Lusin properties in the product space S^n

by

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Abstract. A Lusin set X in a space S is one which is concentrated about every dense subset of S. A ν space is one which is Lusin relative to itself. Some properties of concentration in the product space S^n about certain dense subsets of S^n are examined, giving generalizations of Lusin sets and ν spaces.

1. Introduction. Suppose that S is a Hausdorff space. The subset X of S is *concentrated* about the subset B of S if every open set containing B contains all but countably many points of X. A *Lusin set* X (relative to S) is one which is concentrated about every dense subset of S. A ν *space* is a space which is Lusin relative to itself.

Some well-known relationships concerning these definitions are given. These relationships may be found or easily inferred from [2, Ch. 3, Sec. 40-VII] and [1]. Throughout this paper, the symbol (CH) indicates that the continuum hypothesis is assumed.

1. $X \subset S$ is Lusin relative to S if and only if every nowhere dense subset of S has countable intersection with X. This is often taken as the definition of a Lusin set.

2. S is a ν space if and only if every nowhere dense subset of S is countable. This too, is often taken as a definition.

3. If X is a countable subset of S, then X is Lusin relative to S. The converse is not true however (CH).

4. If $X \subset S$ is Lusin relative to S, then X (as a space) is ν.

5. If S is dense in T and S is a ν space, then S is Lusin relative to T. The premise that S is dense in T may not be removed though (CH).

6. If each of $X_1, X_2, \ldots$ is Lusin relative to S, then so is $\bigcup_{i=1}^{\infty} X_i$. This is not true (even for finite unions) of ν spaces (CH).

7. If $S' \subset S$ and S is a ν space, so is S'.

It is also known that the property of being concentrated about some countable dense subset of S is weaker than being a ν space (CH). Relative to the property of being concentrated about some countable dense subset of S, Michael shows [3, Lemma 6.1] that if (CH) is assumed, then if N is a positive integer, there is an uncountable subspace S of the line such that S contains the rationals, Q, and such that if $n \leq N$, then S^n is concentrated about the "grid" $S^n-(S-Q)^n$. The purpose of this paper is to examine questions of concentration of spaces S^n about *every* dense "grid", thus giving generalizations of Lusin sets and ν spaces.

2. Definitions. Let S be a topological space and n be a positive integer.

DEFINITION 1. S is ν^n (resp. *strongly* ν^n) means that if B is dense in S (resp. each of $B_1, \ldots, B_n$ is dense in S) and O is open in S^n containing $S^n-(S-B)^n$ (resp. $S^n-((S-B_1)\times \ldots \times(S-B_n))$), then S^n-O is countable.

DEFINITION 2. S is ν^∞ (resp. *strongly* ν^∞) means that S is ν^n (resp. strongly ν^n) for every n.

DEFINITION 3. The subset X of S is L^n (relative to S) means that if B is dense in S and O is open in S^n containing $S^n-(S-B)^n$, then X^n-O is countable.

DEFINITION 4. The subset X of S is L^∞ (relative to S) means that X is L^n (relative to S) for every n.

We shall omit the phrase "relative to S" when no confusion arises.

Remark. Another way of saying that S (resp. $X\subset S$) is ν^n (resp. L^n) is that if B is dense in S and M is an uncountable closed set in S^n (resp. with $M\cap X^n$ uncountable), then M intersects $S^n-(S-B)^n$.

3. Notation. Again, let S be a topological space. Since we will often be working with finite product spaces, the n-tuple $(x_1, \ldots, x_n)$ is sometimes denoted by $\langle x\rangle$. If s is an increasing finite subsequence of the positive integers, then $s=(i_1, \ldots, i_k)$ and if A is a subset of S^n (with $n\geq i_k$), we let $\pi_s(A)=\{(a_{i_1}, \ldots, a_{i_k})\colon (a_1, \ldots, a_n)\in A\}$. π_s^{-1} is the inverse of π_s. Also, we sometimes have occasion to talk about a "face" of S^n. To do this, we replace S by S_i for each i, so that $S^n=\prod_{i=1}^{n} S_i$ and the s-face of S^n is $\prod_{j\in s} S_j = S_{i_1}\times\ldots\times S_{i_k}$. If A is a subset of S, then we let

$$\prod_{j\in s} A_j = \{(a_{i_1}, \ldots, a_{i_k})\colon \text{for each } j,\ a_{i_j}\in A\}.$$

The set $S^n-(S-B)^n$ is sometimes called the *B-grid in S^n* (or perhaps simply the *B-grid*).

The interval $[0,1]$ is denoted by I. If O is open in I^n, then $\beta(O)$ is the boundary of O, and we let $\alpha(O)=\{\langle p\rangle\in\beta(O)\colon$ there is an integer k, $1\leq k\leq n$ and numbers a and b such that if x is a number between a and b, then $(p_1, \ldots, p_{k-1}, x, p_{k+1}, \ldots, p_n)$ is also in $\beta(O)\}$. We then let $\gamma(O)=\beta(O)-\alpha(O)$. Notice that if $n=1$, $\gamma(O)=\beta(O)$. The diagonal in I^n is written $\mathrm{diag}(I^n)$.

Finally, if σ is an ordinal number, then $[\sigma]=\sigma$ if σ is finite, and if σ is transfinite, then $\sigma=\lambda+k$ where λ is a limit ordinal and $0\leq k<\omega_0$ and we let $[\sigma]=k$.

4. Preliminary theorems.

THEOREM 1. *If $n\geq 2$ is an integer and S is a ν^n space, then S is a ν^{n-1} space.*

Proof. Let B be dense in S and let M be an uncountable closed set in S^{n-1}. Let x be a point of $S-B$ (which must exist, else M clearly intersects $S^{n-1}-(S-B)^{n-1}$). Since $\{x\}\times M$ must intersect the B-grid in S^n, M must intersect the B-grid in S^{n-1}. Thus S is ν^{n-1}.

THEOREM 2. *If $n\geq 2$ is an integer and $X\subset S$ is L^n, then X is L^{n-1}.*

Proof. Reword the proof of Theorem 1 to say that M has uncountable intersection with X^{n-1}, then pick x in $X-B$.

THEOREM 3. *If S is countable, then S is ν^∞ and S is L^∞ (relative to any space containing S).*

Proof. Obvious.

THEOREM 4. *If S is a σ-compact Hausdorff space, and $X\subset S$ is L^1, then X is L^∞.*

Proof. If X is not L^n for some n, then let B be dense in S and M be closed in S^n missing $S^n-(S-B)^n$ and such that $M\cap X^n$ is uncountable. Now, for each i, $1\leq i\leq n$, $\pi_i(M)$ is of the first category in S since it is σ-compact and misses B. This implies that $\pi_i(M)\cap X$ is countable since X is L^1 and hence that

$$M\cap X^n\subset \prod_{i=1}^{n}(\pi_i(M)\cap X),$$

which is a contradiction since we have an uncountable set lying in a countable set. This concludes the theorem.

Several other facts are readily seen. Obviously the word "strongly" is appropriately used in the sense that strongly ν^n spaces are ν^n spaces. Also, it is possible to modify the proof of Theorem 1 to show that strongly ν^n spaces are strongly ν^{n-1}. However, it is unnecessary to do the latter (in second countable, Hausdorff spaces), because of the following theorem.

Let space mean a Hausdorff space that satisfies the second axiom of countability.

THEOREM 5. *If n is a positive integer and S is ν^n, then S is strongly ν^n.*

Proof. Notice that if S is ν^1, then S is strongly ν^1. Suppose that $n>1$ is an integer and that for all $k<n$, second countable, Hausdorff spaces are ν^k if and only if they are strongly ν^k. Suppose that S is ν^n (hence ν^{n-1} and strongly ν^{n-1}), but that S is not strongly ν^n. Let $B_1, \ldots, B_n$ be dense subsets of S and O be open in S^n containing $S^n-\prod_{i=1}^{n}(S-B_i)$ such that the boundary of O, $\beta(O)=S^n-O$ is uncountable.

Let G be a countable basis for S, and let $G'=\{g_1\times\ldots\times g_n\colon g_1, \ldots, g_n$ are mutually exclusive members of $G\}$. G' forms a basis for $S^n-\bigcup_{i\neq j} H_{ij}$, where $H_{ij}=\{(x_1, \ldots, x_n)\in S^n\colon x_i=x_j\}$.

Next we observe that if $i \neq j$, $\beta(O) \cap H_{ij}$ is countable. To see this, consider the homeomorphism from H_{ij} onto $\prod_{k\neq j} S_k$ (which is homeomorphic to S^{n-1}) that takes $(x_1, \ldots, x_n) \to (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Let U be the image of $O \cap H_{ij}$ and U covers $\prod_{k\neq j} S_k - \prod_{k\neq j} (S-B_k)$. Therefore, $\prod_{k\neq j} S_k - U$ is countable, so $\beta(O) \cap H_{ij}$ is too.

The preceding observation coupled with the fact that G' is countable implies that there is $g_1 \times \ldots \times g_n \in G'$ such that $\beta(O) \cap (g_1 \times \ldots \times g_n)$ is uncountable. Let $O' = S^n - (\beta(O) \cap \mathrm{cl}(g_1 \times \ldots \times g_n))$, which is open in S^n. Furthermore, $S^n - O'$ is uncountable.

Now let

$$B = \bigcup_{i=1}^{n} (B_i \cap g_i) \cup \Big(S - \bigcup_{i=1}^{n} \mathrm{cl}(g_i)\Big).$$

B is dense in S, yet O' covers the B-grid. That is,

(i) if $(x_1, \ldots, x_n) \in S^n$ and $x_i \notin \mathrm{cl}(g_i)$ for some i, then $\langle x \rangle$ is in O' (whether or not $\langle x \rangle$ is in the B-grid); and

(ii) if $(x_1, \ldots, x_n) \in S^n - (S-B)^n$ and for each i, x_i is in $\mathrm{cl}(g_i)$, then letting j be an integer such that $x_j \in B$, we really have that $x_j \in B_j$, so $\langle x \rangle$ is in

$$S^n - ((S-B_1) \times \ldots \times (S-B_n)),$$

which is covered by O, so $\langle x \rangle$ is in O'.

This contradicts that S is ν^n since O' covers the B-grid but $S^n - O'$ is uncountable. This concludes the proof.

Remark. The definition of a "strongly L^n set" may be easily stated and slight changes in the wording of Theorem 5 yield an analogous theorem.

Hereafter we deal only with separable metric spaces (namely subspaces of I), so we shall not use the word "strongly" again, although it is implied.

5. Examples. Our first example shows that Theorem 4 has no analogue for ν^n spaces. In particular, given a positive integer n, we will exhibit a ν^n subspace of I that is not ν^{n+1}. To prevent it from being ν^{n+1}, we will need a set B, dense in S, and a closed uncountable subset M' of S^{n+1} that misses the B-grid in S^{n+1}. Toward this goal, we prove a lemma.

LEMMA A. *Suppose that $B = \{b\colon b$ is rational, $0<b<1\}$ and that n is a positive integer. There is a closed subset M of I^{n+1} such that:*

(1) M does not intersect B^{n+1}, and

(2) if k is an integer, $1 \leqslant k \leqslant n$, and s is an increasing finite subsequence of $(1, 2, \ldots, n+1)$ with k terms, and F is a first category subset of $\prod_{j\in S} I_j$, then $\pi_S^{-1}(F) \cap M$ is of the first category in M.

Proof. Define $g\colon I^n \to I$ by

$$g(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n}.$$

Notice that $g(B^n) = B$. Let C be a countable dense subset of $(0, 1)$ such that B and C do not intersect, and let h be a homeomorphism from I onto I that maps B onto C. Let $f = h \circ g$, and let M be f. That is, let

$$M = \{(x_1, \ldots, x_n, f(\langle x \rangle))\colon (x_1, \ldots, x_n) \in I^n\}.$$

M is closed since f is continuous, hence M is a Baire space. M satisfies condition (1), for if m is in M and the first n coordinates of m are in B, then the last coordinate of m is in C (thus not in B).

Before verifying that condition (2) is met, let us observe that if U is an open rectangle in the *interior* of I^{n+1} and U intersects M, and $1 \leqslant k \leqslant n$ and $s = (i_1, \ldots, i_k)$ is an increasing finite subsequence of $(1, 2, \ldots, n+1)$, then $\pi_s(U \cap M)$ is open in $\prod_{j\in S} I_j$. To see this, suppose that $(p_{i_1}, \ldots, p_{i_k}) \in \pi_s(U \cap M)$ and that for each positive integer t, $(q^t_{i_1}, \ldots, q^t_{i_k})$ is a point *not* in $\pi_s(U \cap M)$, yet this sequence converges to $(p_{i_1}, \ldots, p_{i_k})$. Let $(p_1, \ldots, p_{n+1})$ be a point of $U \cap M$ such that $\pi_s(\langle p \rangle) = (p_{i_1}, \ldots, p_{i_k})$. Now consider two cases, both of which lead to a contradiction of the fact that $(p_{i_1}, \ldots, p_{i_k})$ is a limit point of non-members of $\pi_s(U \cap M)$.

(i) If $i_k \neq n+1$, let $(r^t_1, \ldots, r^t_n)$ be the n-tuple obtained by letting

$$r^t_m = \begin{cases} q^t_m & \text{if } m \text{ is a term of } s, \\ p_m & \text{if } m \text{ is not a term of } s. \end{cases}$$

Let $r^t_{n+1} = f(r^t_1, \ldots, r^t_n)$. Now $\langle r^t \rangle$ is in M for each t, $\langle r^t \rangle \to \langle p \rangle$, so there is t such that $\langle r^t \rangle \in U \cap M$. But $\pi_s(\langle r^t \rangle) = (q^t_{i_1}, \ldots, q^t_{i_k})$ which is supposedly not in $\pi_s(U \cap M)$.

(ii) If $i_k = n+1$, let z be an integer, $1 \leqslant z \leqslant n$, such that $z \neq i_j$ for any j. (Such a z must exist since $k \leqslant n$.) Now for all m, $1 \leqslant m \leqslant n+1$, except z, let

$$r^t_m = \begin{cases} q^t_m & \text{if } m \text{ is a term of } s, \\ p_m & \text{if } m \text{ is not a term of } s. \end{cases}$$

Let r^t_z be the real number (perhaps not in I) that solves

$$h\left(\frac{r^t_1 + \ldots + r^t_z + \ldots + r^t_n}{n}\right) = r^t_{n+1}.$$

It is true that for sufficiently large t, r^t_z must tend toward p_z which is in the *interior* of I, thus for sufficiently large t, $\langle r^t \rangle$ is in $U \cap M$. Again we have that $\langle r^t \rangle \to \langle p \rangle$ and $\pi_s(\langle r^t \rangle) = (q^t_{i_1}, \ldots, q^t_{i_k})$.

Now, condition (2) is easily verified if we notice that it suffices to show that if F is closed and nowhere dense in $\prod_{j\in S} I_j$, then $\pi_s^{-1}(F) \cap M$ is closed and nowhere dense in M, which must be the case in view of our observation on the projection of open rectangles. This concludes the proof of Lemma A.

Lemma A provides the structure for finding an uncountable closed set, $M' = M \cap S^{n+1}$, in S^{n+1} that misses the B-grid. But some caution must be exercised to keep M' away from it. Condition (1) of Lemma A controls the set B^{n+1}, so we will develop controls to assure that we do not allow points to be in S if they could combine with points of B to form an $(n+1)$-tuple that lies in M — that is, M' must not intersect the B-grid in S^{n+1}. We prove a lemma that will be applied in a recursion argument. The reader is forewarned of the difference between the point $(x_1, \ldots, x_{n+1})$ and the set $\{x_1, \ldots, x_{n+1}\}$.

Notation for Lemma B and Lemma C. Let n and $k \leqslant n$ be positive integers, and let A be a finite number set and let $\{y_1, \ldots, y_k\}$ be a number set that does not intersect A. If m is a positive integer (for application, $m = n$ or $m = n+1$) and φ is an m-tuple from $(\{y_1, \ldots, y_k\} \cup A)^m - A^m$, then if $(x_1, \ldots, x_k)$ is a k-tuple, let $P_\varphi(x_1, \ldots, x_k)$ be the m-tuple obtained from φ by replacing each y_i with x_i.

LEMMA B. *Let B, n, and M be as in Lemma A. If C is a countable set (perhaps empty) that does not intersect B and $(C \cup B)^{n+1} - C^{n+1}$ misses M, then there is a first category subset E of M such that if $(x_1, \ldots, x_{n+1})$ is in $M - E$, then M misses $(\{x_1, \ldots, x_{n+1}\} \cup C \cup B)^{n+1} - (\{x_1, \ldots, x_{n+1}\} \cup C)^{n+1}$.*

Proof. Let $\mathscr{S}$ denote the collection of increasing subsequences of $(1, 2, \ldots, n+1)$ with n or fewer terms. For an arbitrary s in $\mathscr{S}$, let k denote the number of terms of s and let $E_s = \{(x_1, \ldots, x_k) \in \prod_{j \in S} I_j$: there is a finite subset A of $B \cup C$ such that $(\{x_1, \ldots, x_k\} \cup A)^{n+1} - A^{n+1}$ intersects $M\}$. Let $\mathscr{A}$ denote the collection of all *finite* subsets of $C \cup B$ and we have that $E_s = \bigcup_{A \in \mathscr{A}} E_{s,A'}$, where

$$E_{s,A} = \{(x_1, \ldots, x_k) \in \prod_{j \in S} I_j\colon (\{x_1, \ldots, x_k\} \cup A)^{n+1} - A^{n+1} \text{ intersects } M\}.$$

Now, for $A \in \mathscr{A}$, $E_{s,A}$ is closed relative to $\prod_{j \in S} I_j - \prod_{j \in S} A_j$, for if $(p_1, \ldots, p_k)$ is in this set and a limit point of $E_{s,A}$, then (using the notation described earlier with $m = n+1$) there is φ and a sequence $\langle q^1 \rangle, \langle q^2 \rangle, \ldots$ converging to $\langle p \rangle$ such that $P_\varphi(\langle q^i \rangle)$ is in M for each i. Since M is closed, $P_\varphi(\langle p \rangle)$ is in M and hence $\langle p \rangle$ is in $E_{s,A}$. Furthermore, $E_{s,A}$ misses the set $\prod_{j \in S} B_j$ since, by assumption, $(C \cup B)^{n+1} - C^{n+1}$ misses M. Therefore, $E_{s,A}$ is nowhere dense in $\prod_{j \in S} I_j$ and consequently, E_s is of the first category in $\prod_{j \in S} I_j$.

Let $E = \bigcup_{s \in \mathscr{S}} \pi_s^{-1}(E_s) \cap M$ which, by Lemma A, is of the first category in M. Furthermore, if $(x_1, \ldots, x_{n+1})$ is in $M - E$, then

$$(\{x_1, \ldots, x_{n+1}\} \cup C \cup B)^{n+1} - (\{x_1, \ldots, x_{n+1}\} \cup C)^{n+1}$$

can not intersect M, for if $(m_1, \ldots, m_{n+1})$ is such a point of intersection, then there is an integer i such that m_i belongs to $\{x_1, \ldots, x_{n+1}\}$. Letting $(x_{j_1}, \ldots, x_{j_k})$ be the coordinates of $\langle x \rangle$ that appear as coordinates of $\langle m \rangle$, we see that $\langle x \rangle$ is in $\pi_s^{-1}(E_s)$, where $s = (j_1, \ldots, j_k)$, which is a contradiction. This concludes the proof.

Lemmas A and B provide the framework to give us a space that is not ν^{n+1}. But we want the space to be ν^n, so we prove one more lemma. As with Lemma B, the following is to be applied in a transfinite construction, hence the peculiar wording.

LEMMA C. *Let B, n, and M be as in Lemma A. If C is a countable set containing B, and C' is a subset of C (perhaps empty), and O is open in I^n, and $C^n - C'^n$ misses $\gamma(O) - B^n$, then there is a first category subset F of M such that if $(x_1, \ldots, x_{n+1})$ is in $M - F$, then $(\{x_1, \ldots, x_{n+1}\} \cup C)^n - C'^n$ misses $\gamma(O) - B^n$.*

Proof. Let $\mathscr{S}$ denote the collection of increasing subsequences of $(1, 2, \ldots, n+1)$ with n or fewer terms. For an arbitrary $s \in \mathscr{S}$, let k denote the number of terms of s, and let $F_s = \{(x_1, \ldots, x_k) \in \prod_{j \in S} I_j$: there is a finite subset A of C such that $(\{x_1, \ldots, x_k\} \cup A)^n - A^n$ intersects $\gamma(O)\}$. Let $\mathscr{A}$ denote the collection of all *finite* subsets of C and we have that $F_s = \bigcup_{A \in \mathscr{A}} F_{s,A}$, where

$$F_{s,A} = \{(x_1, \ldots, x_k) \in \prod_{j \in S} I_j\colon (\{x_1, \ldots, x_k\} \cup A)^n - A^n \text{ intersects } \gamma(O)\}.$$

We will show that each $F_{s,A}$ is nowhere dense in $\prod_{j \in S} I_j$, so suppose that this is not the case. We then have $A \in \mathscr{A}$ and a rectangle R in $\prod_{j \in S} I_j$ and one n-tuple φ (using the notation described earlier with $m = n$), and a dense subset D of R such that if $(x_1, \ldots, x_k) \in D$, $P_\varphi(x_1, \ldots, x_k)$ is in $\gamma(O)$. This implies that for each $(x_1, \ldots, x_k) \in R$, $P_\varphi(x_1, \ldots, x_k)$ is in $\beta(O)$ which in turn implies that for each $(x_1, \ldots, x_k) \in R$, $P_\varphi(x_1, \ldots, x_k)$ is in $\alpha(O)$ and *not* in $\gamma(O)$. The contradiction means that $F_{s,A}$ is nowhere dense in $\prod_{j \in S} I_j$, so F_s is a first category subset of $\prod_{j \in S} I_j$.

Let $F = \bigcup_{S \in \mathscr{S}} \pi_s^{-1}(F_s) \cap M$ which, by Lemma A, is of the first category in M. Furthermore, if $(x_1, \ldots, x_{n+1}) \in M - F$, then $(\{x_1, \ldots, x_{n+1}\} \cup C)^n - C'^n$ can not intersect $\gamma(O) - B^n$. In fact, if $(y_1, \ldots, y_n)$ is in this set and not in $C^n - C'^n$ (which misses $\gamma(O) - B^n$ by hypothesis), then $(y_1, \ldots, y_n)$ is not even in $\gamma(O)$, for letting $s = (j_1, \ldots, j_k)$, where $(x_{j_1}, \ldots, x_{j_k})$ are the members of $\{x_1, \ldots, x_{n+1}\}$ that appear as coordinates of $\langle y \rangle$, we see that F_s keeps $\langle y \rangle$ out of $\gamma(O)$.

THEOREM 6 (CH). *If n is a positive integer, there is a subspace S of I which is dense in I that is ν^n but not ν^{n+1}.*

Proof. Let B, n, and M be as described in Lemma A. Let $\{u_0, u_1, \ldots\}$ be a countable basis for M. Arrange the dense open subsets of I^n which do not contain any open (relative to $\operatorname{diag}(I^n)$) subset of $\operatorname{diag}(I^n)$ as boundary points, into a transfinite sequence: $\{O_\theta\}$, $\theta < \omega_1$.

Before we do a transfinite construction, let us show that our process does work. Thus, we ask the reader to *assume* that for each $\sigma<\omega_1$, $\langle x_\sigma\rangle = (x_\sigma^1, \ldots, x_\sigma^{n+1})$ is a point of $u_{[\sigma]}$ such that

(i) if $\tau<\omega_1$, then $(\bigcup_{\sigma\leqslant\tau}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B)^{n+1}-(\bigcup_{\sigma\leqslant\tau}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\})^{n+1}$ does not intersect M; and

(ii) if $\tau<\omega_1$ and $\theta\leqslant\tau$, then $(\bigcup_{\sigma\leqslant\tau}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B)^{n}-(\bigcup_{\sigma<\theta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\})^{n}$ does not intersect $\gamma(O_\theta)-B^n$; and

(iii) if $\tau<\omega_1$, then $\langle x_\tau\rangle$ is not in $\bigcup_{\sigma<\tau}\{\langle x_\sigma\rangle\}$.

Let $S = \bigcup_{\sigma<\omega_1}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B$, which is obviously dense in I.

S is ν^n, for if B' is dense in S and O is open in S^n containing the B'-grid, then there exists $\theta<\omega_1$ such that $O_\theta\cap S^n = 0$. That is, letting O' be open in I^n such that $O'\cap S^n = 0$, we see that O' is dense in I^n, and furthermore, $\beta(O')$ can not contain a "piece" of $\operatorname{diag}(I^n)$, else there is $(b, b, \ldots, b)$ in B'^n that is not covered by O. Thus we are assured that $O' = O_\theta$ for some $\theta<\omega_1$. Next we observe that $\alpha(O_\theta)\cap S^n$ is empty, for if $(p_1, \ldots, p_n)$ is in $\alpha(O_\theta)\cap S^n$, then each p_i is in S and since $\langle p\rangle$ is in $\alpha(O_\theta)$ (and since one coordinate of $\langle p\rangle$ is allowed to "move" without getting outside $\beta(O_\theta)$), we see that there is a point of the B'-grid that is not covered by O_θ. The contradiction implies that $\alpha(O_\theta)\cap S^n$ is empty and hence that $\beta(O_\theta)\cap S^n = \gamma(O_\theta)\cap S^n$, which is countable by condition (ii). Therefore, S^n-O is countable, and S is ν^n.

S is not ν^{n+1} though, for B is dense in S and we let $M' = M\cap S^{n+1}$, which is uncountable by (iii). M' does not intersect B^{n+1} since M does not. Furthermore, if $(x_1, \ldots, x_{n+1})\in M$ and at least one coordinate is in B (but of course, not all coordinates are in B), then the coordinates of $\langle x\rangle$ which are not in B violate condition (i) imposed upon each $\langle x_\sigma\rangle$, meaning that such coordinates can not be in S. Therefore, M' does not intersect the B-grid in S^{n+1} and S is not ν^{n+1}.

Now, to complete the proof, we need to construct the sequence of points satisfying (i)-(iii). An extra condition, (*), will be introduced to make Lemma B applicable. Condition (i) comes from application of Lemma B, and (ii) comes from application of Lemma C.

Let D be the intersection of M and the B-grid in I^{n+1}. $D = \bigcup_{i=1}^{n+1}\pi_i^{-1}(B)\cap M$, so D is of the first category in M. Apply Lemma B with $C = \emptyset$ to get E_0 (E in Lemma B). Apply Lemma C with $C = B$, $C' = \emptyset$, and $O = O_0$ to get F_0 (F in Lemma C). Let $(x_0^1, \ldots, x_0^{n+1})\in(M-(D\cup E_0\cup F_0))\cap u_0$, and notice that

(*) $\{x_0^1, \ldots, x_0^{n+1}\}$ misses B;

and

(i) $(\{x_0^1, \ldots, x_0^{n+1}\}\cup B)^{n+1}-\{x_0^1, \ldots, x_0^{n+1}\}^{n+1}$ misses M; and

(ii) $(\{x_0^1, \ldots, x_0^{n+1}\}\cup B)^{n}$ misses $\gamma(O_0)-B^n$.

Now suppose that $\tau<\omega_1$ and that

(*) if $\sigma<\tau$, $\langle x_\sigma\rangle$ is a point of $(M-D)\cup u_{[\sigma]}$;

and

(i) if $\delta<\tau$, then $(\bigcup_{\sigma\leqslant\delta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B)^{n+1}-(\bigcup_{\sigma\leqslant\delta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\})^{n+1}$ does not intersect M; and

(ii) if $\delta<\tau$ and $\theta\leqslant\delta$, then $(\bigcup_{\sigma\leqslant\delta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B)^{n}-(\bigcup_{\sigma<\theta}\{x_\sigma^1, \ldots, x_\sigma^{r+1}\})^{n}$ does not intersect $\gamma(O_\theta)-B^n$; and

(iii) if $\delta<\tau$, then $\langle x_\delta\rangle$ is not in $\bigcup_{\sigma<\delta}\{\langle x_\sigma\rangle\}$.

Let $C = \bigcup_{\sigma<\tau}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}$ which is countable and does not intersect B (by (*)). Furthermore, $(C\cup B)^{n+1}-C^{n+1}$ does not intersect M. To see this, consider the cases (1) that $\tau-1 = \delta$ is included in the hypothesis, and (2) τ is a limit ordinal, in which case there is $\delta<\tau$ for which it is true that a supposed point of $M\cap\cap((C\cup B)^{n+1}-C^{n+1})$ belongs to

$$(\bigcup_{\sigma\leqslant\delta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}\cup B)^{n+1}-(\bigcup_{\sigma\leqslant\delta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\})^{n+1}.$$

Since both cases may be ruled out, we apply Lemma B to get the set E_τ (E in Lemma B). Now let $C = \bigcup_{\sigma<\tau}\{x_\omega^1, \ldots, x_\sigma^{n+1}\}\cup B$ which is countable and contains B. For each $\theta\leqslant\tau$, apply Lemma C with $C' = \bigcup_{\sigma<\theta}\{x_\sigma^1, \ldots, x_\sigma^{n+1}\}$ and $O = O_\theta$ to get F_θ' (F in Lemma C). Since there are only countably many $\theta\leqslant\tau$, let $F_\tau = \bigcup_{\theta\leqslant\tau}F_\theta'$, and F_τ is of the first category in M.

Let $\langle x_\tau\rangle$ be a point of $(M-(D\cup E_\tau\cup F_\tau))\cap u_{[\tau]}$. Furthermore, to assure that condition (iii) is met, pick $\langle x_\tau\rangle$ outside $\bigcup_{\sigma<\tau}\{\langle x_\sigma\rangle\}$. It remains to verify that conditions (*), (i), (ii), and (iii) are satisfied, which is easily done.

COROLLARY. *There is a dense subspace S of I that is L^∞ (relative to I) but not ν^2.*

This comes from the theorem when $n = 1$ and since S is ν^1 and dense in I, then (Property 5 from the introduction) S in L^1 so (Theorem 4) S is L^∞.

Finally, we show the existence of a ν^∞ space, and in fact, we exhibit a very "fragile" ν^∞ space. More precisely, from properties stated in the introduction, we see that if S is dense in I and S is a ν^1 space, then if B is a ν^1 space and dense in I (in particular, if B is countable), then $S\cup B$ is also a ν^1 space. Also, it was noted that being a ν^1 space is a hereditary property. These properties are not true of ν^∞ spaces though.

THEOREM 7 (CH). *There exists a subspace S of I which is dense in I such that S is ν^∞ and such that there are countable sets B and C in I for which neither $S\cup B$ nor $S-C$ is ν^2.*

Proof. Let $B = \{b\colon b \text{ is rational, } 0<b<1\}$, and let Z denote the integers. Let h be a homeomorphism from I onto I such that if $b \in B$, $h(b) \notin B$. Let $h_0(x) = x$, and if n is a positive integer, let $h_n(x) = h(h_{n-1}(x))$, and if n is a negative integer, let $h_n(x) = h^{-1}(h_{n+1}(x))$. If $x \in I$, let $H(x) = \bigcup_{n \in Z} \{h_n(x)\}$. Let $D = \bigcup_{b \in B} H(b)$.

If n is a positive integer and O is open in I^n, let $\alpha'(O) = \{\langle p\rangle \in \beta(O)\colon$ there is an integer k, $1 \leqslant k \leqslant n$, and a subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$ and a function $\varrho\colon \{1, \ldots, k\} \to Z$ and two numbers a and b such that if x is a number between a and b, then $(p_1, \ldots, p_{i_1-1}, h_{\varrho_1}(x), p_{i_1+1}, \ldots, p_{i_k-1}, h_{\varrho_k}(x), p_{i_k+1}, \ldots, p_n)$ is also in $\beta(O)\}$. Let $\gamma'(O) = \beta(O) - \alpha'(O)$.

For each positive integer n, well order the dense open sets, O, in I^n for which it is true that if a and b are two numbers, there is a number x between a and b such that $(H(x))^n$ does not intersect $\beta(O)$, $\{O_\theta^n\}$, $\theta < \omega_1$. Let $\{u_0, u_1, \ldots\}$ be a countable basis for I.

We want to generate, for each $\sigma < \omega_1$, a point x_σ from $u_{[\sigma]}$ such that

(i) if $\tau < \omega_1$ and $\theta \leqslant \tau$ and n is a positive integer, then $(\bigcup_{\sigma \leqslant \tau} H(x_\sigma))^n - (\bigcup_{\sigma < \theta} H(x_\sigma))^n$ does not intersect $\gamma'(O_\theta^n)$; and

(ii) if $\tau < \omega_1$, x_τ is not in $\bigcup_{\sigma < \tau} \{x_\sigma\}$.

So we first let, for each positive integer n, $F_0^n = \{x \in I\colon (H(x))^n \text{ intersects } \gamma'(O_0^n)\}$. For each n, F_0^n is of the first category in I. That is, for each function ϱ from $\{1, \ldots, n\}$ into Z, let $F_0^n(\varrho) = \{x \in I\colon (h_{\varrho_1}(x), \ldots, h_{\varrho_n}(x)) \text{ is in } \gamma'(O_0^n)\}$. Now, $F_0^n(\varrho)$ is nowhere dense in I by virtue of the restrictions placed upon each O_0^n. Since there are only countably many such ϱ, we see that F_0^n is of the first category in I. Let $F_0 = \bigcup_{n=1}^{\infty} F_0^n$, and pick x_0 from $u_0 - (D \cup F_0)$.

Now suppose that $\tau < \omega_1$ and that if $\sigma < \tau$, x_σ is a point of $u_{[\sigma]} - D$. Suppose further that

(i) if $\delta < \tau$ and $\theta \leqslant \delta$ and n is a positive integer, then $(\bigcup_{\sigma \leqslant \theta} H(x_\sigma))^n - (\bigcup_{\sigma < \theta} H(x_\sigma))^n$ does not intersect $\gamma'(O_\theta^n)$; and

(ii) if $\delta < \tau$, x_δ is not in $\bigcup_{\sigma < \delta} \{x_\sigma\}$.

Let $C = \bigcup_{\sigma < \tau} H(x_\sigma)$, and for each $\theta \leqslant \tau$, let $C_\theta = \bigcup_{\sigma < \theta} H(x_\sigma)$, both of which are countable. For each positive integer n, let $E_\theta^n = \{x \in I\colon (H(x) \cup C)^n - (C_\theta)^n \text{ intersects } \gamma'(O_\theta^n)\}$. E_θ^n is of the first category in I, for E_θ^n can be written as a countable union of sets of the form $E_\theta^n(\varrho, A) = \{x \in I\colon (\{h_{\varrho_1}(x), \ldots, h_{\varrho_n}(x)\} \cup A)^n - (C_\theta)^n \text{ intersects } \gamma'(O_\theta^n)\}$, where ϱ is a function from $\{1, \ldots, n\}$ into Z, and A is a *finite* subset of C; and $E_\theta^n(\varrho, A)$ is nowhere dense in I by reasoning very similar to the reasoning that showed (in Lemma C) that $F_{s,A}$ is nowhere dense. We let $E_\theta = \bigcup_{n=1}^{\infty} E_\theta^n$ and $F_\tau = \bigcup_{\theta \leqslant \tau} E_\theta$, so F_τ is of the first category in I. Pick x_τ from $u_{[\tau]} - (D \cup F_\tau)$. To satisfy (ii), also pick x_τ outside $\bigcup_{\sigma < \tau} \{x_\sigma\}$. We therefore generate a transfinite sequence of points with the desired properties, (i) and (ii). Let $S = \bigcup_{\sigma < \omega_1} H(x_\sigma)$, which is dense in I since, for each σ, $x_\sigma \in u_{[\sigma]}$.

Next we show that S is ν^∞. Suppose that n is an arbitrary positive integer and B' is dense in S and O is open in S^n containing $S^n - (S - B')^n$. Let O' be open in I^n such that $O' \cap S^n = 0$, and we will show that $O' = O_\theta^n$ for some θ. O' is clearly dense in I^n. Furthemore, if $a < b$, we will find a number x between a and b such that $(H(x))^n$ does not intersect $\beta(O')$. For each function ϱ from $\{1, \ldots, n\}$ into Z, let $K_\varrho = \{x \in (a, b)\colon (h_{\varrho_1}(x), \ldots, h_{\varrho_n}(x)) \text{ is in } \beta(O')\}$. K_ϱ is nowhere dense in (a, b), for it is closed relative to (a, b), and can contain no point q such that $h_{\varrho_1}(q) \in B'$, else $h_{\varrho_i}(q)$ is in S for each i, so $(h_{\varrho_1}(q), \ldots, h_{\varrho_n}(q))$ is in the B'-grid in S^n, yet not in O. Taking the union of K_ϱ over all such ϱ, we only get a first category subset of (a, b), thus we pick x outside this first category set, and $(H(x))^n$ must miss $\beta(O')$. Therefore, there exists $\theta < \omega_1$ such that $O_\theta^n \cap S^n = 0$.

$\alpha'(O_\theta^n) \cap S^n$ is empty, because if $\langle p\rangle \in \alpha'(O_\theta^n) \cap S^n$, then there is k, $1 \leqslant k \leqslant n$, and a subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$ and a function $\varrho\colon \{1, \ldots, k\} \to Z$ and numbers a and b such that if x is between a and b, then

$$(p_1, \ldots, p_{i_1-1}, h_{\varrho_1}(x), p_{i_1+1}, \ldots, p_{i_k-1}, h_{\varrho_k}(x), p_{i_k+1}, \ldots, p_n)$$

is in $\beta(O_\theta^n)$. As before, let q be a point between a and b such that $h_{\varrho_1}(q) \in B'$, and we get $(p_1, \ldots, h_{\varrho_1}(q), \ldots, h_{\varrho_k}(q), \ldots, p_n)$ in S^n and the B'-grid, so this point is supposed to be in O, yet it is not in O_θ^n (which is a contradiction). Therefore, $\alpha'(O_\theta^n) \cap S^n$ is empty, so $\beta(O_\theta^n) \cap S^n = \gamma'(O_\theta^n) \cap S^n$, which is countable by construction, hence $S^n - O$ is countable and S is ν^n. Since n was arbitrary, we have that S is ν^∞.

$S \cup B$ is not ν^2 however, for let M be the closure in $(S \cup B)^2$ of

$$\{(x_\tau, h(x_\tau))\colon \tau < \omega_1\}.$$

M is a subset of the graph of h. M is uncountable, since we conveniently picked a new point x_τ for each τ. M does not intersect the B-grid in $(S \cup B)^2$ though. That is, if $b \in B$ and s is a number such that (b, s) is in M, then (i) s is not in B since $h(b)$ is not, and (ii) $s \neq h_n(x_\tau)$ for any pair (n, τ), else x_τ is in D, so s is not a member of $S \cup B$. Similar reasoning takes care of (s, b). This implies that M misses the B-grid in $(S \cup B)^2$, so $S \cup B$ is not ν^2.

Finally, to show that there is a countable set C such that $S - C$ is not ν^2, let M be the previously defined set (which actually lies in S^2 since it misses the B-grid). Let N denote the fixed points of h. N is nowhere dense in I since it is closed and does not intersect B. Recalling that $\{u_0, u_1, \ldots\}$ is a basis for I, let b_0 be a point of $u_0 - N$, and for each positive integer n, let b_n be a point of

$$u_n - \Big(N \cup \big(\bigcup_{j=0}^{n-1} \{h(b_j)\}\big) \cup \big(\bigcup_{j=0}^{n-1} \{h^{-1}(b_j)\}\big)\Big).$$

Let $C = \bigcup_{j=0}^{\infty} (\{h(b_j)\} \cup \{h^{-1}(b_j)\})$ and $B' = \{b_0, b_1, \ldots\}$. Notice that B' and C do not intersect. For example, if $b_n = h(b_j)$, then $n \neq j$, else $b_n \in N$, and $n \not> j$, else $b_n \in \{h(b_j)\}$, and $n \not> j$, else $b_j \in \{h^{-1}(b_n)\}$. B' is dense in $S-C$ though, for it is dense in I, and $M \cap (S-C)^2$ is closed (relative to $(S-C)^2$) and still uncountable, but it is forced to miss the B'-grid in $(S-C)^2$.

The author is grateful to J. B. Brown and P. L. Zenor for their helpful suggestions and advice, and especially to Zenor for first raising the question of the existence of a ν^∞ space.

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Accepté par la Rédaction le 23. 1. 1978

Dimension of free L-spaces

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Abstract. We introduce the class of free L-spaces which is countably productive and hereditary. The class is an intermediate class between that of L-spaces and that of M_1-spaces. The class has excellent features in dimension theory a part of which is clarified in this paper.

0. Introduction. In a previous paper [6] we introduced the notion of L-spaces which constitute an intermediate class between that of Lašnev spaces and that of M_1-spaces. As was noted there the class of L-spaces is not even finitely productive. In this paper we introduce the notion of free L-spaces in Section 1 which generalizes the notion of L-spaces. The class of free L-spaces is not only hereditary and countably productive but also has many excellent features in dimension theory. In Section 2 we show that even the dimension-raising theorem is valid for the class of free L-spaces. As trivial corollaries of this theorem there are the decomposition theorem and the coincidence theorem for two basic dimensions. A characterization theorem for a free L-space X with $\dim X = n$ is also presented. Our characterization assures the existence of equi-dimensional G_δ-envelopes as in Theorem 2.8 below. In Section 3 we show that the universal space for free L-spaces is the countable product of almost polyhedral spaces. As a special case we prove, in Theorem 3.8 below, that each space X is a free L-space with $\dim X \leqslant 0$ if and only if it is embedded in the countable product of almost discrete spaces. Thus a role played by Baire's 0-dimensional spaces in the theory of metric spaces is done by the countable products of almost discrete spaces in the theory of free L-spaces.

In this paper all spaces are assumed to be Hausdorff topological spaces, maps to be continuous onto, and images to be those under maps. The letter N denotes the positive integers. For undefined terminology refer to [2] and [6].

1. Definition of free L-spaces.

1.1. DEFINITION. Let X be a space, F a closed set of X, and $\mathcal{U}$ an anti-cover of F. If S is a subset of X, $\mathcal{U}(S)$ denotes the star $\bigcup\{U \in \mathcal{U}\colon U \cap S \neq \emptyset\}$. $\mathcal{U}^i(S)$ is defined inductively by the formulae: $\mathcal{U}^1(S) = \mathcal{U}(S)$ and $\mathcal{U}^i(S) = \mathcal{U}(\mathcal{U}^{i-1}(S))$. A set V of X is said to be a *canonical neighborhood* of F (with respect to $\mathcal{U}$) if V is an open neighborhood of F such that, for each i, $\mathrm{Cl}\,\mathcal{U}^i(X-V)$ does not meet F.