Pinning for pairs of countable ordinals

by

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Abstract. The set of all pairs (α, δ) of countable ordinals of which α can be pinned to δ is determined, where α can be pinned to δ means there is a function from α into δ such that the image of any subset of α of order type α has order type δ. This relation was introduced by E. Specker to aid in the study of partition relations.

§ 1. Introduction. Let A and D be well-ordered sets. A function $f\colon A \to B$ is called a *pinning* map in case, for every subset $X \subseteq A$ which is order-isomorphic to A, its image $f''X$ is order-isomorphic to B. If α and β are ordinals, we say α *can be pinned to* β, in symbols $\alpha \longrightarrow \beta$, if there is a pinning map from α into β. Clearly, if A and B have order-type α and β respectively, then $\alpha \longrightarrow \beta$ if and only if there is a pinning map from A into B.

E. Specker introduced this notion in [8], where he studies partition relations of the form $\alpha \to (\alpha, n)^2$ where α is an ordinal and n a cardinal. (See [2] for a definition of this partition relation.) To rule out trivial cases, we may assume that $\alpha > 1$ and $n \geqslant 3$. Positive partition relations of this sort have been proved for only three infinite countable ordinals. Ramsey's Theorem [6] says that $\omega \to (\omega, \omega)^2$. Specker [8] showed that $\omega^2 \to (\omega^2, n)^2$ for every $n < \omega$. Chang [1] showed that $\omega^\omega \to (\omega^\omega, 3)^2$ and E. C. Milner (unpublished) generalized Chang's result by showing that $\omega^\omega \to (\omega^\omega, n)^2$ for every $n < \omega$. (See [4] for a proof of this result.)

Specker [8] observed that if $\alpha \longrightarrow \beta$ and $\alpha \to (\alpha, n)^2$, then $\beta \to (\beta, n)^2$. He proved that $\omega^3 \to (\omega^3, 3)^2$ and that $\omega^m \longrightarrow \omega^3$ for $3 \leqslant m < \omega$, thus proving that $\omega^m \to (\omega^m, 3)^2$ for $3 \leqslant m < \omega$. F. Galvin and the author [3] characterized the set of countable ordinals α which can be pinned to ω^3 as those of the form $\alpha = \omega^a$ where $3 \leqslant a < \omega_1$ and a is additively decomposable. So far all the countably infinite additively indecomposable ordinals known to satisfy $\alpha \to (\alpha, 3)^2$ also satisfy $\alpha \longrightarrow \omega^3$.

B. Rotman [7] has also worked on pinning countable ordinals, showing that if $\alpha < \beta < \omega_1$, then $\alpha \not\longrightarrow \beta$. The question is still open for uncountable ordinals, namely are there α and β with $\alpha < \beta$ and $\alpha \longrightarrow \beta$? The notation $\alpha \longrightarrow \beta$ is due to Rotman.

The following lemma, which is easy to prove, reduces the problem of characterizing pinning to the problem of characterizing it for additively indecomposable (AI) ordinals. (An ordinal α is AI if whenever $\alpha = \beta + \gamma$, then $\alpha \leqslant \beta$ of $\alpha \leqslant \gamma$. In fact, $\alpha \leqslant \gamma$ if $\gamma \neq 0$. Even more is true of AI ordinals. See [5].)

LEMMA 1. *Let* $\alpha = a_0+a_1+\ldots+a_{m-1}$ *and* $\delta = d_0+d_1+\ldots+d_{n-1}$ *be the unique decompositions of* α *and* δ *into* AI *factors with* $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_{m-1} \geqslant 1$ *and* $d_0 \geqslant d_1 \geqslant \ldots \geqslant d_{n-1} \geqslant 1$. *Then* $\alpha \to \delta$ *if and only if there is a one-to-one function* $j\colon n \to m$ *so that for all* $i<n$, $a_{j(i)} \to d_i$.

The next lemma shows every AI ordinal can be expressed as a product of multiplicatively indecomposable (MI) ordinals. (An ordinal α is MI if whenever $\alpha = \beta\cdot\gamma$, then $\alpha \leqslant \beta$ or $\alpha \leqslant \gamma$. In fact, $\alpha \leqslant \gamma$ if $\gamma \neq 1$.)

LEMMA 2. *If* $\alpha>1$ *is* AI, *then* α *can be expressed uniquely* $\alpha = \alpha_0\cdot\alpha_1\cdot\ldots\cdot\alpha_m$ *as the product of* MI *factors where* $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_m > 1$.

Proof. It is well-known that the AI ordinals are those of the form $\alpha = \omega^a$. Now $a = a_0+a_1+\ldots+a_m$ can be uniquely expressed as the sum of AI factors where $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_m \geqslant 1$. For $i \leqslant m$, let $\alpha_i = \omega^{a_i}$. Then $\alpha = \omega^a = \alpha_0\cdot\alpha_1\cdot\ldots\cdot\alpha_m$, and $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_m \geqslant \omega > 1$. Suppose $\alpha = \beta_0\cdot\beta_1\cdot\ldots\cdot\beta_n$, where $\beta_0, \beta_1, \ldots, \beta_m$ are MI and $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_n$. Let b_i be such that $\beta_t = \omega^{b_i}$. Since $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_n \geqslant \omega$ it follows that $b_0 \geqslant b_1 \geqslant \ldots \geqslant b_n \geqslant 1$. Since $\beta_0, \ldots, \beta_n$ are MI, it follows that $b_0, b_1, \ldots, b_n$ are AI. So $a = b_0+\ldots+b_n = a_0+\ldots+a_m$. Since $a = a_0+\ldots+a_m$ is unique, it follows that $\alpha = \alpha_0\cdot\alpha_1\cdot\ldots\cdot\alpha_m$ is unique.

The above lemma allows us to make the next definitions.

DEFINITION. Assume α, β are AI. If $\alpha>1$, then express it uniquely $\alpha = \alpha_0\cdot\alpha_1\cdot\ldots\cdot\alpha_m$ as the product of MI factors where $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_m > 1$. If $\beta>1$, express it similarly as $\beta = \beta_0\cdot\beta_1\cdot\ldots\cdot\beta_n$. Then α *does not mix with* β if and only if either $\alpha = 1$ or $\beta = 1$ or $\alpha_m \geqslant \beta_0$.

DEFINITION. For MI $\alpha<\omega_1$, let $P(\alpha) = \{1\} \cup \{\omega^t\colon \alpha = \omega^{s\cdot t}$ where s, t are AI and s does not mix with t$\}$. Let $Q(1) = \{1\}$, $Q(\omega) = \{1, \omega\}$, and for MI α with $\omega<\alpha<\omega_1$, let $Q(\alpha) = \{\omega^2\} \cup P(\alpha)$.

Now we can state the main theorem.

THEOREM 4(a). *If* $\alpha<\omega_1$ *is* MI, *then* $\alpha \to \delta$ *if and only if* $\delta \in Q(\alpha)$.

(b) *If* $1<\alpha<\omega_1$ *is* AI, $\alpha = a_0\cdot a_1\cdot\ldots\cdot a_{m-1}$ *is expressed uniquely as the product of* MI *factors where* $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_{m-1}$, *then* $\alpha \to \delta$ *if and only if* $\delta = d_0\cdot d_1\cdot\ldots\cdot d_{m-1}$ *is the product of ordinals where* $d_0 \in Q(a_0)$ *and for all* i *with* $0<i<m$, $d_i \in P(a_i)$.

For convenience extend the definition of $Q(\alpha)$ to AI α so that the above theorem becomes $\alpha \to \delta$ if and only if $\delta \in Q(\alpha)$. The proof of this theorem consists of Theorems 9 and 17 below.

One consequence of the theorem is that the set of ordinals to which a given ordinal can be pinned is finite. Every AI ordinal $\alpha \geqslant \omega^2$ can be pinned to 1, ω, ω^2 and α. The collection of AI ordinals $\alpha \geqslant \omega^2$ which can be pinned to no other ordinals is the collection of such ordinals which are exponentially indecomposable (EI), that is, ordinals which cannot be expressed non-trivially as $\alpha = c^d$. These ordinals are of the form ω^a where a is MI. Thus the theorem shows that for EI ordinals $\alpha \geqslant \omega^2$, pinning sheds no light on the question of whether or not the partition relation $\alpha \to (\alpha, 3)^2$ holds.

In Section 2, we show the desired pinning maps exist. In Section 3, we show that the relation holds only where stated.

Our set theoretic notation is standard. We define an ordinal to be the set of its predecessors.

If A, B are subsets of an ordered set X, we write $A<B$ if every element of A precedes every element of B in the ordering of X.

We write $\operatorname{tp} X$ for the order-isomorphism type of X. Since we shall only be considering order types of well-ordered sets, we shall write $\operatorname{tp} X = \alpha$, rather than $\operatorname{tp} X = \operatorname{tp}\alpha$, where α is an ordinal.

We write $f\restriction A$ for f restricted to A, $f''A$ for the image of the set A under the function f. We write $\sum \alpha_n$ for the sum of a sequence of ordinals, and $\lim a_n$ for the limit. Note that for an increasing sequence of countable AI ordinals, these two notions coincide.

If α is a successor ordinal, $\alpha = \beta+1$, then $\alpha \dot{-} 1 = \beta$, while if α is a limit ordinal $\alpha \dot{-} 1 = \alpha$.

Notice that if $f\colon X \to Y$ is any function between sets X and Y with $\operatorname{tp} X = \alpha$ and $\operatorname{tp} Y = \delta$, then, since the ordinals are well-founded, there is a set $Z \subseteq X$ of type α so that $f\restriction Z$ is a pinning map onto its image. Of course, all one can say immediately about $\operatorname{tp} f''Z$ is that $\operatorname{tp} f''Z \leqslant \delta$.

Since all the ordinals under consideration in this paper are countable, we will not always explicitly state that they are countable.

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§ 2. **The pinning maps.** The problem of proving that the required maps exist falls into two parts, namely proving that the required maps exist for MI ordinals, and proving that such maps can be put together to define functions for products of MI ordinals. For the latter we introduce the notion of strong pinning.

DEFINITION. For AI α, we say α *can be strongly pinned to* δ and write $\alpha \Rightarrow \delta$ if there is a function $f\colon \alpha \to \delta$ so that whenever $\langle X_n\colon n<\omega\rangle$ is a sequence of subsets of α with $\lim \operatorname{tp} X_n = \alpha$, then also $\lim \operatorname{tp} f''X_n = \delta$. We call such a function a *strong pinning map*.

LEMMA 5. *Assume* β, $\gamma>1$ *are* AI *and* β *does not mix with* γ. *Express* $\beta = \beta_0\cdot\beta_1\cdot\ldots\cdot\beta_{n-1}$ *uniquely as the product of* MI *ordinals with* $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{n-1}$.

(a) *If* $\beta \to \eta$, *then* $\beta\cdot\gamma \Rightarrow \eta$.

(b) *If* $\beta \to \eta$ *and* $\beta_{n-1} \geqslant \gamma$, *then there is a function* $h\colon \beta\cdot\gamma \to \eta$ *so that for every* $X \subseteq \beta\cdot\gamma$, *if* $\operatorname{tp} X > \beta$, *then* $\operatorname{tp} h''X = \eta$.

Proof. Let $\pi\colon \beta\cdot\gamma \to \beta\times\gamma$ be an isomorphism where $\beta\times\gamma$ is ordered anti-lexicographically. Let $f\colon \beta\cdot\gamma \to \beta$ be defined by $f(x) = u$ where $\pi(x) = (u, v)$. Let $g\colon \beta \to \eta$ be a pinning map. Let $h\colon \beta\cdot\gamma \to \eta$ be defined by $h(x) = g(f(x)) = g(u)$ where $\pi(x) = (u, v)$.

Express $\gamma = \gamma_0\cdot\gamma\cdot\ldots\cdot\gamma_{m-1}$ uniquely as the product of MI ordinals with

$\gamma_0 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_{m-1}$. Since β does not mix with γ, it follows that $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{n-1} \geqslant \gamma_0 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_{m-1}$. If $\gamma > \beta_{n-1}$, then $\gamma_0 = \beta_{n-1}$ and $m > 0$.

The proof of the lemma divides on whether $\gamma > \beta_{n-1}$ or $\gamma \leqslant \beta_{n-1}$. If $\gamma \leqslant \beta_{n-1}$, then we must show that every subset X of type $> \beta$, has image $h''X$ of type η. If $\gamma > \beta_{n-1}$, then it suffices to show that every subset X of type $> \beta \cdot \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-2}$ has image $h''X$ of type η. So if $\gamma \leqslant \beta_{n-1}$, let X be a set of type $> \beta$, and if $\gamma > \beta_{n-1}$, let X be a set of type $> \beta \cdot \gamma_0 \cdot \ldots \cdot \gamma_{m-2}$. If $\operatorname{tp} f''X = \beta$, then since g is a pinning map, $\operatorname{tp} g''f''X = \operatorname{tp} h''X = \eta$, and we are done. Thus it is enough to show $\operatorname{tp} f''X = \beta$. Assume by way of contradiction, that $\operatorname{tp} f''X = \xi < \beta$. Then $X \subseteq Y = \{x : f(x) \in f''X\}$, and $Y = \pi^{-1}(f''X \times \gamma)$ has type $\xi \cdot \gamma$. Now $\xi \leqslant \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \varepsilon$ for some $\varepsilon < \beta_{n-1}$. Thus if $\gamma \leqslant \beta_{n-1}$, we get the desired contradiction from this string of inequalities:

$$\beta < \operatorname{tp} X \leqslant \operatorname{tp} Y = \xi \cdot \gamma \leqslant (\beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \varepsilon) \cdot (\beta_{n-1}) = \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \beta_{n-1} = \beta .$$

On the other hand, if $\gamma > \beta_{n-1}$, we get the contradiction from the following inequalities:

$$\begin{aligned} \beta \cdot \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-2} &< \operatorname{tp} X \leqslant \operatorname{tp} Y \\ &= \xi \cdot \gamma \leqslant \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \varepsilon \cdot \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-1} \\ &= \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-1} \\ &\leqslant \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-2} \cdot \beta_{n-1} \cdot \gamma_0 \cdot \ldots \cdot \gamma_{m-2} \\ &= \beta \cdot \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-2} . \end{aligned}$$

Here we used the fact that $\beta_{n-1} \geqslant \gamma_0 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_{m-2} \geqslant \gamma_{m-1}$. So in either case the lemma follows.

LEMMA 6. *Assume* β, γ *are* AI *and* β *does not mix with* γ. *If* $\beta \to \eta$, *then* $\beta \cdot \gamma \to \eta \cdot \omega$.

Proof. Express $\beta = \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-1}$ and $\gamma = \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{m-1}$ uniquely as the products of MI ordinals where $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{n-1}$ and $\gamma_0 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_{m-1}$. Since β does not mix with γ, it follows that $\beta_0 \geqslant \ldots \geqslant \beta_{n-1} \geqslant \gamma_0 \geqslant \ldots \geqslant \gamma_{m-1}$. In view of the previous lemma, it is enough to show that $\beta \cdot \gamma_0 \to \eta \cdot \omega$. So let $\theta = \gamma_0$.

Let $\chi \colon \eta \times \omega \to \eta \cdot \omega$ be an isomorphism where $\eta \times \omega$ is ordered anti-lexicographically. If $\theta = \omega$, then express $\beta \cdot \omega = \bigcup B_k$ as the disjoint union of sets of type β with $B_0 < B_1 < \ldots$ Using the hypothesis $\beta \to \eta$, let $g \colon \beta \cdot \omega \to \eta \times \omega$ be a function with the property that for each $k < \omega$, g restricted to B_k is a pinning map into $\eta \times \{k\}$. Let $h \colon \beta \cdot \omega \to \eta \cdot \omega$ be defined by $h(x) = \chi(g(x))$. If $X \subseteq \beta \cdot \omega$, and $\operatorname{tp} X = \beta \cdot \omega$, then since β is AI, for infinitely many k, $X \cap B_k$ has type β, so $\operatorname{tp} h''X = \eta \cdot \omega$, and the lemma follows.

If $\theta > \omega$, let $\langle \theta_k \colon k < \omega \rangle$ be an increasing sequence of AI ordinals whose limit is θ, and express $\beta \cdot \theta = \bigcup B_k$ as the disjoint union of sets where $\operatorname{tp} B_k = \beta \cdot \theta_k$ and $B_0 < B_1 < \ldots$ Since $\beta_{n-1} \geqslant \gamma_0$, it follows that $\beta_{n-1} > \theta_k$. For each $k < \omega$, let $g_k \colon B_k \to \eta \times \{k\}$ be a pinning map as described in part (b) of Lemma 5, so that for every $X \subseteq B_k$, if $\operatorname{tp} X > \beta$, then $\operatorname{tp} g_k''K = \eta$. Let $g = \bigcup g_k$, and let $h \colon \beta \cdot \theta \to \eta \cdot \omega$ be defined by $h(x) = \chi(g(x))$. Suppose $X \subseteq \beta \cdot \theta$ and $\operatorname{tp} X = \beta \cdot \theta$. Then for infinitely many k, we have $\operatorname{tp} X \cap B_k > \beta$, since otherwise for some $r < \omega$, we have $\beta \cdot \theta = \operatorname{tp} X \leqslant \beta \cdot \theta_0 + \ldots + \beta \cdot \theta_r + \beta \cdot \omega < \beta \cdot \theta$. So using the choice of the g_k, we see that $\operatorname{tp} h''X = \eta \cdot \omega$ and the lemma follows.

LEMMA 7. *If* $c > 1$ *is* MI, $d > 0$ *is* AI, *then* $c^d \Rightarrow \omega^d$. *In fact, there is a function* $f \colon c^d \to \omega^d$ *so that for all* e *with* $0 < e \leqslant d$, *for all* $X \subseteq c^d$ *of type* $\geqslant c^e$, $f''X$ *has type* $\geqslant \omega^e$.

Proof. Let c be any given countable MI ordinal. Define by recursion a sequence of functions $\langle f_a \colon a < \omega_1 \rangle$ where $f_a \colon c^a \to \omega^a$ for all $a < \omega_1$. Now if $c = \omega$, then choosing all of these functions to be the identity proves the lemma. So assume $c > \omega$, and decompose $c = \bigcup_{n<\omega} C_n$ as the disjoint union of countably many sets where $C_0 < C_1 < \ldots$ and $\operatorname{tp} C_0 < \operatorname{tp} C_1 < \ldots$ Define $r \colon c \to \omega$ by $r(x) = n$ where $x \in C_n$. Notice that r is a strong pinning map of c to ω.

If $a = 0$, then $c^0 = 1 = \omega^0$, and we set f_1 equal to the identity.

If $a = b+1$, then let $\pi \colon c^a \to c^b \times c$ and $\chi \colon \omega^b \times \omega \to \omega^a$ be isomorphisms where the cartesian products are ordered anti-lexicographically. Define $f_a \colon c^a \to \omega^a$ by $f_a(x) = \chi(f_b(u), r(v))$ where $\pi(x) = (u, v)$.

If a is a limit ordinal, $a = \lim_{n<\omega} a(n)$, then $c^a = \lim_{n<\omega} c^{a(n)}$, so we can write $c^a = \bigcup D_n$ where $D_0 < D_1 < \ldots$ and for all $n < \omega$, $\operatorname{tp} D_n = c^{a(n)}$. We assume without loss of generality that $\langle a(n) \colon n < \omega \rangle$ is an increasing sequence. We can also write $\omega^a = \bigcup E_n$ where $E_0 < E_1 < \ldots$ and for all $n < \omega$, $\operatorname{tp} E_n = \omega^{a(n)}$. For each $n < \omega$, let $\pi_n \colon C_n \to C^{a(n)}$ and $\chi_n \colon \omega^{a(n)} \to E_n$ be isomorphisms. Define $f_a \colon c^a \to \omega^a$ by $f_a(x) = \chi_n\big(f_{a(n)}(\pi_n(x))\big)$ where $x \in C_n$.

Having defined the maps by recursion, we prove by induction that $f_a \colon c^a \to \omega^a$ is a strong pinning map for a AI by showing that for all $a < \omega$, for all $e \leqslant a$, if $X \subseteq c^a$ has type $\geqslant c^e$, then $f_a''X$ has type $\geqslant \omega^e$.

For $a = 0$, the induction hypothesis is clearly true.

Suppose $a = b+1$, $e \leqslant a$ and $X \subseteq c^a$ has type $\geqslant c^e$. Let $U_n = \{u \colon \exists v (u, v) \in \pi''X$ and $r(v) = n\}$ for $n < \omega$. If for some $n < \omega$, $\operatorname{tp} U_n \geqslant c^e$, then $\operatorname{tp} f_b'' U_n \geqslant \omega^e$ by the induction hypothesis, so from the definition of f_a it follows that $\operatorname{tp} f_a''X \geqslant \omega^e$. So we may assume that $\operatorname{tp} U_n < c^e$ for $n < \omega$. If e is a limit ordinal, then $c^e = \sum c^{e(n)}$ where $\langle e(n) \colon n < \omega \rangle$ is an increasing sequence. If for each $n < \omega$, there is an $m < \omega$ with $\operatorname{tp} U_m \geqslant c^{e(n)}$, then arguing as above, we see that $\operatorname{tp} f_a''X \geqslant \sum c^{e(n)} = c^e$. If for some n, there is no $m < \omega$ with $\operatorname{tp} U_m \geqslant c^{e(n)}$, then we have the contradiction that

$$\pi''X = \bigcup_{v \in c} \{(u, w) \in \pi''X \colon w = v\}$$

has type $\leqslant c^{e(n)} \cdot c = c^{e(n)+1} < c^e$. Thus if e is a limit ordinal, the induction statement holds. If e is a successor ordinal, $e = s+1$, then $c^e = c^{s+1} = c^s \cdot c > c^s \cdot \omega$. If there are infinitely many n with $\operatorname{tp} U_n \geqslant c^s$, then by the induction hypothesis, $f_a''X$ has type $\omega^s \cdot \omega = \omega^{s+1} = c^a$. We have already assumed that $\operatorname{tp} U_n < c^e$ for $n < \omega$. Suppose for all $n > k_0$, that $\operatorname{tp} U_n < c^s$. Let $Y_n = \{(u, v) \in \pi''X\colon r(v) = n\}$. For $n > k_0$, $\operatorname{tp} Y_n \leqslant \operatorname{tp} U_n \cdot \operatorname{tp} C_n < c^s$. So $\bigcup_{k_0 < n} Y_n$ has type $\leqslant c^s < c^e$. So $\pi''X = Y_0 \cup Y_1 \cup \ldots \cup Y_{k_0} \cup \bigcup_{k_0 < n} Y_n$ has type $< c^e$, since c^e is indecomposable. But that contradicts the facts that X has type $\geqslant c^e$ and π is an isomorphism. So there are infinitely many n with $\operatorname{tp} U_n \geqslant c^s$, and the induction statement follows.

Finally, suppose a is a limit ordinal, $e \leqslant a$ and $X \subset c^a$ has type $\geqslant c^e$. If $X \cap D_n$ has type $\geqslant c^e$ for some $n < \omega$, then the induction hypothesis insures $f''X$ has type $\geqslant \omega^e$. Otherwise as in the previous case, either $e = \lim e(n)$, and for every $n < \omega$, there is an $m < \omega$ with $\operatorname{tp} X \cap D_m \geqslant c^{e(n)}$ guaranteeing that $\operatorname{tp} f_a'' X \geqslant \sum \omega^{e(n)} = \omega^e$, or $e = s+1$, and for infinitely many $n < \omega$, $\operatorname{tp} X \cap D_n \geqslant c^s$, guaranteeing that $\operatorname{tp} f_a X'' \geqslant \sum \omega^s = \omega^{s+1} = \omega^e$.

LEMMA 8. *Assume β and $d > 1$ are* AI, $\gamma = c^d$ *and c are* MI, *and β does not mix with γ. If $\beta \Rightarrow \eta$, then $\beta \cdot \gamma \Rightarrow \eta \cdot \omega^d$.*

Proof. Let $\theta = \omega^d$. Let $\pi\colon \beta \cdot \gamma \to \beta \times \gamma$ and $\chi\colon \eta \times \theta \to \eta \cdot \theta$ be isomorphisms where the cartesian products are ordered anti-lexicographically. Let $f\colon \beta \to \eta$ be a strong pinning map. Let $g\colon \gamma \to \theta$ be the strong pinning map of Lemma 1, and recall that we proved in that lemma that if $X \subseteq \gamma$ and $\operatorname{tp} X \geqslant c^e$ for $e \leqslant d$, then $\operatorname{tp} g''X \geqslant \omega^e$. Define $h\colon \beta \cdot \gamma \to \eta \cdot \theta$ by $h(x) = \chi(f(u), g(v))$, where $\pi(x) = (u, v)$.

Suppose $\langle X_n\colon n < \omega \rangle$ is a sequence of subsets of $\beta \cdot \gamma$ with $\lim \operatorname{tp} X_n = \beta \cdot \gamma$. Now d is AI and $d > 1$, so d is the limit of an increasing sequence, $d = \lim d(n)$. It is enough to show that if $\operatorname{tp} X_k \geqslant \beta \cdot c^{d(n)}$, then $\operatorname{tp} h'' X_k \geqslant \eta \cdot \omega^{d(n)-1}$. So fix k, n and assume $\operatorname{tp} X_k \geqslant \beta \cdot c^{d(n)}$. Let $s = d(n)$, and let $Y = \pi'' X_k$.

Suppose $J = \{v\colon \operatorname{tp} Y \cap \beta \times \{v\} = \beta\}$ has type $\geqslant c^{d(n)}$. Let

$$Z = \bigcup_{v \in J} \{x \in X_k\colon \exists u\, \pi(x) = (u, v)\}.$$

Then $\operatorname{tp} Z \geqslant \beta \cdot c^{d(n)}$, and given the facts that f, g are strong pinning maps, it follows that $h''Z$ and thus $h''X$ have type $\geqslant \eta \cdot \omega^{d(n)}$. So we may assume J has type $< c^{d(n)}$. Then $\operatorname{tp} Z < \beta \cdot c^{d(n)}$, so $\operatorname{tp}(X_k - Z) = \beta \cdot c^{d(n)}$, and we may assume without loss of generality, that J is empty, that is, for all v, $Y \cap \beta \times \{v\}$ has type $< \beta$.

Since $\operatorname{tp} Y = \beta \cdot c^{d(n)}$, we can express $Y = \bigcup_{\xi < c^s} Y(\xi)$ as the disjoint union of sets of type β where if $\xi < \xi'$, then $Y(\xi) < Y(\xi')$. Let $\langle \beta(n)\colon n < \omega \rangle$ be a strictly increasing sequence of AI ordinals whose limit is β.

CLAIM. *For all $\xi < c^s$, for all $k < \omega$, there is $v < c^d$ so that $\{(u, w) \in Y(\xi)\colon w = v\}$ has type $\geqslant \beta(k)$.*

Proof of Claim. Suppose for some ξ and some n the statement fails to hold, i.e., $\operatorname{tp}\{(u, w) \in Y(\xi)\colon w = v\} < \beta(k)$ for all $v < c^d = \gamma$. Since $Y(\xi) < Y(\xi+1)$ and we have ordered $\beta \times \gamma$ anti-lexicographically, there is some $\bar{v} \in \gamma$ so that if $(u, w) \in Y(\xi)$, then $w < \bar{v}$. So $Y(\xi) = \bigcup_{v \in \bar{v}} \{(u, w) \in Y(\xi)\colon w = v\}$ has type $\leqslant \beta(k) \cdot \bar{v}$. To prove the claim, it suffices to show that $\beta(k) \cdot \bar{v} < \beta$, since then we have the contradiction $\beta = \operatorname{tp} Y(\xi) \leqslant \beta(k) \cdot \bar{v} < \beta$.

Now β does not mix with γ and γ is MI, so $\beta \cdot \gamma = \omega^\varrho$ where $\varrho = \varrho_0 + \varrho_1 + \ldots + \varrho_n$, $\varrho_0 \geqslant \varrho_1 \geqslant \ldots \geqslant \varrho_n$, $\varrho_0, \varrho_1, \ldots, \varrho_n$ AI and for $\sigma = \varrho_0 + \varrho_1 + \ldots + \varrho_{n-1}$, $\tau = \varrho_n$, we have $\beta = \omega^\sigma$ and $\gamma = \omega^\tau$. Since $\beta(k) < \beta$, we have $\beta(k) < \omega^{\tau'}$ where $\tau' = \varrho_0 + \varrho_1 + \ldots + \varrho_{n-2} + \varepsilon$ and $\varepsilon < \varrho_{n-1}$. Since $\bar{v} < \gamma$, we have $\bar{v} = \omega^{\sigma'}$ where $\sigma' < \varrho_{n-1}$. Since ϱ_{n-1} is AI, $\varepsilon + \sigma' < \varrho_{n-1}$, so $\tau' + \sigma' < \varrho_0 + \varrho_1 + \ldots + \varrho_{n-1}$, and $\beta(k) \cdot \bar{v} \leqslant \omega^{\tau'} \cdot \omega^{\sigma'} = \omega^{\tau' + \sigma'} < \beta$.

Enumerate the ordinals less than $c^s = c^{d(n)}$ in order type ω so that each ordinal appears infinitely often, as $\langle \xi(j)\colon j < \omega \rangle$. For each $j < \omega$, let $Z(j) \subseteq Y(\xi(j))$ be a set of type $\beta(j)$, with the properties that for some v, $Z(j) \subseteq \beta \times \{v\}$, and if $l < j$ and $Z(l) \subseteq \beta \times \{w\}$, then $v \neq w$. Such a sequence must exist by the above claim and the assumption that $\operatorname{tp} Y \cap \beta \times \{x\} < \beta$ for all x. Let $Z = \bigcup_{j < \omega} Z(j)$. Let

$$W = \{v\colon \exists j < \omega\, Z(j) \subseteq \beta \times \{v\}\}.$$

Since for each $\xi < c^s = c^{d(n)}$, we have $\xi = \xi(j)$ for infinitely many j, and $\xi < \xi'$ implies $Y(\xi) < Y(\xi')$, we know that $\operatorname{tp} W \geqslant c^{d(n)}$. So $\operatorname{tp} g''W \geqslant \omega^{d(n)}$. Notice that if $N \subseteq \omega$ is infinite, then $\operatorname{tp} \bigcup_{j \in N} \pi^{-1} Z(j) = \operatorname{tp} \bigcup_{j \in N} Z(j) \geqslant \beta$ and $\operatorname{tp} \bigcup_{j \in N} h''\pi^{-1} Z(j) \geqslant \eta$. So $\operatorname{tp} h''\pi^{-1} Z' \geqslant \eta \cdot \omega^{d(n) \dot- 1}$. Thus $\operatorname{tp} h'' X_k \geqslant \eta \cdot \omega^{d(n) \dot- 1}$, and the lemma follows.

THEOREM 9. (a) *If $\alpha < \omega_1$ is* MI *and $\delta \in Q(\alpha)$, then $\alpha \to \delta$.*

(b) *If $\alpha = \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_{n-1}$ is expressed uniquely as the product of* MI *ordinals with $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{n-1}$, and $\delta = d_0 \cdot d_1 \cdot \ldots . d_{n-1}$ is the product of ordinals where $d_0 \in Q(\beta_0)$ and for i with $0 < i < n$, $d_i \in P(\beta_i)$, then $\alpha \to \delta$.*

Proof. (a) It is easy to see that $1 \to 1$, $\omega \to 1$ and $\omega \to \omega$. So assume $\omega < \alpha < \omega_1$. It is not hard to show $\alpha \to \omega^2$. See [3] or [8]. If $\delta \in Q(\alpha)$ and $\delta \neq \omega^2$, then by Lemma 7, $\alpha \to \delta$.

(b) If $\mu < \nu$, then $\mu \cdot \nu = \nu$, so we may rewrite $\delta = \mu_0 \cdot \mu_1 \cdot \ldots \cdot \mu_{n-1}$ where $\mu_0 \in Q(\beta_0)$, $\mu_i \in P(\beta_i)$ for i with $0 < i < n$ so that $\mu_j = 1$ if there is a k with $j < k < n$ and $d_k > \mu_j$. If $\mu_0 = \omega^2$, then for all i with $0 < i < n$, we have $\mu_i \in \{1, \omega\}$, so using the result that $\beta_0 \to \omega^2$ for $\beta_0 > \omega$ and Lemmas 5 and 6, we see that $\alpha \to \delta$. Otherwise, $\mu_0 \in P(\beta_0)$. Thus by using induction, and Lemmas 5, 6, 7 and 8, we can show that for all $i < n$, $\beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_i \to \mu_0 \cdot \mu_1 \cdot \ldots \cdot \mu_i$, thus completing the proof of the theorem.

§ 3. Otherwise, no pinning maps.

In this section, we show that pinning maps exist only in the cases given in the previous section. The first lemma shows some bounds on the pinning maps that exist for MI ordinals.

LEMMA 10. *If $\alpha < \omega_1$ is* MI *and $\alpha \to \delta$, then either $\delta = \omega^2$ or δ* MI.

Proof. By Lemma 1, we know that δ is AI. If $\delta = 1$ or $\delta = \omega$, then δ is MI. If $\delta = \omega^2$, then the lemma is satisfied. So assume by way of contradiction, that $\delta > \omega^2$ and δ is not MI. Then $\delta = \omega^d$ where d is decomposable, so by a theorem of [3], $\delta \to \omega^3$. Since pinning is transitive, it follows that $\alpha \to \omega^3$. But by a theorem of [3], $\alpha \nrightarrow \omega^3$. This contradiction gives the lemma.

The next lemma addresses the problem of pinning and products of ordinals.

LEMMA 11. *Assume β and γ are* AI, *γ is* MI, *$\beta>1$, $\gamma>1$ and β does not mix with γ. Further assume the inductive hypothesis that for all* AI *$\alpha'<\alpha=\beta\cdot\gamma$, if $\alpha'\to\delta'$, then $\delta'\in Q(\alpha')$. If $\delta=\eta\cdot\theta$ where $\beta\to\eta$ and $\theta\in P(\gamma)$, then $\delta\in Q(\beta\cdot\gamma)$.*

Proof. Write $\beta\cdot\gamma=a_0\cdot a_1\cdot\ldots\cdot a_{n-1}$ where $a_0, a_1, \ldots, a_{n-1}$ are all MI and $a_0\geqslant a_1\geqslant\ldots\geqslant a_{n-1}>1$. Since $\beta>1$, $\gamma>1$, γ is MI and β does not mix with γ, we must have $\beta=a_0\cdot a_1\cdot\ldots\cdot a_{n-2}$ and $\gamma=a_{n-1}$. So $n\geqslant 2$. If $n=2$, then $a_0=\beta$, $a_1=\gamma$, $d_0=\eta$, $d_1=\theta$ show $\delta=\eta\cdot\theta\in Q(\beta\cdot\gamma)$. So suppose $n>2$. Since $\alpha>\beta\cdot\gamma$, by the inductive hypothesis, if $\beta\to\delta'$, then $\delta'\in Q(\beta)$. Thus the set of δ' to which β can be pinned is finite. So we can find $d_0, d_1, \ldots, d_{n-2}$ such that $\eta=d_0\cdot d_1\cdot\ldots\cdot d_{n-2}$, $d_0\in Q(a_0)$ and for all i with $0<i<n-1$, $d_i\in P(a_i)$. Set $d_{n-1}=\theta$. Then $a_0, a_1, \ldots, a_{n-1}$, and $d_0, d_1, \ldots, d_{n-1}$ show $\delta\in Q(\beta\cdot\gamma)$.

Since our proof proceeds by induction, we look at increasing sequences of ordinals and derived sequences of ordinals to which the first ones can be pinned. The next two lemmas show that the later sequences must be fairly nice if the former sequences are chosen with care.

LEMMA 12. *Assume γ is* MI, *$\gamma>\omega$ and $\gamma\neq\varkappa^\omega$ for any ordinal $\varkappa$. Then there is an increasing sequence $\langle c_k\colon k<\omega\rangle$ of* MI *ordinals whose limit is γ so that for every infinite increasing sequence $\langle u_k\colon k\in N\rangle$ of ordinals with the property that $u_k\in Q(c_k)$ for $k\in N$, the limit $\mu=\lim u_k\in P(\gamma)$.*

Proof. Since γ is MI, we have $\gamma=\omega^a$ where a is AI. Since $\gamma>\omega$, we know that $a>1$. Express $a=s\cdot t$ uniquely as the product of two AI ordinals, where $t>1$ is MI and s does not mix with t. Then $\gamma=\omega^a=\omega^{s\cdot t}=(\omega^s)^t$. Since $\gamma\neq\varkappa^\omega$ for any ordinal $\varkappa$, it follows that $t>\omega$. Since t is MI, we have $t=\omega^\lambda$ where λ is AI. So $\lambda=\omega^l$. Since $t>\omega$, we have $l>0$. Either l is a successor ordinal and $t=\tau^\omega$ for some MI ordinal τ, or l is a limit ordinal, and t is the limit of an increasing sequence of MI ordinals, $\langle t(n)\colon n<\omega\rangle$. If $t=\tau^\omega$, set $t(k)=\tau^k$. Then $\langle t(k)\colon k<\omega\rangle$ is an increasing sequence whose limit is t. For $k<\omega$, set $c_k=\omega^{s\cdot t(k)}$. Then $\langle c_k\colon k<\omega\rangle$ is an increasing sequence of MI ordinals whose limit is γ. Suppose $\langle u_k\colon k\in N\rangle$ is an infinite increasing sequence of ordinals with $u_k\in Q(c_k)$ for $k\in N$. Let μ be the limit of the sequence. Since the sequence is increasing and the only elements of $Q(c_k)$ less than ω^2 are 1 and ω, we may assume without loss of generality that $u_k>\omega^2$ for $k\in N$. So each $u_k\in P(c_k)$ for $k\in N$. Now $c_k=\omega^{s\cdot t(k)}$. So either $t=\tau^\omega$ and $u_k=\omega^{t(l)}$ for some $l\leqslant k$, or there are $\varrho(k)$, $\sigma(k)$ so that $s=\varrho(k)\cdot\sigma(k)$, $\varrho(k)$ does not mix with $\sigma(k)$ and $u_k=\omega^{\sigma(k)\cdot t(k)}$. Since $\langle u_k\colon k\in N\rangle$ is increasing, we have one of the following two cases. Either $t=\tau^\omega$ and for all $k\in N$, there is $l\leqslant k$ so that $u_k=\omega^{t(l)}$. Or there is k_0 so that for all $k\in N$ with $k\geqslant k_0$ there are $\varrho(k)$, $\sigma(k)$ as above with $u_k=\omega^{\sigma(k)\cdot t(k)}$.

If $t=\tau^\omega$ and for all $k\in N$, there is $l\leqslant k$ so that $u_k=\omega^{t(l)}$, then since $\langle u_k\colon k\in N\rangle$ is increasing, it follows that $\mu=\omega^t\in P(\gamma)$.

So we may assume we are in the other case. Fix k_0. Since there are only finitely many possibilities for $\varrho(k)$, $\sigma(k)$ and $\langle u_k\colon k<\omega\rangle$ is increasing, there are $k_1\geqslant k_0$, ϱ and σ so that for all $k\in N$ with $k\geqslant k_1$, both $\varrho=\varrho(k)$ and $\sigma=\sigma(k)$. Then $\mu=\lim\omega^{\sigma\cdot t(k)}=\omega^{\sigma\cdot t}\in P(\gamma)$. So in either case the lemma follows.

LEMMA 13. *Assume $\varkappa$ is* MI, *$N\subseteq\omega$ infinite and let $\langle\theta_k\colon k\in N\rangle$ be an infinite increasing sequence with the property that for all $k\in N$, $\theta_k\in Q(\varkappa^k)$. Let θ be the limit of $\langle\theta_k\colon k\in N\rangle$.*

(a) *If θ is* MI, *then $\theta\in P(\varkappa^\omega)$.*

(b) *Otherwise, there are ordinals μ, ν, an infinite set $K\subseteq N$ and a sequence $\langle\nu_k\colon k\in K\rangle$ so that $\theta=\mu\cdot\nu$, $\nu\in P(\varkappa^\omega)$, ν is the limit of $\langle\nu_k\colon k\in K\rangle$ and for all $k\in K$, $\theta_k=\mu\cdot\nu_k$.*

Proof. Enumerate $P(\varkappa)\sim\{1\}$ in decreasing order as $r_0, r_1, r_2, \ldots, r_j=\omega$. Since θ_k is the product of elements of $P(\varkappa)$, using the properties of ordinal multiplication, we can write $\theta_k=r_0^{t(0,k)}\cdot r_1^{t(1,k)}\cdot\ldots\cdot r_j^{t(j,k)}$ where for each $i\leqslant j$, we have $0\leqslant t(i,k)<\omega$. If for each $i\leqslant j$, there were a bound $u(i)$ for the sequence $\langle t(i,k)\colon k\in N\rangle$, then there would only be a finite set of ordinals that θ_k could be chosen from, contradicting the fact that $\langle\theta_k\colon k\in N\rangle$ is an increasing sequence. So at least one sequence must be unbounded. Let I be the least $i\leqslant j$ so that $\langle t(i,k)\colon k\in N\rangle$ is unbounded. Choose $u(0), u(1), \ldots, u(I-1)$ in turn so that $K=\{k\colon \forall i<I,\ t(i,k)=u(i)\}$ is infinite. Let $\mu=r_0^{u(0)}\cdot r_1^{u(1)}\cdot\ldots\cdot r_{I-1}^{u(I-1)}$. Then for $k\in K$, set $\nu_k=r_I^{t(I,k)}\cdot r_{I+1}^{t(I+1,k)}\cdot\ldots\cdot r_j^{t(j,k)}$, so that $\theta_k=\mu\cdot\nu_k$. Since $\langle\theta_k\colon k\in N\rangle$ is an increasing sequence, we have

$$\theta=\lim_{k\in N}\theta_k=\lim_{k\in K}\theta_k=\lim_{k\in K}\mu\cdot\nu_k=\mu\cdot\lim_{k\in K}\nu_k=\mu\cdot r_I^\omega.$$

We leave it to the reader to check that since $r_I\in P(\varkappa)$, also $\nu=r_I^\omega\in P(\varkappa^\omega)$.

Next we define what it means for a function to be plodding, a concept used several times in this section.

DEFINITION. Assume $\beta<\omega_1$ is MI, $B\subseteq\alpha$ has type β and $f\colon B\to\gamma$ is a function. Then f is *plodding on B* if and only if there is a decomposition of $B=\bigcup_{n<\omega}B_n$ satisfying all of the following conditions:

(a) $\operatorname{tp}B_0, \operatorname{tp}B_1, \operatorname{tp}B_2, \ldots$ *are all* AI,

(b) $B_0<B_1<B_2<\ldots$,

(c) $f''B_0<f''B_1<f''B_2<\ldots$,

(d) *if $\gamma=\omega$, then $|f''B_i|=1$ for all $i<\omega$.*

Note that if $\beta=\gamma=\omega$, then plodding and one-to-one coincide.

LEMMA 14. *If $\beta<\omega_1$ is* MI, *$A\subseteq\alpha$ has type β and $f\colon A\to\omega$ is a function, then there is a set $B\subseteq A$ of type β so that either $f\restriction B$ is constant, one-to-one, or plodding.*

Proof. If $\beta=1$ or $\beta=\omega$, then it is easy to see that the lemma is true. So assume $\beta>\omega$. Since β is MI, we can find $\langle\beta_k\colon k<\omega\rangle$, an increasing sequence of AI ordinals less than β, and decompose $A=\bigcup_{k<\omega}A_k$ into a countable disjoint union of sets where $A_0 < a_1 < A_2 < \ldots$, and for all $k<\omega$, $\operatorname{tp} A_k = \beta_k^2$. Since for each $k<\omega$, we have $\operatorname{tp} A_k = \beta_k^2$, we may express $A_k = \bigcup_{\gamma<\beta_1} A_k(\gamma)$ as the disjoint union of β_i sets each of type β_i so that if $\gamma<\delta<\beta_i$, then $A_i(\gamma)<A_i(\delta)$.

Case 1. There is an infinite set $I\subseteq\omega$, so that for all $i\in I$, there is $\gamma_i<\beta_i$ with $f''A_i(\gamma_i)$ finite.

First, for each $i\in I$, let $C_i\subseteq A_i(\gamma_i)$ be a set of type β_i so that f restricted to C_i is constant. Such a set must exist since β_i is AI. Next, define $g\colon I\to\omega$ by $g(i)=f(\delta)$ for $\delta\in C_i$. Let $J\subseteq I$ be infinite so that either $g\restriction J$ is constant or one-to-one. Finally, let $B=\bigcup_{j\in J} C_i$. If $g\restriction J$ is constant, then $f\restriction B$ is constant. If $g\restriction J$ is one-to-one, then $f\restriction B$ is plodding on B.

Case 2. There is an i_0 so that for all $i>i_0$, for all $\gamma<\beta i$, $f''A_i(\gamma)$ is infinite.

Enumerate the set of all $A_i(\gamma)$ for $i>i_0$ and $\gamma<\beta_i$ as $\langle R_k\colon k<\omega\rangle$. By recursion define $C=\{C_k\colon k<\omega\}$ so that $C_k\in R_k$ and $f(C_k)>f(C_{k-1})$ for $k>0$. Then C has type β and $f\restriction C$ is one-to-one.

The following notion is needed for the next lemma.

Definition. Assume α, β are countable AI ordinals, $\operatorname{tp} A=\alpha$, $\operatorname{tp} B=\beta$ and $f\colon A\to B$ is a function. We say f is *essentially cofinal on* A if for every subset $C\subseteq A$ of type α, the image $f''C$ is unbounded in B.

Lemma 15. *Let β, γ, δ be countable* AI *ordinals. Assume γ is* MI, *$\gamma\geqslant\omega$, β does not mix with γ and for all* AI *$\alpha'<\beta\cdot\gamma$, if $\alpha'\to\delta'$, then $\delta'\in Q(\alpha)$. If $\beta\cdot\gamma\to\delta$ and $f\colon \beta\cdot\gamma\to\delta$ attests to it, then either*

(a) *$\delta\in Q(\beta\cdot\gamma)$ or*

(b) *$\beta>1$, $\gamma=\varkappa^\omega$, $\delta\neq d\cdot\omega$ for any d, and there is a set $X\subseteq\beta\cdot\gamma$ of type $\beta\cdot\gamma$ so that $f\restriction X$ is a plodding map.*

Proof. Express $\beta\cdot\gamma=\bigcup_{c\in\gamma} B(c)$ as the disjoint union of γ sets each of type β with the property that if $c<c'$, then $B(c)<B(c')$. Express $\delta=\bigcup_{n\in\omega} D_n$ as the disjoint union of a collection of sets whose order types are AI so that $D_0<D_1<D_2<\ldots$ and either $\operatorname{tp} D_0=\operatorname{tp} D_1=\operatorname{tp} D_2=\ldots$ or $\operatorname{tp} D_0<\operatorname{tp} D_1<\operatorname{tp} D_2<\ldots$ Enumerate γ as $\langle c_n\colon n<\omega\rangle$ in type ω. For each $c<\gamma$, with $c=c_n$, let $B'(c)\subseteq B(c)$ be a set of type β so that f restricted to $B'(c)$ is a pinning map onto its image, and either f restricted to $B'(c)$ is essentially cofinal in δ and $f''B'(c)\cap\bigcup_{k\leqslant n} D_k=\emptyset$, or for some $m<\omega$, $f''B'(c)\subseteq D_m$. For each $c<\gamma$, let $q(c)=\operatorname{tp} f''B'(c)$. Find q and $C\subseteq\gamma$, a set of type γ so that for all $c\in C$, $q(c)=q$ and either for all $c\in C$, f restricted to $B'(c)$ is essentially cofinal, or for all $c\in C$, f restricted to $B'(c)$ is bounded in δ.

Case 1. For all $c\in C$, f restricted to $B'(c)$ is essentially cofinal.

Let $X=\bigcup_{c\in C} B'(c)$. Then $\operatorname{tp} X=\beta\cdot\gamma$. Since for $c\in C$, we have $\operatorname{tp} f''B'(c)=q$, it follows that $\operatorname{tp} f''X\geqslant q$. We shall show $\operatorname{tp} f''X=q$. If $c=c_n\in C$, then $f''B'(c)\cap D_k=\emptyset$ for $k\leqslant n$ and $\operatorname{tp} f''B'(c)\cap D_k<q$ for $k>n$, since f restricted to $B'(c)$ is essentially cofinal. So for $k<\omega$, we have $\operatorname{tp} f''X\cap D_k<q$. So $\operatorname{tp} f''X\leqslant q$. So $\delta=q\in Q(\beta)\subseteq Q(\beta\cdot\gamma)$.

Now we may assume that for all $c\in C$, f restricted to $B'(c)$ is bounded, and in fact that $f''B'(c)\subseteq D_k$ for some k. This defines a function $p\colon C\to\omega$. Using Lemma 14, find a set $C'\subseteq C$ of type γ so that p restricted to C' is either constant, one-to-one, or plodding. Let $X=\bigcup_{c\in C'} B'(c)$. Then $\operatorname{tp} X=\beta\cdot\gamma$.

Case 2. p restricted to C' is constant.

Here $f''X\subseteq D_k$ where k is the constant value of p on C'. So $\operatorname{tp} f''X<\delta$, contradicting the fact that f is a pinning map. Thus this case cannot occur.

Case 3. p restricted to C' is one-to-one.

In this case, $\operatorname{tp} f''X=q\cdot\omega=\delta$, where $q\in Q(\beta)$. Thus $q\cdot\omega\in Q(\beta\cdot\gamma)$, and the lemma follows.

Case 4. p restricted to C' is plodding and $\delta\neq d\cdot\omega$.

Since plodding and one-to-one coincide in the case $\gamma=\omega$ and the lemma has been proved if p is one-to-one, we may assume $\gamma>\omega$. Express $C'=\bigcup_{n\in\omega} G(n)$ as in the definition of plodding. Since $\gamma>\omega$, we must have $\operatorname{tp} G(0)<\operatorname{tp} G(1)<\operatorname{tp} G(2)<\ldots$ For each $n<\omega$, let $X(n)=\bigcup_{c\in G(n)} B'(n)$. Then $\operatorname{tp} X(n)=\beta\cdot\operatorname{tp} G(n)$ is AI. Also $\operatorname{tp} X(0)<\operatorname{tp} X(1)<\ldots$ So f restricted to X is plodding. Thus if $\beta>1$ and $\gamma=\varkappa^\omega$ for some $\varkappa$, and $\delta\neq d\cdot\omega$ then part (b) of the lemma follows. Otherwise we can find $Y=\bigcup_{n\in\omega} Y(n)\subseteq X$ of type $\beta\cdot\gamma$, so that f restricted to Y is plodding, $Y=\bigcup_{n\in\omega} Y(n)$ is a decomposition which shows it, and for all $n\in\omega$, f restricted to $Y(n)$ is a pinning map onto its image and for all $n\in\omega$, $\operatorname{tp} Y(n)=\beta\cdot\varkappa^n$ if $\gamma=\varkappa^\omega$ and otherwise $\operatorname{tp} Y(n)=\beta\cdot\gamma_n$ where γ_n is as prescribed in Lemma 12. If $\beta=1$, $\gamma=\varkappa^\omega$ and $\delta\neq d\cdot\omega$, then by Lemmas 1 and 13, it follows that $\operatorname{tp} f''Y=\delta\in P(\gamma)=P(\beta\cdot\gamma)$. If $\gamma\neq\varkappa^\omega$ and $\delta\neq d\cdot\omega$, then from the induction hypothesis and Lemmas 11 and 12, it follows that $\operatorname{tp} f''Y=\delta\in P(\beta\cdot\gamma)$. Thus the only remaining case is $\delta=d\cdot\omega$.

For each $k<\omega$, D_k has type d. If $d=d'\cdot\omega$ where d' is AI, then set $d(0)=d'=d(1)=d(2)=\ldots$ Otherwise, let $\langle d(i)\colon i<\omega\rangle$ be an increasing sequence of AI ordinals with limit d. For each $k<\omega$, express $D_k=\bigcup_{l\in\omega} D_k(l)$ where for $l<\omega$, $\operatorname{tp} D_k(l)=d(l)$ and $D_k(0)<D_k(1)<\ldots$

Use the decomposition $C'=\bigcup G(n)$ and the function $p\colon C'\to\omega$ to define $r\colon \omega\to\omega$ by $r(n)=p(c)$ for $c\in G(n)$.

For each $n<\omega$, we repeat what we did with $\beta\cdot\gamma$, δ and $f\colon \beta\cdot\gamma\to\delta$, this time with $X(n)$, $D_{r(n)}$ and $f\colon X(n)\to D_{r(n)}$. So for each $c\in G(n)$ with $c=c_m$, we find $B''(c)\subseteq B'(c)$ of type β so that either f restricted to $B''(c)$ is essentially cofinal in $D_{r(n)}$ and $f''B''(c)\cap\bigcup_{k\leqslant m} D_{r(n)}(k)=\emptyset$, or for some $k<\omega$, $f''B''(c)\subseteq D_{r(n)}(k)$. For each $n<\omega$, let $G'(n)\subseteq G(n)$ be a set with $\operatorname{tp} G'(n)=\operatorname{tp} G(n)$ so that either for all $c\in G'(n)$, f restricted to $B''(c)$ is essentially cofinal, or for all $c\in G'(n)$, f is bounded on $B''(c)$.

Let $N \subseteq \omega$ be infinite so that either for all $n \in N$, for all $c \in G'(n)$, f restricted to $B''(c)$ is essentially cofinal, or for all $c \in G'(n)$, f is bounded on $B''(c)$. Note that from earlier considerations, we know that f restricted to $B''(c)$ is a pinning map onto a set of type q.

Subcase a. For all $n \in N$ for all $c \in G'(n)$ f restricted to $B''(c)$ is essentially cofinal in $D_{r(n)}$.

Let $Z = \bigcup_{n \in N} \bigcup_{C \in G'(n)} B''(c)$. Then $\operatorname{tp} Z = \beta \cdot \gamma$. As we saw in Case 1, for each $n \in N$, $\operatorname{tp} f''Z(n) = q$, where $Z(n) = \bigcup_{c \in G'(n)} B''(c)$. It follows that $\operatorname{tp} f''Z = q \cdot \omega \in Q(\beta \cdot \gamma)$, and the lemma is proved.

Thus we may assume that for all $n \in N$, $s_n = s\colon G'(n) \to \omega$ is defined with $f''B''(c) \subseteq D_{r(n)}(s(c))$ for $c \in G'(n)$. For each $n \in N$, let $G''(n) \subseteq G'(n)$ be a set with $\operatorname{tp} G''(n) = \operatorname{tp} G'(n)$ so that s restricted to $G''(n)$ is either constant, one-to-one or plodding. Let $M \subseteq N$ be infinite so that either for all $n \in M$, s restricted to $G''(n)$ is constant or for all $n \in M$, s restricted to $G''(n)$ is one-to-one, or for all $n \in M$, s restricted to $G''(n)$ is plodding.

Subcase b. For all $n \in M$, s restricted to $G''(n)$ is constant.

Let $Z = \bigcup_{n \in M} \bigcup_{C \in G''(n)} B''(c)$, and let $Z(n) = \bigcup_{C \in G''(n)} B''(c)$ for $n \in M$. Then $\operatorname{tp} Z = \beta \cdot \gamma$, and for $n \in m$, $\operatorname{tp} f''Z(n) < d = \operatorname{tp} D_{r(n)}$, so $\operatorname{tp} f''Z \leqslant d < d \cdot \omega$. In other words f is not a pinning map, so this case cannot occur.

Subcase c. For all $n \in M$, s restricted to $G''(n)$ is plodding.

For each $n \in M$, express $G''(n) = \bigcup_{k \in \omega} G''(n, k)$ as in the definition of plodding. Since $\gamma > \omega$, we known $\operatorname{tp} G''(0) < \operatorname{tp} G''(1) < \ldots$ Without loss of generality we may assume $\omega < \operatorname{tp} G''(0)$. So for $n \in M$, $\operatorname{tp} G''(n, 0) < \operatorname{tp} G''(n) < \ldots$ Thus we can find a function $t\colon M \to \omega$ so that $\operatorname{tp} G''(n, t(n)) > \operatorname{tp} G''(n-1)$. (Assume $0 \notin M$.) Thus $\operatorname{tp} \bigcup_{n \in M} G''(n, t(n)) = \gamma$. For $n \in M$, let $Z(n) = \bigcup \{B''(c)\colon c \in G\colon (n, t(n))\}$. Let $Z = \bigcup_{n \in M} Z(n)$. Then $\operatorname{tp} Z = \beta \cdot \gamma$. Since for all $n \in M$, we have $f''Z(n) \subseteq D_{r(n)}(s(c))$ for $c \in G''(n, t(n))$, it follows that $\operatorname{tp} f''Z(n) < d$. So $\operatorname{tp} f''Z \leqslant d < d \cdot \omega$. In other words f is not a pinning map, so this case cannot happen.

Subcase d. For all $n \in M$, s restricted to $G''(n)$ is one-to-one.

For each $n \in M$, let $Z(n) = \{B''(c)\colon c \in G''(n)\}$. Let $Z = \bigcup_{n \in \omega} Z(n)$. Then $\operatorname{tp} Z = \beta \cdot \gamma$. For $n \in M$, since r restricted to $G''(n)$ is one-to-one, it follows that $\operatorname{tp} f''Z(n) = q \cdot \omega$. Therefore, $\operatorname{tp} f''Z = q \cdot \omega \cdot \omega = q \cdot \omega^2$. If $\beta = 1$, then $q = 1$, so $\delta = q \cdot \omega^2 = \omega^2 \in P(\gamma) = P(\beta \cdot \gamma)$. So we may assume $\beta > 1$. To prove the lemma, we must show that $\beta \to q \cdot \omega$.

Suppose that for every $b < \beta$, every $n \in M$ and every $c \in G''(n)$, there is a set $U \subseteq B''(c)$ of type $\geqslant b$ so that $\operatorname{tp} f''U < q$. Enumerate $\bigcup_{n \in M} G''(n) = \langle g(k)\colon k < \omega \rangle$ and let $\langle \beta_k\colon k < \omega \rangle$ be an increasing sequence of ordinals cofinal in β. For each $k < \omega$, let $W_k \subseteq B''(g(k))$ be a set of type $\geqslant \beta_k$ so that $\operatorname{tp} f''W_k < q$. Let $W = \bigcup_{k \in \omega} W_k$.

Since the limit of any infinite subsequence of $\langle \beta_k\colon k < \omega \rangle$ is β, and $\omega \cdot \gamma = \gamma$, we have $\operatorname{tp} W = \beta \cdot \gamma$. For any $n \in M$, since r restricted to $G''(n)$ is one-to-one, and for all $c \in G''(n)$ with $c = g(k)$, we have $f''W_k < d$ it follows that

$$f''(\bigcup \{W_k\colon g(k) \in G''(n)\}) = \bigcup \{f''W_k\colon g(k) \in G''(n)\}$$

has type $\leqslant q$. Thus

$$f''Z = f''(\bigcup_{n \in M} \bigcup \{W_k\colon g(k) \in G''(n)\})$$

has type $\leqslant q \cdot \omega < q \cdot \omega^2$. So we have reached a contradiction to the fact that f is a pinning map.

Therefore, for some $b < \beta$, for some $n \in M$, and some $c \in G''(a)$, for every set $U \subseteq B''(c)$ of type $\geqslant b$, we have $\operatorname{tp} f''U \geqslant q$. Partition $B''(c) = \bigcup_{k < \omega} A_k$ into countably many sets so that $A_0 < A_1 < \ldots$; either $\operatorname{tp} A_0 = \operatorname{tp} A_1 = \ldots < \beta$ or $\operatorname{tp} A_0 < \operatorname{tp} A_1 < \ldots$; $\operatorname{tp} A_0, \operatorname{tp} A_1, \ldots$ are all AI and $\operatorname{tp} A_0 \geqslant b$. Then if $S \subseteq B''(c)$ is a set of type β, we must have $\operatorname{tp} S \cap A_i \geqslant b$ for infinitely many i. Define $\varrho\colon B''(c) \to f''B''(c) \times \omega$ by $\varrho(x) = (f(x), i)$ where $x \in A_i$. Since $f''B''(c)$ has type q, the set $f''B''(c) \times \omega$ has type $q \cdot \omega$ when ordered antilexicographically. Thus ϱ is a pinning map of a set of type β into a set of type $q \cdot \omega$. Thus $\delta = q \cdot \omega \cdot \omega \in P(\beta \cdot \gamma)$, and the lemma follows.

LEMMA 16. *Assume for all $\alpha < \beta$, if $\alpha \to \delta$, then $\delta \in Q(\alpha)$. Further, assume β, λ AI, β does not mix with λ, $\beta \to \eta$, U is a set of type $\beta \cdot \lambda^2$, V a set of type $\eta \cdot \mu \cdot \nu$ and $f\colon U \to V$. Then there is a set $X \subseteq U$ of type $\beta \cdot \lambda$ so that either*

(a) *$\operatorname{tp} f''X \leqslant a \cdot \mu \leqslant \eta \cdot \mu \cdot \nu$ where $\beta \to a$ or,*

(b) *$a \leqslant \operatorname{tp} f''X \leqslant a \cdot \nu$ where $\beta \to a$.*

Proof. Since U has type $\beta \cdot \lambda^2$, we can express $U = \bigcup_{m, l < \lambda} U(l, m)$ as the disjoint union of a family of sets each of type β indexed by pairs of ordinals less than λ where if $(l, m) < (l', m')$ in the antilexicographic ordering, then $U(l, m) < U(l', m')$. We can write $V = \bigcup \{V(u, v)\colon u < \mu \,\&\, v < \nu\}$ analogously. For each pair (l, m) of ordinals less than λ, let $A(l, m) \subseteq U(l, m)$ be a set of type β so that f restricted to $A(l, m)$ is a pinning map onto its image and either there is some $v(l, m)$ so that $f''A(l, m) \subseteq \bigcup_{u < \mu} V(u, v(l, m))$, or for all $v \in \nu$, the set $A(l, m) \cap f^{-1}(\bigcup_{u < \mu} V(u, v))$ has type less than β. Suppose there is a set $P \subseteq \lambda \times \lambda$ of type λ and $a \in P(\beta)$ so that for all $p \in P$, $\operatorname{tp} f''A(p) = a$ and $A(p) \cap f^{-1}(\bigcup_{u < \mu} V(u, v))$ has type $< \beta$ for $v \in \nu$.

Enumerate $P = \langle p_n\colon n < \omega \rangle$ and $\nu = \langle v_n\colon n < \omega \rangle$. Let $B(p_n) \subseteq A(p_n)$ be a set of type β so that $f''B(p_n) \subseteq \bigcup_{n \leqslant k < \omega} \bigcup_{u < \mu} V(u, v_k)$. Let $X = \bigcup_{n < \omega} B(p_n)$. Then $\operatorname{tp} X = \beta \cdot \lambda$, since $\operatorname{tp} P = \lambda$. Clearly $\operatorname{tp} X \geqslant a$. Since for each $k < \omega$, $f''B(p_n) \cap \bigcup_{u < \mu} V(u, v_k) \neq \varnothing$ for only finitely many n, it follows that $\operatorname{tp} f''X \cap \bigcup_{u < \mu} V(u, v_k) < a \cdot \omega$, so $\operatorname{tp} f''X \leqslant a \cdot \nu$. And in this case, the lemma follows.

So suppose there is no such set P. Let $Q\subseteq\lambda\times\lambda$ be the set of pairs (l, m) with $f''A(l,m)\subseteq\bigcup_{u<\mu}V(u, v(l,m))$. Suppose there are $l_0\in\lambda$, $v_0\in v$ and a set $R\subseteq\lambda$ of type λ so that for all $r\in R$, $f''A(r, l_0)\subseteq\bigcup_{u<\mu}V(u, v_0)$. Then

$$X=\bigcup\{A(r, l_0)\colon r\in R \text{ and } (r, l_0)\in Q\}$$

satisfies condition (a) of the lemma. So we may assume for each $l\in\lambda$, that the set $\{v(r,l)\colon (r,l)\in Q\}$ is empty or infinite. By the inductive hypothesis, we know that β can be pinning to only finitely many different ordinals. So we can find an a so that $\beta\to a$, and a set $L\subseteq\lambda$ of type λ so that for all $l\in L$, the set $\{v(r,l)\colon (r,l)\in Q$ and $\operatorname{tp}f''A(r,l)=a\}$ is infinite. Enumerate $L=\langle l_n\colon n<\omega\rangle$. Define by recursion $\langle r_n\colon n<\omega\rangle$ so that $\langle r_n, l_n\rangle\in Q$ and if $m\neq n$, then $v(r_m, l_m)\neq v(r_n, l_n)$. Let $X=\bigcup_{n<\omega}A(r_n, l_n)$. Then $\operatorname{tp}X=\beta\cdot\lambda$, and $a\leqslant\operatorname{tp}f''X\leqslant a\cdot v$. So condition (b) of the lemma holds.

THEOREM 17. *If $\alpha<\omega_1$ is* AI *and $\alpha\to\delta$, then $\delta\in Q(\alpha)$.*

Proof. The proof is by induction on the countable AI ordinals. If $\alpha=1$, then clearly α can only be pinned to 1. If $\alpha=\omega$, then it is easy to see that α can only be pinned to 1 and to ω. So assume $\alpha>\omega$. If α is MI, then $\alpha=\beta\cdot\gamma$ where $\beta=1$ and $\gamma=\alpha$, so from Lemma 15 it follows that if $\alpha\to\delta$, then $\delta\in Q(\alpha)$. So assume α is not MI. Then $\alpha=\beta\cdot\gamma$ where β is AI, γ is MI, $\beta>1$, $\gamma>1$ and β does not mix with γ. Suppose $\alpha\to\delta$. Again from Lemma 15 it follows that $\delta\in Q(\alpha)$ except for the case $\gamma=\varkappa^\omega$ and $\delta\neq d\cdot\omega$ for any d. Let $f\colon\alpha\to\delta$ be a pinning map. Using Lemma 15, we may assume we have a set $X\subseteq\alpha$ of type α so that f restricted to X is plodding. Let $X=X_n$ be the decomposition which shows f is plodding on X. Now $\langle\operatorname{tp}X_n\colon n<\omega\rangle$ is an increasing sequence. Without loss of generality, we may assume $\operatorname{tp}X_n=\beta\cdot\varkappa^{2n}$, since if these equations fail to hold, we can construct a subset of X of type α which has a decomposition showing f is plodding in which these equations hold. For each $n<\omega$, let $Y_n\subseteq X_n$ be a set of type $\beta\cdot\varkappa^{2n}$ so that f restricted to Y_n is a pinning map onto its image. By the induction hypothesis, not only is the set of ordinals to which β can be pinned finite, but also $\operatorname{tp}f''Y_n=b_n\cdot\theta_n$ where $\beta\to b_n$ and θ_n is the product of $2n$ elements of $P(\varkappa)$. Using the fact that the set of ordinals to which β can be pinned is finite, find $b<\omega$ and an infinite set $N\subseteq\omega$ so that for all $n\in N$, $b_n=b$, and so that the function φ defined by $\varphi(n)=\theta_n$ is either constant or increasing. Let $Y=\bigcup_{n\in N}Y_n$. Then $\operatorname{tp}Y=\alpha$. If φ is constantly t on N, then

$$\operatorname{tp}f''Y=\sum_{n\in N}\operatorname{tp}f''Y_n=\sum_{n\in N}b\cdot t=b\cdot t\cdot\omega, =\delta$$

contrary to our assumption that $\delta\neq d\cdot\omega$ for any d. So φ is increasing on N. Let $\theta=\lim_{n\in N}\theta_n$. Then $\operatorname{tp}f''Y=b\cdot\theta=\delta$. If θ is MI, then by Lemma 13, $\theta\in P(\varkappa^\omega)$, so by Lemma 11, we have $b\cdot\theta\in Q(\alpha)$. So assume θ is not MI. Let $K\subseteq N$, μ, v, $\langle v_k\colon k\in K\rangle$ be as in Lemma 13. For each $k\in K$, use Lemma 16 to find $a(k)$ and $Z_k\subseteq Y_k$ a set of type $\beta\cdot\varkappa^n$ so that $B\to a(k)$ and either

(a) $\operatorname{tp}f''Z_k\leqslant a(k)\cdot\mu\leqslant b\cdot\mu\cdot v_k$ or

(b) $a(k)\leqslant\operatorname{tp}f''Z_k\leqslant a(k)\cdot v_k$.

Let $a<\omega_1$ and $M\subseteq K$, an infinite set, be such that for all $k\in M$, $a(k)=a$ and either for all $k\in M$, $\operatorname{tp}f''Z_k\leqslant a\cdot\mu\leqslant b\cdot\mu\cdot v_k$ or for all $k\in M$, $a\leqslant\operatorname{tp}f''Z_k\leqslant a\cdot v_k$.

Let $Z=\bigcup_{k\in M}Z_k$. Then $\operatorname{tp}Z=\alpha$. If for all $k\in M$, $\operatorname{tp}f''X_k\leqslant a\cdot\mu\leqslant b\cdot\mu\cdot v_k$, then $\operatorname{tp}f''X\leqslant a\cdot\mu\cdot\omega\leqslant\delta$. Since $\delta\neq d\cdot\omega$ for any d and δ is AI by Lemma 1, it follows that $a\cdot\mu\cdot\omega<\delta$, so f is not a pinning map. This contradiction shows that we must assume that for all $k\in M$, $a\leqslant\operatorname{tp}f''Z_k\leqslant a\cdot v_k$. Thus $a\cdot\omega\leqslant\operatorname{tp}f''Z\leqslant a\cdot v$. Since f is a pinning map, $\operatorname{tp}f''Z=\delta=b\cdot\mu\cdot v$. So $a\cdot\omega\leqslant b\cdot\mu\cdot v\leqslant a\cdot v$. Since $v\in P(\varkappa^\omega)$, it follows that v is MI. So $\delta=b\cdot\mu\cdot v=a\cdot v$, and by Lemma 11, $\delta=a\cdot v\in Q(\beta\cdot\gamma)$. So the theorem follows.

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