \mathscr{F}\})$, with each $\mathscr{F}_i$ discrete, be a free L-structure of X. Set $\mathscr{F}_i = \{F(i, \alpha): \alpha \in A_i\}$. Let $\{U(i, \alpha): \alpha \in A_i\}$ and $\{V(i, \alpha): \alpha \in A_i\}$ be discrete collections of open sets such that

$$F(i, \alpha) \subset V(i, \alpha) \subset \operatorname{Cl} V(i, \alpha) \subset U(i, \alpha) \quad \text{for each } \alpha \in A_i .$$

Set

$$H_i = \bigcup \{F(i, \alpha): \alpha \in A_i\} ,$$

$$U_i = \bigcup \{U(i, \alpha): \alpha \in A_i\} ,$$

and

$$V_i = \bigcup \{V(i, \alpha): \alpha \in A_i\} .$$

Then $H_i \subset V_i \subset \overline{V}_i \subset U_i$. By the perfect normality of X there exists an anti-cover $\mathscr{V}_i$ of H_i consisting of $X - \overline{V}_i$ and of open subsets of $U_i - H_i$ such that V_i is a canonical neighborhood of H_i with respect to $\mathscr{V}_i$. Set

$$\mathscr{W}_i = \{X - \overline{V}_i\} \cup \big(\bigcup \{\mathscr{U}_{F(i,\alpha)}|(U(i, \alpha) - F(i, \alpha)) \wedge \mathscr{V}_i|(U(i, \alpha) - F(i, \alpha)): \alpha \in A_i\}\big).$$

Then $\mathscr{W}_i$ is an anti-cover of H_i. Let $\bigcup_{j=1}^{\infty} \mathscr{W}_{ij}$ be an anti-cover of H_i refining $\mathscr{W}_i$ such that each $\mathscr{W}_{ij}$ is discrete in $X - H_i$ and $\bigcup_{j=1}^{\infty} \mathscr{W}_{ij}$ is locally finite in $X - H_i$. Set $\mathscr{W}_{ij} = \{W_{ij\beta}: \beta \in B_{ij}\}$ and $W_{ij} = \bigcup \{W_{ij\beta}: \beta \in B_{ij}\}$. Set $W_{ij} = \bigcup_{k=1}^{\infty} K_{ijk}$, where each K_{ijk} is a closed set of X.

By Leĭbo [1] there exist a metric space ϱX with $\dim \varrho X \leqslant \dim X$ and a contraction (i.e. a one-one map) ϱ: $X \to \varrho X$ such that the images of all K_{ijk} and $X - W_{ij}$

under ϱ are closed in ϱX. Well order the collection of all disjoint pairs $(X-W_{ij}, K_{ijk})$ as follows:

$$\{(X-W_{ij}, K_{ijk}):\ i, j, k\in N\} = \{(P_i, Q_i):\ i\in N\}\,.$$

Let $\mathscr{G}_i = \{G_{i\lambda}:\ \lambda\in\Lambda_i\}$, $i\in N$, be a sequence of locally finite open covers of ϱX satisfying the following four conditions for each $i\in N$:

a) $\operatorname{mesh}\mathscr{G}_i\leqslant 1/i$,

b) $\overline{\mathscr{G}}_{i+1}<\mathscr{G}_i$,

c) $\operatorname{order}\mathscr{G}_i\leqslant n+1$,

d) $\mathscr{G}_i<\{\varrho X-\varrho P_i, \varrho X-\varrho Q_i\}$.

Let π_i^{i+1}: $\Lambda_{i+1}\to\Lambda_i$ be a transformation such that $\pi_i^{i+1}(\lambda)=\mu$ yields $\overline{G}_{i+1,\lambda}\subset G_{i\mu}$, and consider the inverse system $\{\Lambda_i, \pi_i^{i+1}\}$, where each Λ_i is endowed with the discrete topology. Let σZ be the aggregate of all points (λ_i) of $\varprojlim\Lambda_i$ with $\bigcap_{i=1}^{\infty} G_{i\lambda_i}\neq\emptyset$. Let g: $\sigma Z\to\varrho X$ be a transformation defined by: $g((\lambda_i)) = \bigcap_{i=1}^{\infty} G_{i\lambda_i}$. Then g is a closed map onto with order $g\leqslant n+1$ and $\dim\sigma Z\leqslant 0$ by Nagami [2], Theorem 12.6. Let π_j: $\sigma Z\to\Lambda_j$ be the restriction of the projection of $\varprojlim\Lambda_i$ to Λ_j. Consider the diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X\\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\varrho}\\ \sigma Z & \xrightarrow{\ g\ } & \varrho X\end{array}$$

Let the set Z be identical with ϱZ, σ the identity transformation of Z onto σZ and f: $Z\to X$ the transformation such that $g\sigma = \varrho f$. Give Z the minimal topology among those which enable both f and σ to be continuous. By an argument which is essentially the same as in Nagami [4], Theorem 6, we can see that f is a closed map onto. Since σ is a contraction, Z is Hausdorff. Since order f = order $g\leqslant n+1$, Z is a paracompact space as a perfect preimage of a paracompact space X.

Let us construct a free L-structure of Z. Set $\mathscr{H} = \{\pi_i^{-1}(\lambda):\ \lambda\in\Lambda_i, i\in N\}$. Since $\mathscr{H}$ is σ-discrete and each element of $\mathscr{H}$ is open and closed, $\sigma^{-1}(\mathscr{H})$ is σ-discrete and each element of $\sigma^{-1}(\mathscr{H})$ is open and closed. Set $\mathscr{K} = \sigma^{-1}(\mathscr{H})\wedge f^{-1}(\mathscr{F})$. Then $\mathscr{K}$ is a closed collection which is σ-discrete in Z. Let K be a generic element of $\mathscr{K}$. Then $K = \sigma^{-1}\pi_i^{-1}(\lambda)\cap f^{-1}(F)$ for some i, some $\lambda\in\Lambda_i$, and some $F\in\mathscr{F}$. Set

$$\mathscr{L}_K = \{Z-\sigma^{-1}\pi_i^{-1}(\lambda)\}\cup\big(f^{-1}(\mathscr{U}_F)|\ \sigma^{-1}\pi_i^{-1}(\lambda)\big)\,.$$

Then $\mathscr{L}_K$ is an anti-cover of K.

To see that $\mathscr{P} = (\mathscr{K}, \{\mathscr{L}_K:\ K\in\mathscr{K}\})$ is a free L-structure of Z let z be an arbitrary point of Z and U an arbitrary open neighborhood of z. Then

$$z\in\sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}(V)\subset U$$

for some j, some $\mu\in\Lambda_j$, and some open neighborhood V of $f(z)$. Choose a finite subcollection $\{F_1, \ldots, F_k\}$ of $\mathscr{F}$ and canonical neighborhood D_i of F_i, $i = 1, \ldots, k$, such that

$$f(z)\in\bigcap_{i=1}^{k} F_i\subset\bigcap_{i=1}^{k} D_i\subset V\,.$$

Set $K_i = \sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}(F_i)$. Then $\sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}(D_i)$ is a canonical neighborhood of K_i with respect to $\mathscr{L}_{K_i}$. Since

$$\begin{aligned} z\in\bigcap_{i=1}^{k} K_i &\subset\bigcap_{i=1}^{k}\big(\sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}(D_i)\big)\\ &= \sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}\Big(\bigcap_{i=1}^{k} D_i\Big)\\ &\subset\sigma^{-1}\pi_j^{-1}(\mu)\cap f^{-1}(V)\subset U\,,\end{aligned}$$

then $\mathscr{P}$ is a free L-structure of Z.

To prove the final inequality $\dim Z\leqslant 0$ we need some assertions.

ASSERTION 1. *$f^{-1}(P_i)$ and $f^{-1}(Q_i)$ can be separated in Z by the empty set.*

Proof. Set $\Lambda_j' = \{\lambda\in\Lambda_j:\ G_{j\lambda}\cap\varrho P_j\neq\emptyset\}$, $j\in N$. Then by the condition d), $g^{-1}\varrho P_j\subset\pi_j^{-1}(\Lambda_j')\subset\sigma Z-g^{-1}\varrho Q_j$, $\sigma^{-1}g^{-1}\varrho P_j\subset\sigma^{-1}\pi_j^{-1}(\Lambda_j')\subset Z-\sigma^{-1}g^{-1}\varrho Q_j$, and hence $f^{-1}(P_j)\subset\sigma^{-1}\pi_j^{-1}(\Lambda_j')\subset Z-f^{-1}(Q_j)$. Obviously $\sigma^{-1}\pi_j^{-1}(\Lambda_j')$ is open and closed. That proves the assertion.

ASSERTION 2. *Let D_i be a canonical neighborhood of H_i with respect to $\mathscr{W}_i$ with $D_i\subset V_i$. Then $f^{-1}(H_i)$ and $Z-f^{-1}(D_i)$ can be separated by the empty set.*

Proof. Set $E_i = \mathscr{W}_i^5(X-D_i)$ and $\mathscr{D}_i = \{f^{-1}(D_i)-f^{-1}(H_i), f^{-1}(E_i)\}$. Then $\mathscr{D}_i$ is a binary open cover of the subspace $Z-f^{-1}(H_i)$. For each j, $k\in N$, there exists, by Assertion 1, an open and closed set R_{ijk} such that $f^{-1}(K_{ijk})\subset R_{ijk}\subset f^{-1}(W_{ij})$. Set $S_{ijk\beta} = f^{-1}(W_{ij\beta})\cap R_{ijk}$. Then $S_{ijk\beta}$ is an open and closed set. Set

$$\mathscr{S}_{ijk} = \{S_{ijk\beta}:\ \beta\in B_{ij}\},\qquad \mathscr{S}_i = \bigcup\{\mathscr{S}_{ijk}:\ j, k\in N\}\,.$$

Then $\mathscr{S}_{ijk}$ is discrete, covers R_{ijk}, and refines $f^{-1}(\mathscr{W}_{ij})$. Since

$$\begin{aligned} Z-f^{-1}(H_i) &= \bigcup\{f^{-1}(K_{ijk}):\ j, k\in N\}\\ &\subset\bigcup\{R_{ijk}:\ j, k\in N\}\\ &\subset\bigcup\{f^{-1}(W_{ij}):\ j\in N\} = Z-f^{-1}(H_i)\,,\end{aligned}$$

$\mathscr{S}_i$ is a σ-discrete cover of $Z-f^{-1}(H_i)$ whose elements are open and closed. Since $\mathscr{S}_i<\bigcup_{j=1}^{\infty} f^{-1}(\mathscr{W}_{ij})<f^{-1}(\mathscr{W}_i)$, there exists, by Lemma 2.1, a set D, being open and closed in $Z-f^{-1}(H_i)$, such that

$$Z-f^{-1}(E_i\cup H_i)\subset D\subset f^{-1}(D_i)-f^{-1}(H_i)\,.$$

Since $Z-f^{-1}(E_i)$ is a canonical neighborhood of $f^{-1}(H_i)$ with respect to $f^{-1}(\mathscr{W}_i)$, $D\cup f^{-1}(H_i)$ is a canonical neighborhood of $f^{-1}(H_i)$ with respect to $f^{-1}(\mathscr{W}_i)$. The latter fact implies that $D\cup f^{-1}(H_i)$ is open and closed in Z. The inequalities $f^{-1}(H_i)\subset Z-f^{-1}(E_i)\subset D\cup f^{-1}(H_i)\subset f^{-1}(D_i)$ proves the assertion.

ASSERTION 3. *For each point $x\in X$ and each open neighborhood U of x there exist finite elements $F(i,\alpha_i)\in\mathscr{F}_i$, $i\in M$, and an open and closed set V of Z such that*

$$f^{-1}(x)\subset \bigcap_{i\in M} f^{-1}(F(i,\alpha_i))\subset V\subset\Big(\bigcap_{i\in M} f^{-1}(V(i,\alpha_i))\Big)\cap f^{-1}(U).$$

Proof. Choose a finite set $M\subset N$, elements $F(i,\alpha_i)\in\mathscr{F}_i$, $i\in M$, and canonical neighborhoods G_i of $F(i,\alpha_i)$ with respect to $\mathscr{U}_{F(i,\alpha_i)}$, $i\in M$, such that

$$x\in \bigcap_{i\in M} F(i,\alpha_i)\subset \bigcap_{i\in M} G_i\subset U.$$

Notice that

$$\mathscr{W}_i|(U(i,\alpha_i)-F(i,\alpha_i))<\mathscr{U}_{F(i,\alpha_i)}|(U(i,\alpha_i)-F(i,\alpha_i))$$

and $V(i,\alpha_i)$ is a canonical neighborhood of $F(i,\alpha_i)$ with respect to

$$\mathscr{W}_i|(U(i,\alpha_i)-F(i,\alpha_i)),$$

then $G_i\cap V(i,\alpha_i)$ is a canonical neighborhood of $F(i,\alpha_i)$ with respect to $\mathscr{W}_i|(U(i,\alpha_i)-F(i,\alpha_i))$. Therefore, if we set

$$S_i=(G_i\cap V(i,\alpha_i))\cup\big(\bigcup\{V(i,\alpha):\ \alpha\neq\alpha_i\}\big),$$

S_i is a canonical neighborhood of H_i with respect to $\mathscr{W}_i$. By Assertion 2 there exists an open and closed set T_i of Z with $f^{-1}(H_i)\subset T_i\subset f^{-1}(S_i)$. If we set $T_i'=T_i\cap f^{-1}(V(i,\alpha_i))$, T_i' is still an open and closed set of Z with $f^{-1}(F(i,\alpha_i))\subset T_i'\subset f^{-1}(G_i\cap V(i,\alpha_i))$. Set $V=\bigcap_{i\in M} T_i'$. Then

$$\begin{aligned} f^{-1}(x)&\subset \bigcap_{i\in M} f^{-1}(F(i,\alpha_i))\subset V\\ &\subset \bigcap_{i\in M} f^{-1}(G_i\cap V(i,\alpha_i))\\ &\subset\Big(\bigcap_{i\in M} f^{-1}(V(i,\alpha_i))\Big)\cap f^{-1}(U).\end{aligned}$$

That proves the assertion.

ASSERTION 4. *Let U be an open set of X. Then $f^{-1}(U)$ admits a cover $\mathscr{W}$ which is σ-discrete in Z and each element of which is open and closed.*

Proof. Set

$$\begin{aligned} N^*&=\{M\subset N:\ |M|<\infty\},\\ A_M&=\prod\{A_i:\ i\in M\},\quad M\in N^*,\\ F_\gamma&=\bigcap\{F(i,\alpha_i):\ i\in M\},\quad \gamma=(\alpha_i:\ i\in M)\in A_M,\ M\in N^*,\\ V_\gamma&=\bigcap\{V(i,\alpha_i):\ i\in M\},\quad \gamma=(\alpha_i:\ i\in M)\in A_M,\ M\in N^*.\end{aligned}$$

Let B_M be the aggregate of all indices $\gamma\in A_M$ such that $f^{-1}(F_\gamma)\subset W_\gamma\subset f^{-1}(V_\gamma\cap U)$ for some open and closed set W_γ. Set

$$\mathscr{W}=\{W_\gamma:\ \gamma\in B_M,\ M\in N^*\}.$$

By Assertion 3, $\{f^{-1}(F_\gamma):\ \gamma\in B_M,\ M\in N^*\}$ covers $f^{-1}(U)$ and hence $\mathscr{W}$, consisting of open and closed sets, covers $f^{-1}(U)$. Since $\{V_\gamma:\ \gamma\in A_M\}$ is discrete in X for each $M\in N^*$ and N^* is countable, $\{f^{-1}(V_\gamma):\ \gamma\in B_M,\ M\in N^*\}$ is σ-discrete in Z and hence $\mathscr{W}$ is σ-discrete in Z. That proves the assertion.

ASSERTION 5. *Let G be an open set of Z. Then G has a cover $\mathscr{G}$ which is σ-discrete in Z and each element of which is open and closed.*

Proof. For each i and each $\lambda\in A_i$ let U_λ be the largest open set of X such that $\sigma^{-1}\pi_i^{-1}(\lambda)\cap f^{-1}(U_\lambda)\subset G$. Then $G=\bigcup\{\sigma^{-1}\pi_i^{-1}(\lambda)\cap f^{-1}(U_\lambda):\ \lambda\in A_i,\ i\in N\}$. By Assertion 4 there exists a cover $\mathscr{W}_\lambda$ of $f^{-1}(U_\lambda)$ which is σ-discrete in Z and each element of which is open and closed. Set

$$\mathscr{G}_\lambda=\mathscr{W}_\lambda|\sigma^{-1}\pi_i^{-1}(\lambda)\cap f^{-1}(U_\lambda).$$

Then $\mathscr{G}_\lambda$ is still σ-discrete in Z and each element of $\mathscr{G}_\lambda$ is open and closed. Set

$$\mathscr{G}=\bigcup\{\mathscr{G}_\lambda:\ \lambda\in A_i,\ i\in N\}.$$

Then $\mathscr{G}$ is a cover of G consisting of open and closed sets. Since $\sigma^{-1}(\mathscr{H})$ is σ-discrete, $\mathscr{G}$ is also σ-discrete. That proves the assertion.

By Assertion 5 each binary open cover of Z can be refined by a cover which is σ-discrete in Z and each element of which is open and closed. That implies $\dim Z\leqslant 0$ by Lemma 2.1. The proof of the theorem is thus completed.

From the above argument and from the first half of Theorem 1.3 it can easily be seen that the restriction of $\{(K_{ijk},X-W_{ij}):\ i,j,k\in N\}$ to an arbitrary subset S of X determines $\operatorname{Ind}S$. Thus we have the following which is a generalization of Nagami [6], Lemma 3.5.

2.4. THEOREM. *A free L-space X admits a countable collection of disjoint pairs of closed sets determining* Ind *of all subsets of X.*

2.5. DEFINITION. Let X be a space. Let $\mathscr{F}_i=\{F(i,\alpha):\ \alpha\in A_i\}$ be a discrete collection of closed sets of X. Let $\mathscr{V}_i=\{V(i,\alpha):\ \alpha\in A_i\}$ be a discrete collection of open sets with $F(i,\alpha)\subset V(i,\alpha)$, $\alpha\in A_i$. Let $\mathscr{U}_i$ be an anti-cover of

$$H_i=\bigcup\{F(i,\alpha):\ \alpha\in A_i\}.$$

Then $(\mathscr{F}=\bigcup\mathscr{F}_i,\bigcup\mathscr{V}_i,\bigcup\mathscr{U}_i)$ is said to be a *free L-mesh* of X if the following two conditions are satisfied:

a) $\bigcup\{V(i,\alpha):\ \alpha\in A_i\}$ is a canonical neighborhood of H_i with respect to $\mathscr{U}_i$.

b) For each point $x\in X$ and each open neighborhood U of x there exist finite elements $F(i,\alpha_i)\in\mathscr{F}$, $i\in M$, and canonical neighborhoods V_i of H_i with respect to $\mathcal{U}_i$, $i \in M$, such that

$$x \in \bigcap_{i \in M} F(i, \alpha_i) \subset \bigcap_{i \in M} (V_i \cap V(i, \alpha_i)) \subset U .$$

A free L-mesh is said to be a *strict free L-mesh* if it satisfies one more condition:

c) The intersection of all canonical neighborhoods of H_i with respect to $\mathcal{U}_i$ is H_i.

2.6. Lemma. *A paracompact space X is a free L-space if and only if X admits a (strict) free L-mesh.*

The necessity is essentially proved in Theorem 2.3. The sufficiency is an easy exercise.

2.7. Theorem. *A paracompact space X is a free L-space with $\dim X \leqslant n$ if and only if X admits a (strict) free L-mesh $(\bigcup \mathcal{F}_i, \bigcup \mathcal{V}_i, \bigcup \mathcal{U}_i)$ such that each $\mathcal{U}_i$ is locally finite in $X - H_i$ and $\dim \partial U \leqslant n-1$ for each element U of $\bigcup \mathcal{U}_i$.*

This is also an easy exercise by virtue of Lemmas 2.1 and 2.6, if we apply an analogous argument to that in Assertion 4 of Theorem 2.3.

2.8. Theorem. *Let X be a free L-space and Y a subset of X with $\dim Y \leqslant n$. Then there exists a G_δ-set S with $Y \subset S$ and $\dim S \leqslant n$.*

Proof. By Theorem 2.3 there exist subsets Z_i, $i = 1, \ldots, n+1$, of Y with $Y = \bigcup_{i=1}^{n+1} Z_i$ and with $\dim Z_i \leqslant 0$. Let $(\bigcup \mathcal{F}_i, \bigcup \mathcal{V}_i, \bigcup \mathcal{U}_i)$ be a strict free L-mesh of X, where each $\mathcal{U}_i$ is locally finite in $X - H_i$. Set $\mathcal{U}_i = \{U(i, \beta)\colon \beta \in B_i\}$. Let $\mathcal{P}_i = \{P(i, \beta)\colon \beta \in B_i\}$ be a closed cover of $X - H_i$ with $P(i, \beta) \subset U(i, \beta)$ for each $\beta \in B_i$. Since the mesh is strict, $\mathrm{Cl}\, U(i, \beta) \cap H_i = \emptyset$ and hence $P(i, \beta)$ is closed in X. Since $\dim Z_1 \leqslant 0$, there exists an open set $Q(i, \beta)$ such that $P(i, \beta) \subset Q(i, \beta) \subset \mathrm{Cl}\, Q(i, \beta) \subset U(i, \beta)$ and $\partial Q(i, \beta) \cap Z_1 = \emptyset$. Set

$$R_i = \bigcup \{\partial Q(i, \beta)\colon \beta \in B_i\} .$$

Then R_i is closed in $X - H_i$. Since H_i is G_δ, R_i is an F_σ-set of X. Set $S_1 = X - \bigcup_{i=1}^{\infty} R_i$. Then S_1 is a G_δ-set of X with $Z_1 \subset S_1$. Set $\mathcal{Q}_i = \{Q(i, \beta)\colon \beta \in B_i\}$. Let $\mathcal{T} = (\bigcup \mathcal{F}'_i, \bigcup \mathcal{V}'_i, \bigcup \mathcal{Q}'_i)$ be the restriction of $(\bigcup \mathcal{F}_i, \bigcup \mathcal{V}_i, \bigcup \mathcal{Q}_i)$ to S_1. Then $\mathcal{T}$ is a strict free L-mesh of S_1 such that $\partial Q' = \emptyset$ for each element Q' of $\bigcup \mathcal{Q}'_i$. Thus $\dim S_1 \leqslant 0$ by Theorem 2.6.

Let S_i, $i = 2, \ldots, n+1$, be G_δ-sets of X with $\dim S_i \leqslant 0$ and $Z_i \subset S_i$. Set $S = \bigcup_{i=1}^{n+1} S_i$. Then S is G_δ-set of X with $\dim S \leqslant n$ and $Y \subset S$. That completes the proof.

2.9. Theorem. *For a separable free L-space X the following four conditions are equivalent.*

(1) $\dim X \leqslant n$.

(2) *X is the image of a separable free L-space Z with $\dim Z \leqslant 0$ under a closed map of order $\leqslant n+1$.*

(3) $\mathrm{Ind}\, X \leqslant n$.

(4) $\mathrm{ind}\, X \leqslant n$.

Proof. For a separable X, ϱX in the diagram in Theorem 2.2 has to be separable. Then we can assume that σZ is separable. In that case the pullback Z in the diagram has to be separable too. Thus the implications $(1) \to (2) \to (3) \to (1)$ are true. When X is separable, X is Lindelöf. Hence for such X, as is well known, $\mathrm{ind}\, X = \mathrm{Ind}\, X$. That completes the proof.

2.10. Theorem. *Let X be a (separable) free L-space. Then X is the perfect image of a (separable) free L-space Z with $\dim Z \leqslant 0$.*

This is essentially proved in Theorem 2.3, where we have to assume that g in the diagram is merely perfect. In that case f in the diagram has to be perfect.

2.11. Problem. Is the perfect image of a free L-space again a free L-space?

2.12. Problem. Let X be a Lašnev space. Is X the perfect image of a Lašnev space Z with $\dim Z \leqslant 0$? If moreover $\dim X \leqslant n$, is X the image of a Lašnev space Z with $\dim Z \leqslant 0$ under a closed map f of order $\leqslant n+1$?

3. Embedding theorems for free L-spaces.

3.1. Definition. Let X be a space. The set of all points of X which have metric neighborhoods is said to be the *metric part* of X. The complement of the metric part is said to be the *nonmetric part*. X is said to be an *almost metric space* if the following three conditions are satisfied:

a) X is perfectly normal and paracompact.

b) The collection of points of the nonmetric part X_0 is discrete.

c) X_0 has an anti-cover approaching to X_0.

An almost metric space is said to be an *almost discrete space* if its metric part is discrete as a relative space.

3.2. Lemma. *An almost metric space X is an L-space.*

Proof. X is of course a σ-space. Let X_0 be the nonmetric part of X. Let $\mathcal{U}$ be an anti-cover of X_0 approaching to X_0. Let F be an arbitrary closed set of X. Let U be an open set of X with $X_0 - F \subset U \subset \overline{U} \subset X - F$. Let $\mathcal{V}$ be an open cover of $X - X_0 \cup F$ approaching to $F - X_0$ in $X - X_0$. Set $\mathcal{W} = \mathcal{U} \wedge \mathcal{V}$. Then $\mathcal{W}$ is an open cover of $X - X_0 \cup F$. Set $\mathcal{G} = \mathcal{W} \cup \{U\}$. Then $\mathcal{G}$ is an anti-cover of $X - F$.

To prove that $\mathcal{G}$ is approaching to F let G be an arbitrary open neighborhood of F. Since $\mathcal{U}$ is approaching to X_0, there exists an open neighborhood D of $F \cap X_0$ such that $D \cap \mathcal{U}(X - G) = \emptyset$ and hence $D \cap \mathcal{W}(X - G) = \emptyset$. Since $\mathcal{V}$ is approaching to $F - X_0$ in $X - X_0$, there exists an open set E of $X - X_0$ with $F - X_0 \subset E$ and $E \cap \mathcal{V}(X - G) = \emptyset$. Then $E \cap \mathcal{W}(X - G) = \emptyset$. Set $T = D \cup E - \overline{U}$. Then T is an open neighborhood of F with $T \cap \mathcal{G}(X - G) = \emptyset$, which implies that $\mathcal{G}$ is approaching to F. That completes the proof.

3.3. Definition. In this paper polyhedra are simplicial polyhedra with the metric topology. Let K be a polyhedron and A the vertex-set of K. Let the star of

$\alpha \in A$, say $\mathrm{St}(\alpha)$, be the set of all points of K whose barycentric weights on α are positive. The *star-cover* of K is $\{\mathrm{St}(\alpha)\colon \alpha \in A\}$. An almost metric space X is said to be an *almost polyhedral space* if the following two conditions are satisfied:

a) The metric part of X is a polyhedron K.

b) The star-cover of K is approaching to the nonmetric part of X.

3.4. Theorem. *For a space X the following four statements are equivalent.*

(1) *X is a free L-space.*

(2) *X is embedded in the countable product of almost polyhedral spaces.*

(3) *X is embedded in the countable product of almost metric spaces.*

(4) *X is embedded in the countable product of L-spaces.*

Proof. That (2) implies (3) is clear. The implications $(3)\to(4)$ and $(4)\to(1)$ are assured to be true respectively by Lemma 3.2 and by Theorem 1.3.

To prove that (1) implies (2) let $(\bigcup \mathscr{F}_i, \bigcup \mathscr{V}_i, \bigcup \mathscr{U}_i)$ be a strict free L-mesh of X, where

a) $\mathscr{F}_i = \{F(i,\alpha)\colon \alpha \in A_i\}$,

b) $\mathscr{V}_i = \{V(i,\alpha)\colon \alpha \in A_i\}$,

c) $\mathscr{U}_i = \{U(i,\beta)\colon \beta \in B_i\}$,

d) $H_i = \bigcup \{F(i,\alpha)\colon \alpha \in A_i\}$,

e) $\mathscr{U}_i$ is locally finite in $X - H_i$,

f) H_i is the countable intersection of its canonical neighborhoods with respect to $\mathscr{U}_i$,

g) $\{\mathscr{U}_i(V(i,\alpha))\colon \alpha \in A_i\}$ is discrete.

Let K_i be the nerve of $\mathscr{U}_i$ and $f_i\colon X - H_i \to K_i$ be a Kuratowski map defined as follows: The vertex-set of K_i is B_i. $\varphi_\beta\colon X - H_i \to I$ is a map such that $\varphi_\beta(x) > 0$ if and only if $x \in U(i,\beta)$ and such that $\{\varphi_\beta\colon \beta \in B_i\}$ is a partition of unity. $f_i(x)$ is the point of K_i whose barycentric weight on $\beta \in B_i$ is $\varphi_\beta(x)$, i.e. $f_i(x) = \sum\{\varphi_\beta(x)\beta\colon \beta \in B_i\}$. Introduce K_i the metric topology $\mathscr{D}_i$. Then as is well known f_i is continuous.

Let X_i be the disjoint sum of K_i and A_i. Set $\mathscr{S}_i = \{\mathrm{St}(\beta)\colon \beta \in B_i\}$. Let $g_i\colon X \to X_i$ be a transformation defined by:

$$g_i | X - H_i = f_i, \qquad g_i(x) = \alpha, \qquad x \in F(i,\alpha).$$

To give X_i a suitable topology we need some notations. Set

$$B(E) = \{\beta \in B_i\colon U(i,\beta) \subset E\}, \qquad E \subset X,$$

$$[E] = \big(\bigcup \{U(i,\beta)\colon \beta \in B(E)\}\big) \cup (E \cap H_i).$$

Let $\mathscr{E}_i = \{E_\lambda\colon \lambda \in \Lambda_i\}$ be the collection of all canonical neighborhoods of H_i with $E_\lambda = [E_\lambda]$. Set

$$G_\lambda = A_i \cup \big(\bigcup \{\mathrm{St}(\beta)\colon \beta \in B(E_\lambda)\}\big),$$

$$\mathscr{G}_i = \{G_\lambda\colon \lambda \in \Lambda_i\},$$

$$W_\alpha = \{\alpha\} \cup \Big(\bigcup \{\mathrm{St}(\beta)\colon \beta \in B\big(V(i,\alpha)\big)\}\Big), \qquad \alpha \in A_i,$$

$$\mathscr{W}_i = \{W_\alpha\colon \alpha \in A_i\},$$

$$\mathscr{B}_i = \mathscr{D}_i \cup (\mathscr{G}_i \wedge \mathscr{W}_i).$$

Give X_i the topology having $\mathscr{B}_i$ as a base. Since $g_i^{-1}(G_\lambda) = E_\lambda$, $\lambda \in \Lambda_i$, and $g_i^{-1}(W_\alpha) = [V(i,\alpha)]$, $\alpha \in A_i$, then g_i is continuous.

Assertion 1. *If E is a canonical neighborhood of H_i, then $[E]$ is also canonical.*

Proof. The assertion follows at once from the inequalities $X - \mathscr{U}_i(X-E) \subset [E] \subset E$.

Assertion 2. *$\mathscr{W}_i$ is discrete.*

Proof. Assume that there exist distinct elements $\alpha, \alpha' \in A_i$ and $\mathrm{St}(\beta)$ such that $\mathrm{St}(\beta) \cap W_\alpha \cap W_{\alpha'} \neq \emptyset$. Then there exist $\beta_1 \in B\big(V(i,\alpha)\big)$ and $\beta_2 \in B\big(V(i,\alpha')\big)$ such that $\mathrm{St}(\beta) \cap \mathrm{St}(\beta_1) \neq \emptyset$ and $\mathrm{St}(\beta) \cap \mathrm{St}(\beta_2) \neq \emptyset$. Thus $\{\beta, \beta_1\}$ and $\{\beta, \beta_2\}$ span 1-simplices of K_i and hence $U(i,\beta) \cap U(i,\beta_1) \neq \emptyset$ and $U(i,\beta) \cap U(i,\beta_2) \neq \emptyset$. These two inequalities imply that $\mathscr{U}_i(V(i,\alpha)) \cap V(i,\alpha') \neq \emptyset$, which contradicts to g).

Assertion 3. *For each element $E_\mu \in \mathscr{E}_i$, there exists an element $E_\lambda \in \mathscr{E}_i$ with $\mathscr{U}_i(E_\lambda) \cup H_i \subset E_\mu$.*

Proof. Set $E_\lambda = [X - \mathrm{Cl}\,\mathscr{U}_i(X - E_\mu)]$. Then E_λ is the required.

Assertion 4. *The inequality $\mathscr{U}_i(E_\lambda) \cup H_i \subset E_\mu$, $\lambda, \mu \in \Lambda_i$, implies that $\mathscr{S}_i(G_\lambda) \cup A_i \subset G_\mu$ and hence $\overline{G}_\lambda \subset G_\mu$.*

Proof. Let $\beta \in B_i$ be an index with $\mathrm{St}(\beta) \cap G_\lambda \neq \emptyset$. Then there exists $\beta_1 \in B(E_\lambda)$ with $\mathrm{St}(\beta) \cap \mathrm{St}(\beta_1) \neq \emptyset$. Hence $U(i,\beta) \cap U(i,\beta_1) \neq \emptyset$, which implies that $U(i,\beta) \subset \mathscr{U}_i(E_\lambda)$ and hence $\beta \in B(E_\mu)$. Thus $\mathrm{St}(\beta) \subset G_\mu$.

Assertion 5. *X_i is Hausdorff.*

The assertion can be seen by f), Assertions 1, 2, 3, 4 and by the fact that the topology of X_i does not disturb the topology of K_i.

Assertion 6. *A_i is a G_δ-set of X_i.*

Proof. By f) there exists a sequence E_{λ_j}, $j \in N$, of elements of $\mathscr{E}_i$ such that $H_i = \bigcap_{j=1}^{\infty} E_{\lambda_j}$. By Assertion 3 we can pick $E_{\mu_j} \in \mathscr{E}_i$ with $\mathscr{U}_i(E_{\mu_j}) \subset E_{\lambda_j}$. Let $\mathrm{St}(\beta)$ be an arbitrary element of $\mathscr{S}_i$. Then there exists k with $U(i,\beta) - E_{\lambda_k} \neq \emptyset$ and hence with $U(i,\beta) \cap E_{\mu_k} = \emptyset$. Thus for each $\beta' \in B(E_{\mu_k})$, $U(i,\beta) \cap U(i,\beta') = \emptyset$. Therefore $\mathrm{St}(\beta) \cap \mathrm{St}(\beta') = \emptyset$ for each $\beta' \in B(E_{\mu_k})$ and hence $\mathrm{St}(\beta) \cap G_{\mu_k} = \emptyset$. That proves the equality $A_i = \bigcap_{j=1}^{\infty} G_{\mu_j}$.

Assertion 7. *X_i is paracompact and perfectly normal.*

Proof. Let $\mathscr{U}$ be an arbitrary open cover of X_i. For each $\alpha \in A_i$ choose $U_\alpha \in \mathscr{U}$ with $\alpha \in U_\alpha$, and then choose $\lambda(\alpha) \in \Lambda_i$ with $W_\alpha \cap G_{\lambda(\alpha)} \subset U_\alpha$. Set

$$E = \bigcup \{g_i^{-1}(W_\alpha \cap G_{\lambda(\alpha)})\colon \alpha \in A_i\}.$$

Then E is a canonical neighborhood of H_i. By Assertions 2 and 3, there exist elements $\mu, \nu \in A_i$ with $E_\mu \subset E$ and $\bar{G}_\nu \subset G_\mu$. Let $\mathscr{V}$ be a locally finite open cover of $X_i - G_\nu$ refining $\mathscr{U}$. Set

$$\mathscr{W} = \{W_\alpha \cap G_{\lambda(\alpha)}\colon \alpha \in A_i\}, \qquad \mathscr{D} = \mathscr{W} \cup \mathscr{V} | (X - \bar{G}_\nu).$$

Then by Assertion 2, $\mathscr{D}$ is a locally finite open cover of X_i refining $\mathscr{U}$. Thus X_i is a paracompact space by Assertion 5 and hence normal.

Let U be an arbitrary open set of X_i. By Assertion 4 there exists a sequence $G_j, j \in N$, of open sets of X_i with $A_i = \bigcap_{j=1}^{\infty} G_j$. Let $F_{jk}, k \in N$, be a sequence of closed sets of X_i such that $U - G_j = \bigcup_{k=1}^{\infty} F_{jk}$. Then

$$U = \Big(\bigcup_{j,k=1}^{\infty} F_{jk}\Big) \cup (U \cap A_i).$$

Since $U \cap A_i$ is closed, U is an F_σ-set of X_i.

Assertion 8. *$\mathscr{S}_i$ is approaching to A_i.*

This is clear from Assertions 3 and 4.

Assertion 9. *Let $g\colon X \to \prod_{i=1}^{\infty} X_i$ be a transformation defined by: $g(x) = (g_i(x))$. Then g is an embedding.*

Proof. Since each g_i is continuous, g is continuous. To prove g is one-one let x, y be distinct points of X. Then for some $j \in N$, some canonical neighborhood V_j of H_j with respect to $\mathscr{U}_j$, and some $\alpha \in A_j$,

$$x \in F(j, \alpha) \subset V_j \cap (V(j, \alpha)) \subset X - \{y\}.$$

When $y \notin V(j, \alpha)$, $g(y) \notin W_\alpha$ and hence $g_j(y) \neq \alpha$. Since $g_j(x) = \alpha$, $g_j(x) \neq g_j(y)$ and hence $g(x) \neq g(y)$.

Consider the case when $y \notin V_j$. Choose $\lambda \in A_j$ such that $E_\lambda \subset V_j$. Since $g_j^{-1}(G_\lambda) = E_\lambda$, $g_j(y) \notin G_\lambda$. Since $g_j(x) = \alpha \in G_\lambda$, $g_j(x) \neq g_j(y)$ and hence $g(x) \neq g(y)$.

To prove the continuity of g^{-1} let x be an arbitrary point of X and U an arbitrary open neighborhood of x. Choose finite elements $F(i, \alpha_i) \in \bigcup \mathscr{F}_j$, $j \in M$, and elements $\lambda_i \in A_i$, $i \in M$, such that

$$x \in \bigcap_{i \in M} F(i, \alpha_i) \subset \bigcap_{i \in M} \big(E_{\lambda_i} \cap [V(i, \alpha_i)]\big) \subset U.$$

Set

$$W = \prod_{i \in M} (G_{\lambda_i} \cap W_{\alpha_i}) \times \prod_{i \in N - M} X_i.$$

Since

$$g_i^{-1}(G_{\lambda_i} \cap W_{\alpha_i}) = E_{\lambda_i} \cap [V(i, \alpha_i)], \quad i \in M, \quad g^{-1}(W) \subset U.$$

Thus g^{-1} is continuous.

Assertion 10. *X_i is embedded in the countable product of almost polyhedral spaces.*

Proof. If X_i is not first-countable at any $\alpha \in A_i$, then X_i itself is almost polyhedral and there is no problem. Let A_i' be the set of all points $\alpha \in A_i$ at which X_i is not first-countable. Set $A_i'' = A_i - A_i'$. Then it can easily be seen that $K_i \cup A_i''$ is metric and A_i' is the nonmetric part of X_i. Let $\{P(j, \alpha)\colon \alpha \in A_i'', j \in N\}$ be a collection of open sets of X_i such that $\{P(j, \alpha)\colon j \in N\}$ forms a neighborhood base of $\alpha \in A_i''$, $P(j, \alpha) \supset \mathrm{Cl}\, P(j+1, \alpha)$ for each $j \in N$ and each $\alpha \in A_i''$, and $\{\mathrm{Cl}\, P(1, \alpha)\colon \alpha \in A_i''\}$ forms a collection of subsets of $X_i - A_i'$ which is discrete in X_i. Let $\mathscr{T}_{ij}$ be an anti-cover of A_i' such that $\mathscr{T}_{ij} < \mathscr{S}_i \cup \{P(j, \alpha)\colon \alpha \in A_i''\}$, $\mathscr{T}_{ij}$ is locally finite in $X_i - A_i'$, and mesh $\mathscr{T}_{ij} < 1/j$. Let L_{ij} be the nerve of $\mathscr{T}_{ij}$ and $g_{ij}\colon X_i - A_i' \to L_{ij}$ be a Kuratowski map. Let X_{ij} be the disjoint sum of L_{ij} and A_i'. Define $h_{ij}\colon X_i \to X_{ij}$ by:

$$h_{ij}|(X - A_i') = g_{ij}, \quad h_{ij}(\alpha) = \alpha, \quad \alpha \in A_i'.$$

Give X_{ij} the topology as follows. An open set of L_{ij} is open in X_{ij} and a set V of X_{ij} meeting A_i' is open if $h_{ij}^{-1}(V)$ is open in X_i and if $V - A_i'$ is the sum of some sub-collection of the star-cover $\mathscr{S}_{ij}$ of L_{ij}.

Set $P = \bigcup \{\mathrm{Cl}\, P(1, \alpha)\colon \alpha \in A_i''\}$. Let α be an arbitrary element of A_i' and $\{Q_k\colon k \in N\}$ be an arbitrary collection of open neighborhoods of α in X_{ij} such that $Q_k \supset \bar{Q}_{k+1}$ $(k \in N)$, $\bar{Q}_1 \cap A_i' = \{\alpha\}$, and $\bar{Q}_1 \cap P = \emptyset$. Since X_i is not first-countable at α and $\mathscr{S}_i$ is approaching to A_i, there exists an open neighborhood U of α such that $\mathrm{Cl}\, \mathscr{S}_i(U) \cap A_i = \{\alpha\}$, $\mathscr{S}_i(U) \cap P = \emptyset$, and $h_{ij}^{-1}(Q_k) - \mathscr{S}_i(U) \neq \emptyset$ for any $k \in N$. Set

$$U' = \big(\bigcup \{T \in \mathscr{T}_{ij}\colon T \subset \mathscr{S}_i(U)\}\big) \cup \{\alpha\}.$$

Then there exists a set Q of X_{ij} such that $h_{ij}^{-1}(Q) = U'$ and $Q - A_i'$ is the sum of some subcollection of $\mathscr{S}_{ij}$. Since

$$\mathscr{T}_{ij}|(X_i - P) < \mathscr{S}_i|(X_i - P), \qquad U \subset U' \subset \mathscr{S}_i(U)$$

and hence U' is an open neighborhood of α in X_i. Thus Q is an open neighborhood of α in X_{ij} with

$$h_{ij}^{-1}(Q_k) - h_{ij}^{-1}(Q) = h_{ij}^{-1}(Q_k) - U' \supset h_{ij}^{-1}(Q_k) - \mathscr{S}_i(U) \neq \emptyset.$$

Therefore $Q_k - Q \neq \emptyset$ for any $k \in N$ and $\{Q_k\colon k \in N\}$ cannot be a neighborhood base of α in X_{ij}. Since the first-countability does not hold at any $\alpha \in A_i'$ in X_{ij}, the nonmetric part of X_{ij} is exactly A_i'.

Using the fact that $\mathscr{S}_i$ is approaching to A_i in X_i and that

$$\mathscr{T}_{ij}|(X_i - P) < \mathscr{S}_i|(X_i - P),$$

it is easy to see that $\mathscr{S}_{ij}$ is approaching to A_i' in X_{ij}. Evidently X_{ij} is paracompact and perfectly normal. Thus X_{ij} is almost polyhedral.

Define $h_i\colon X_i \to Y_i = \prod_{j=1}^{\infty} X_{ij}$ by: $h_i(p) = (h_{ij}(p)\colon j \in N)$. Then by the fact that $\lim_{j\to\infty} \operatorname{mesh} \mathcal{T}_{ij} = 0$, h_i is, as can easily be seen, an embedding.

To complete the proof of the theorem define $\psi_i\colon X \to Y_i$ by: $\psi_i = h_i g_i$, and $\psi\colon X \to \prod_{i=1}^{\infty} Y_i$ by: $\psi = (\psi_i)$. Then ψ is an embedding into the countable product of almost polyhedral spaces. The proof of the theorem is finished.

The author introduced in [3] the notion of μ-spaces. A space is said to be a *σ-metric space* if it is the countable sum of closed metric subsets. A space is said to be a *μ-space* if it is embedded in the countable product of paracompact σ-metric spaces. Since each almost metric space is σ-metric, we get at once the following.

COROLLARY 3.5. *The class of free L-spaces is a subclass of μ-spaces.*

Thus each Lašnev space is a μ-space, which is a new information about Lašnev spaces.

When $\dim X \leqslant 0$, we can assume without loss of generality that order $\mathcal{U}_i \leqslant 1$ and order $\mathcal{T}_{ij} \leqslant 1$ in the proof of Theorem 3.4. Then X_{ij} is an almost discrete space. Moreover if X is separable, the cardinalities of $\mathcal{U}_i$, $\mathcal{T}_{ij}$ and A_i are countable. Thus we get the following.

LEMMA 3.6. *Let X be a free L-space with* $\dim X \leqslant 0$. *Then X is embedded in the countable product of almost discrete spaces. Moreover if X is separable, then X is embedded in the countable product of almost discrete countable spaces.*

LEMMA 3.7 (Nagami [5], Lemma 3). *Let X be the countable product of paracompact Σ-spaces X_i, $i \in N$, with* $\dim X_i \leqslant 0$. *Then* $\dim X \leqslant 0$.

The following is a direct consequence of Theorem 3.5, Lemmas 3.6 and 3.7.

THEOREM 3.8. *A space X is a (separable) free L-space with* $\dim X \leqslant 0$ *if and only if X is embedded in the countable product of almost discrete (countable) spaces.*

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