References

[1] F. Adams, *A variant of E. H. Brown's representability theorem*, Topology 10 (1971), pp. 185–198.
[2] E. Brown, Jr., *Abstract homotopy theory*, Trans. Amer. Math. Soc. 119 (1965), pp. 79–85.
[3] A. Calder and J. Siegel, *Homotopy and Kan Extensions*, Categorical Topology Mannheim 1975, Springer Lecture Notes (54), pp. 152–163.
[4] — *Homotopy and uniform homotopy*, Trans. Amer. Math. Soc. 235 (1978), pp. 245–270.
[5] — *Kan extensions of homotopy functors*, J. Pure Appl. Algebra 12 (1978), pp. 253–269.
[6] A. Deleanu, *The Brown-Adams representability theorem*, J. London Math. Soc. 6 (1973), pp. 561–562.
[7] P. J. Hilton, *On Kan extensions of cohomology theories and Serre classes of groups*, Fund. Math. 73 (1971), pp. 113–165.
[8] — *Localization homology and a construction of Adams*, Trans. Amer. Math. Soc. 179 (1973), pp. 349–362.
[9] A. Dold, *Lectures in Algebraic Topology*, Springer, Berlin–Heidelberg 1972.
[10] C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. 69 (1947), pp. 200–242.
[11] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press 1952.
[12] S. T. Hu, *Homotopy Theory*, Academic Press, 1959.
[13] S. MacLane, *Categories for the Working Mathematician*, Springer, New York 1971.

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The number of countable models of a theory of one unary function

by

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Abstract. If T is a theory in the language of one unary function symbol then T has 1, $\aleph_0$, or $2^{\aleph_0}$ countable models.

§ 1. **Introduction.** Let L^0 denote the language containing equality and one unary function symbol. We prove:

THEOREM 1. *If T is a complete first order theory in L^0, then T has 1, $\aleph_0$, or $2^{\aleph_0}$ countable models.*

The part of the theorem claiming that if T has $>\aleph_0$ countable models then T has $2^{\aleph_0}$ countable models is the first-order Vaught conjecture for L^0. The $L^0_{\omega_1\omega}$ Vaught conjecture was claimed by Burris in [1] but an error was found by Arnold Miller. After writing the first draft of this paper I learned that Miller [5] had already proven Theorem 1 by a different method in a more general setting, and some information about the $L^0_{\omega_1\omega}$ case.

The following theorem of Shelah gives information about the number of uncountable models of a theory in L^0.

THEOREM (Shelah). *If T is a complete first-order theory in L^0 then either T has 2^λ models of power λ for all $\lambda \geqslant \aleph_1$ or T has $\leqslant \beth_n(|\alpha|)$ models of power $\aleph_\alpha$ for some $n<\omega$ and all $\alpha \geqslant \omega$.*

There is a similar theorem for $L^0_{\omega_1\omega}$.

The proof uses general considerations of stability. The problem of the number of countable models of a first-order theory of linear order was solved in Rubin [6].

I am indebted to Mati Rubin for calling my attention to the error in [1], and to him and to Miller for detecting errors in earlier versions of the present paper.

§ 2. **Preliminaries.** We preserve the notation and definitions of [4]. Here is a brief review. (For model-theoretic notation and definitions see [3].) The language contains one unary function symbol f, and equality.

The distance between a and b relative to a set A is $d_A(a, b) = \min\{r$: there are k, l such that $k+l = r$ and there are $x_0, \ldots, x_k, y_0, \ldots, y_l \in A$ such that $a = x_0$, $b = y_0$, $f(x_i) = x_{i+1}$ for $i<k$, $f(y_j) = y_{j+1}$ for $j<l$, and $x_k = y_l\}$. A path from a to b is such a sequence $\langle x_0, \ldots, y_l\rangle$. We say a is *above* b if there is a path from a to b which contains $f(b)$. The set A is *below* b if b is above every element of A. Notice

that there could be a and b such that a is both above and below b. A is *directly below* b if A is the union of sets of the form $A_k = \bigcup_{n<\omega} f^{-n}(a_k)$, where $f(a_k) = b$. (Note $f^0(a) = a$.)

Any model breaks up into the disjoint union of components, a component being the set of points of finite distance from a given point. When we count *occurrences* of components in a model, we count α disjoint copies of the same component as α occurrences.

If $a \in B$ let $\mathrm{Nbd}_B^r a$, the r-neighborhood of a in B, be the set of points in B of distance $\leqslant r$ from a. If A is a set let $\mathrm{Nbd}_B^r A = \bigcup_{a \in A} \mathrm{Nbd}_B^r a$. $A \dot{\cup} B$ indicates disjoint union, i.e., if necessary taking an isomorphic copy of A which is disjoint from B.

Let A be a set, $a_i \in A$, $i = 1, \ldots, n$; $q < \omega$. The q-type of $\langle a_1, \ldots, a_n \rangle$ over A is $\{\psi(x_1, \ldots, x_n) : \psi$ has $\leqslant q$ quantifiers and $A \models \psi(a_1, \ldots, a_n)\}$. We write $\mathrm{Nbd}_A^r \bar{a} \equiv_q \mathrm{Nbd}_B^r \bar{b}$ if the q-type of $\bar{a}$ over $\mathrm{Nbd}_A^r \bar{a}$ is the same as the q-type of $\bar{b}$ over $\mathrm{Nbd}_B^r \bar{b}$.

We use $\equiv$ for elementary equivalence, $\prec$ for elementary submodel, $\cong$ for isomorphism.

The following two results are from [4].

LEMMA 1.1. *For any m and n there are numbers $r = r(m, n)$ and $q = q(m, n)$ such that for any $\psi(\bar{x})$ with n free variables and m quantifiers, and for any model M and every $\bar{a}$, $\bar{b}$ from M of length n, if $\mathrm{Nbd}_M^r \bar{a} \equiv_q \mathrm{Nbd}_M^r \bar{b}$ then $M \models \psi(\bar{a}) \equiv \psi(\bar{b})$.*

1.2. *If $f(a) = b$ and a is not algebraic over b in M (i.e., a does not satisfy any formula $\varphi(x, b)$ which is satisfied by only finitely many members of M) then $M - \bigcup_{n<\omega} f^{-n}(a) \prec M$.*

In addition we need the following facts.

LEMMA 2. *Let $C_1 \equiv C_2$ be components. Then:*

(i) *For any M, $M \dot{\cup} C_1 \equiv M \dot{\cup} C_2$.*

(ii) *Let $b_1 \in C_1$ and $b_2 \in C_2$ realize the same type, F_l is the set of predecessors of $f(b_l)$ which are above b_l, $l = 1, 2$. Let C_1' be the component whose set of elements is $(C_1 - F_1) \dot{\cup} F_2$, and f is defined as in $C_1 - F_1$ and F_2 except that $f(a) = b_1$ if $a \in F_2$ and $f(a) \notin F_2$. Then $C_1' \equiv C_1$.*

Proof. Follows from Lemma 1.

LEMMA 3. *Let A, B be components, $a \in A$, $b \in B$, $M = A \dot{\cup} B$. Then $(M, a) \equiv (M, b)$ iff $(A, a) \equiv (B, b)$.*

Proof. We use the Ehrenfeucht game criterion. See [2]. We shall give informal descriptions of the winning strategies.

$\Rightarrow$: We must show that player *II* has a winning strategy in $G_n((A, a), (B, b))$ for $n < \omega$. We know that *II* wins in every $G_n((M, a), (M, b))$; we just have to show that he can choose elements in the appropriate components. So assume *I* chooses $x_1 \in A$. *II* consults his strategy for $G_{d(a,x_1)+3^n}((M, a), (M, b))$ and finds a suitable $y_1 \in M$. Obviously $d(x_1, a) = d(y_1, b)$ so $y_1 \in B$. Now let *I* choose $x_2 \in A$ (if *I* chooses $y_2 \in B$ the proof is the same). If $d(a, x_2) < d(a, x_1) + 2 \cdot 3^{n-1}$ *II* can choose $y_2 \in M$ according to $G_{d(a,x_1)+3^n}((M, a), (M, b))$ and again it will follow that $y_2 \in B$. If however $d(a, x_2) \geqslant d(a, x_1) + 2 \cdot 3^{n-1}$ then *II* looks at $G_{d(a,x_2)+3^{n-1}}((M, a), (M, b))$ as if x_2 were *I*'s first move, and then *II* chooses $y_2 \in M$. Again y_2 must be in B. Note that $d(y_1, y_2) \geqslant 2 \cdot 3^{n-1}$. It is clear how *II* continues to play.

$\Leftarrow$: Now assume *II* has a winning strategy in all $G_n((A, a), (B, b))$, and assume *I* chooses x_1 as a first move in $G_n((M, a), (M, b))$. Assume $x_1 \in (M, a)$. If $x_1 \in A$, *II* plays according to $G_n((A, a), (B, b))$. If $x_1 \in B$, let $y_1' = x_1$ and play $G_n((A, a), (B, b))$ getting $x_1' \in A$. Let $y_1 = x_1'$. Now $\langle (a, x_1'), (b, y_1') \rangle$ is a winning position for *II* in $G_n((A, a), (B, b))$ so $\langle (a, x_1), (b, y_1) \rangle$ is a winning position for *II* in $G_n((M, a), (M, b))$. The continuation is clear.

LEMMA 4. *Let A and B be components $a \in A$, $b \in B$.*

(i) *If for every r, q, $\mathrm{Nbd}_A^r a \equiv_q \mathrm{Nbd}_B^r b$, then $A \equiv B$.*

(ii) *If there is $n < \omega$ such that for all r, q there is $a' \in A$, $d(a, a') < n$ such that $\mathrm{Nbd}_A^r a' \equiv_q \mathrm{Nbd}_B^r b$, then $A \equiv B$.*

Proof. (i): Let $M = A \dot{\cup} B$. Since $\mathrm{Nbd}_A^r a \equiv_q \mathrm{Nbd}_B^r b$, and $\mathrm{Nbd}_A^r a = \mathrm{Nbd}_M^r a$, $\mathrm{Nbd}_B^r b = \mathrm{Nbd}_M^r b$, we have by Lemma 1. (i) $(M, a) \equiv (M, b)$. Thus by Lemma 3, $(A, a) \equiv (B, b)$, and $A \equiv B$. (ii): We shall show that there are $a^* \in A$, $b^* \in B$ such for all r, q $\mathrm{Nbd}_A^r a^* \equiv_q \mathrm{Nbd}_B^r b^*$, and thus the result follows from (i). From the hypothesis of (ii) we can find $a^* \in A$, $d(a, a^*) < n$ and $m \leqslant n$ such that for all r, q there is $a' \in A$, $f^m(a') = a^*$, and $\mathrm{Nbd}_A^r a' \equiv_q \mathrm{Nbd}_B^r b$. Now take $b^* = f^m(b)$.

LEMMA 5. *The following conditions satisfy* (iv) $\rightarrow$ (i) $\equiv$ (ii) $\equiv$ (iii). *Let M be a model and C a component.*

(i) *$M \prec M \dot{\cup} C$.*

(ii) *There is $N \equiv M$ in which $\aleph_0$ copies of C occur.*

(iii) *There is $N \equiv M$ with $\aleph_0$ occurrences of components $\equiv C$.*

(iv) *There is $N \equiv M$ such that a different number (including zero) of components $\equiv C$ occur in M and N.*

Proof. (ii) $\equiv$ (iii) is clear by Lemma 2, so we shall prove (i) $\equiv$ (ii) and then (iv) $\rightarrow$ (i).

(i) $\rightarrow$ (ii): It is sufficient to show that if $M \prec M \dot{\cup} C$, then $M \dot{\cup} C \prec M \dot{\cup} C \dot{\cup} C$. Let C_0, C_1 be two copies of C disjoint from M and from each other. Assume $M \cup C_0 \cup C_1 \models \varphi(c, \bar{a}, \bar{b})$ where $\bar{a} \in M$, $\bar{b} \in C_0$, $c \in C_1$. By the Tarski–Vaught test (see [3]) and Lemma 4 it is sufficient to find $d \in M \cup C_0$ "equivalent enough" to c and "far enough" away from $\bar{a} \cup \bar{b}$. But $M \prec M \cup C_0$, so we can even find $d \in M$ which will work.

(ii) $\rightarrow$ (i): Let $M \dot{\cup} C \models \varphi(b, \bar{a})$, $\bar{a} \in M$, $b \in C$. As above it is sufficient to find $c \in M$ equivalent enough to b and far enough away from $\bar{a}$. But since C occurs $\aleph_0$ times in a model $\equiv M$ there are elements $c \in M$ equivalent to b, and arbitrarily far from each other. So some will be far enough away from $\bar{a}$.

(iv) $\to$ (i): First assume that more components $\equiv C$ occur in N than in M. Let $M \cup C_0 \models \varphi(a, \bar{b}, \bar{c})$, ($\bar{b}$ or $\bar{c}$ possibly empty) where C_0 is a copy of C disjoint from M, $a \in C_0$, $\bar{b} \in M$ but the members of $\bar{b}$ do not occur in components $\equiv C$, $\bar{c} \in M$ and the members of $\bar{c}$ do occur in components $\equiv C$. We must find $d \in M$ far from $\bar{b}$ and $\bar{c}$ and equivalent to a. Since there are more components $\equiv C$ in N we can find a' and $\bar{c}'$ in N which are equivalent enough to a and $\bar{c}$ respectively, and a' is not in any component which contains a member of $\bar{c}'$. Thus in M we can find d which is equivalent enough to a and arbitrarily far from $\bar{c}$. Now if all these d were close to an element $b \in \bar{b}$ then by Lemma 4(ii) $C \equiv$ the component of b, contradiction.

Now assume that M contains more occurrences of components $\equiv C$ than does N. So by the above proof $N \prec N \dot{\cup} C$. Thus by (i) $\to$ (ii) there is a model $\equiv N$ in which $\aleph_0$ copies of C occur. Thus there is such a model $\equiv M$ (since $M \equiv N$). Thus by (ii) $\to$ (i) $M \prec M \dot{\cup} C$. ■

§ 3. From one to $\aleph_0$ countable models. We use a result of Shishmarev [7]. He defines T as *limited* if there is $n<\omega$ such that

$$T \vdash \forall x\Big(\bigvee_{l,m=1}^{n} f^l(x) = f^{l+m}(x)\Big),$$

i.e., every component is of diameter $\leqslant n$.

THEOREM 3 (Shishmarev [7]). *T is $\aleph_0$-categorical iff*

(i) *T is limited, and*

(ii) *If $M \models T$ then there are only a finite number of non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(a)$ in M.*

THEOREM 4. *If T has $\geqslant 2$ non-isomorphic countable models, then T has infinitely many non-isomorphic countable models.*

Proof. Assume T has only finitely many countable models, and let M_0 be the prime model of T. Since T is not $\aleph_0$-categorical at least one of (i), (ii) from Theorem 3 does not hold. Thus it is easily seen: (iii) The components of M_0 are not of bounded diameter, or (iv) There are infinitely many non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(a)$ in M_0.

Case 1. First assume (iii). Consider the following type

$$p = p(x) = \{x \text{ is part of a component } C \text{ (of infinite diameter) disjoint from } M_0 \text{ such that no element of } C \text{ is algebraic in } C\}.$$

It is easy to see how to write p as a set of first-order sentences with parameters from M_0. (Algebraicity in a component can be expressed by a formula referring to the whole model by using Lemma 1; see Corollary 2.2 of [4].) Also p is consistent, so such a component C exists in some model $\equiv M_0$.

CLAIM 1. *M_0 contains <2 copies of C.*

Proof. It is sufficient to prove $C \prec \underbrace{C \dot{\cup} C \dot{\cup} \ldots}_{\text{finite or infinite}}$, since if $M_0 = A \cup C \dot{\cup} C \dot{\cup} \ldots$, where A does not intersect any copy of C, then $A \cup C \prec M_0$, and the two can be isomorphic only if M_0 contained only one copy of C, or $M_0 = A$, with no copies of C.

The proof of $C \prec C \dot{\cup} C \dot{\cup} \ldots$ is similar to the analogous parts of Lemma 5.

CLAIM 2. $M_0 \prec M_0 \dot{\cup} C$.

Proof. Follows from Lemma 5, (iv) $\to$ (i).

Now we can similarly add any finite number of copies of C to M_0, getting $\aleph_0$ non-isomorphic elementary extensions. This proves Theorem 4 in Case 1.

Case 2. (iv) holds. In this case the elements $a \in M_0$ for which the sets $\bigcup_{n<\omega} f^{-n}(a)$ are non-isomorphic all realize different atomic types. So there is a non-atomic 1-type in T. Let b realize this type, and let $D = \big(\bigcup_{n<\omega} f^{-n}(b)\big) \cap N$, where N is some countable model of T containing b. Since $b \notin M_0$, $D \cap M_0 = \emptyset$.

Case 2.1. If there is $m<\omega$ such that $f^m(b) \in M_0$, let m be the minimal one and consider $c = f^{m-1}(b)$. Then c is not algebraic over $f(c)$, simply because $c \notin M_0$. So we may adjoin the set $\bigcup_{n<\omega} f^{-n}(c) \cap N$ onto $f(c)$ any finite number of times and thus get $\aleph_0$ non-isomorphic elementary extensions of M_0. See Lemma 1.2.

Case 2.2. There is no such m. Then no component containing b intersects M_0. Let C be any countable component containing b. Now by Lemma 5, (iv $\to$ i) $M_0 \prec M_0 \dot{\cup} C$ and we again get $\aleph_0$ elementary extensions of M_0.

This completes the proof of Theorem 4.

§ 4. From $\aleph_0$ to $2^{\aleph_0}$ countable models.

THEOREM 5. *If T has $>\aleph_0$ countable models, then T has $2^{\aleph_0}$ countable models.*

Proof. Let M_i, $i<\omega_1$, be countable elementarily equivalent models.

Case 1. There are only $\aleph_0$ non-isomorphic components C_j, $j<\omega$, which occur as components in some M_i's.

CLAIM. *At least one of the following four alternatives holds:*

(A) *There is $i<\omega_1$ and $j_n<\omega$ for $n<\omega$ such that*

(i) $C_{j_m} \equiv C_{j_n}$ for $m, n<\omega$.

(ii) *M_i has $\aleph_0$ occurrences of components $\equiv C_{j_0}$.*

(B) *There is $i<\omega_1$ and $j_n<\omega$ for $n<\omega$ such that*

(i) *$C_{j_m} \not\equiv C_{j_n}$ for $m \neq n$.*

(ii) *M_i has only a finite (or zero) number of occurrences of components $\equiv C_{j_n}$, $n<\omega$.*

(iii) *For all $n<\omega$ there is $i_n<\omega_1$ such that M_{i_n} has a different number (zero, finite, or infinite) of occurrences of components $\equiv C_{j_n}$ than does M_i.*

(C) *There is* $i<\omega_1$ *and* $k_n<\omega$ *for* $n<\omega$ *such that*

(i) $C_{k_m} \not\equiv C_{k_n}$, $m \neq n$.

(ii) M_i *has a finite (or zero) number of occurrences of components* $\cong C_{k_n}$, $n<\omega$.

(iii) *For all* $n<\omega$ *there is* $i_n<\omega_1$ *such that* M_{i_n} *has an infinite number of occurrences of components* $\equiv C_{k_n}$.

(D) *There is* $i<\omega_1$ *and* $l_n<\omega$ *for* $n<\omega$ *such that*

(i) $C_{l_m} \not\equiv C_{l_n}$, $m \neq n$.

(ii) M_i *has a (positive) occurrence of* C_{l_n}, $n<\omega$.

(iii) *For all* $n<\omega$ *there are infinitely many* $j<\omega$ *such that* $C_j \equiv C_{l_n}$.

Proof of Claim. Assume (A) does not hold. Then: (*) if for any $j<\omega$ and $i<\omega_1$ M_i has $\aleph_0$ occurrences of components $\equiv C_j$, then there are only finitely many $k<\omega$ with $C_k \equiv C_j$.

Assume also (B) does not hold. Then: (**) If C_{j_n}, $n<\omega$, are pairwise $\not\equiv$ and there is i_0 such that M_{i_0} has only a finite (or zero) number of occurrences of components $\equiv C_{j_n}$ for all $n<\omega$, then for all but finitely many $n<\omega$, *every* M_i contains the same number of occurrences of components $\equiv C_{j_n}$. Divide the C_j, $j<\omega$, into elementary equivalence classes E_n, $n<\alpha\leqslant\omega$, where $E_n = \{C^n_j\colon j<\alpha_n\leqslant\omega\}$. For all $i<\omega_1$ consider the set $\{\langle n;\ \beta^{n,i}_0, \ldots, \beta^{n,i}_k, \ldots\rangle_{k<\alpha_n}\colon n<\alpha\}$ where $\beta^{n,i}_k$ is the number of copies of C^n_k occurring in M_i.

The following properties hold:

1) (From (*).) If $\alpha_n = \omega$ then for all $i<\omega_1$, $\sum_{k<\alpha_n} \beta^{n,i}_k < \aleph_0$.

2) (From (**).) If there is $i_0<\omega_1$ such that for infinitely many $n<\omega$, $\sum_{k<\alpha_n} \beta^{n,i_0}_k = \gamma_n<\omega$, then for all but finitely many of the above n and for all $i<\omega_1$ $\sum_{k<\alpha_n} \beta^{n,i}_k = \gamma_n$.

Now since there are $\aleph_1$ non-isomorphic M_i there are $\aleph_1$ sets $\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha\}$ satisfying 1) and 2).

Case 1.1. There is only a finite number of $n<\omega$ such that $\alpha_n = \omega$. Then by 1) it follows that there are $\aleph_1$ sets

$$\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n<\omega\}.$$

By 2) it follows that there are $\aleph_1$ sets

$$\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n<\omega \text{ and } \sum_{k<\alpha_n} \beta^{n,i}_k = \aleph_0\}.$$

This set is just

$$\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n<\omega \text{ and } (\exists k<\alpha_n)(\beta^{n,i}_k = \aleph_0)\}.$$

Thus there are $\aleph_1$ sets

$$\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n<\omega \text{ and } (\exists k<\alpha_n)(\beta^{n,i}_k = \aleph_0) \text{ and } (\exists l<\alpha_n)(\beta^{n,i}_l<\aleph_0)\}.$$

Thus (C) holds.

Case 1.2. For infinitely many $n<\omega$, $\alpha_n = \omega$; and there are $\leqslant\aleph_0$ sets $\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n<\omega\}$ satisfying 1) and 2).

Thus there must be $\aleph_1$ sets $\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha \text{ and } \alpha_n = \omega\}$. By 1), for each $i<\omega_1$, $\sum_{k<\alpha_n} \beta^{n,i}_k<\aleph_0$. So there are $\aleph_1$ infinite sets

$$\{\langle n;\ \beta^{n,i}_k\rangle_{k<\alpha_n}\colon n<\alpha,\ \alpha_n = \omega,\ \sum_{k<\alpha_n} \beta^{n,i}_k \neq 0\}.$$

This satisfies (D). The claim is proved. Now whether (A), (B), (C) or (D) holds we can construct $2^{\aleph_0}$ non-isomorphic models $\equiv M_i$ as follows: If (A) holds use Lemma 2.

If (B) holds use (iv) $\to$ (i) of Lemma 5.

If (C) holds use (iii) $\to$ (i) of Lemma 5.

If (D) holds use Lemma 2.

Case 2. There are $\aleph_1$ non-isomorphic components appearing as components in some M_i's.

Case 2.1. Among those $\aleph_1$ components, there are $\aleph_1$ which are pairwise not elementarily equivalent. Thus there are infinitely many not elementarily equivalent to any component of M_0, say. So, by (iv) $\to$ (i) of Lemma 5, we can construct $2^{\aleph_0}$ non-isomorphic elementary extensions of M_0.

Case 2.2. There is a component C such that there are $\aleph_1$ non-isomorphic components $\equiv C$ which appear as components in some M_i's. Thus it is sufficient to prove the following.

Lemma 6. *If there are* $\aleph_1$ *non-isomorphic components* $C_i \equiv C$, $i<\omega_1$, *then there are* $2^{\aleph_0}$ *non-isomorphic components* $\equiv C$.

Proof. If there are not $2^{\aleph_0}$, then only $\aleph_0$ complete types are realized in the C_i; thus we may assume that there is a complete type p which is realized in each C_i, by b_i, say.

Claim 1. (i) *or* (ii) *holds*:

(i) $\aleph_1$ *of the* C_i *are pairwise non-isomorphic from* b_i *down* (*i.e., the isomorphism* $b_i \mapsto b$; *cannot be extended to the set of predecessors of* b_i).

(ii) $\aleph_1$ *of the* C_i *are pairwise non-isomorphic from* b_i *up* (*i.e., the isomorphism* $b_i \mapsto b$; *cannot be extended to* $C_i - \bigcup_{n<\omega} f^{-n}(b_i)$).

Now we prove Lemma 6 in the case that (i) holds; when (ii) holds the proof is similar.

There exists n_1, $1\leqslant n_1<\omega$, such that for $\aleph_1$ of the $i<\omega_1$ there is $1\leqslant k_1(i)\leqslant\omega$ such that for all $j<k_1(i)$, $f^{n_1}(a^1_{i,j}) = b_i$, $a^1_{i,j}$ is not algebraic over $f(a^1_{i,j})$, $\{a^1_{i,j}\colon j<k(i)\} = \{x\colon f^{n_1}(x) = b_i,\ x \text{ is not algebraic over } f(x)\}$, and n_1 is minimal. Thus the sets $\bigcup_{0\leqslant n<n_1} f^{-n}(b_i)$ are elementarily isomorphic for all the above i.

Case 1. There are $\aleph_1$ i such that the sets $\bigcup_{0\leqslant n<n_1+1} f^{-n}(b_i)\cup\bigcup_{\substack{n<\omega\\ j<k_1(i)}} f^{-n}(a^1_{i,j})$ are pairwise nonisomorphic.

Claim 2. *Then* (iii) *or* (iv) *holds*:

(iii) *There are* $\aleph_1$ i *such that there exists* $j(i)$ *such that all the* $f(a^1_{i,j(i)})$ *realize the same type and the sets* $\bigcup_{n<\omega} f^{-n}(f(a^1_{i,j(i)}))$ *are pairwise nonisomorphic.*

(iv) *There are* $\aleph_0$ i *such that there exists* $j(i)$ *such that all the* $f(a^1_{i,j(i)})$ *realize different types.*

Proof of Claim 2. Assume not (iv). Then there is I, $|I|=\aleph_1$, such that the number of types realized in $\{f(a^1_{i,j})\colon j<k_1(i)\}$ is finite for each $i\in I$. In addition we may assume that the number of types realized in $\{f(a^1_{i,j})\colon j<k_1(i),\ i\in I\}$ is finite. Furthermore we may assume that the number of times *each type* is realized in $\{f(a^1_{i,j})\colon j<k_1(i)\}$ is constant for all $i\in I$. So we may assume that there is a certain type q such that the sets $\bigcup\{\bigcup_{n<\omega} f^{-n}(f(a^1_{i,j}))\colon f(a^1_{i,j})$ realizes $q\}$ are pairwise nonisomorphic for all $i\in I$. Thus (iii) follows.

Now we return to the main proof.

Assume (iii) holds. There are then $\aleph_0$ nonisomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(a^1_{i,j(i)})$. Furthermore there is i such that for $\aleph_0$ of the above sets, call them A_l, $l<\omega$, the set $\{j<k(i)\colon \bigcup_{n<\omega} f^{-n}(a^1_{i,j})\cong A_l\}$ is finite or empty. For every $I\subseteq\omega$ construct the model C_I as follows: Start with C_i and form C_I by attaching exactly one (additional) copy of A_l beneath $f(a^1_{i,j(i)})$ iff $l\in I$.

By Lemma 1.2 $C_I\equiv C_i$, and there are $2^{\aleph_0}$ of them which are pairwise nonisomorphic.

Now assume (iv) but not (iii). Notice that all the types realized by the $f(a^1_{i,j(i)})$ in the statement of (iv) are realized in every C_i in $f^{-(n_1-1)}(b_i)$, say by d^i_m, $m<\omega$. Furthermore there is an i such that for $\aleph_0$ of the d^i_m, $m<\omega$, call them d^i_l, the set $\{j<k(i)\colon f(a^1_{i,j})=d^i_l\}$ is finite or empty.

For every $I\subseteq\omega$ we construct the model C_I as follows: Start with C_i and form C_I by attaching behind d^i_l exactly one (additional) set of the form $\bigcup_{n<\omega} f^{-n}(a^1_{i_0,j})$ where i_0 and j are such that $f(a^1_{i_0,j})$ realizes in C_{i_0} the same type as d^i_l in C_i. (If $\{j<k(i)\colon f(a^1_{i,j})=d^i_l\}\neq 0$ then of course we may take $i_0=i$).

Again by Lemma 1.2 $C_I\equiv C_i$, and there are $2^{\aleph_0}$ of them which are pairwise nonisomorphic.

This concludes the proof for Case 1.

Case 2. No $\aleph_1$ of the sets $\bigcup_{0\leqslant n<n_1+1} f^{-n}(b_i)\cup\bigcup_{\substack{n<\omega\\ j<k_1(i)}} f^{-n}(a^1_{i,j})$ are pairwise nonisomorphic.

Thus, there are $\aleph_1$ which are isomorphic. The isomorphism has to break down at some later stage, so there is $n_2>n_1$ such that for $\aleph_1$ i there are $a^2_{i,j}\in C_i$,

$j<k_2(i)\leqslant\omega$, $f^{n_2}(a^2_{i,j})=b_i$, $a^2_{i,j}$ not algebraic over $f(a^2_{i,j})$, $a^2_{i,j}\notin\bigcup_{\substack{n<\omega\\ j<k_1(i)}} f^{-n}(a^1_{i,j})$, and n_2 is minimal (after n_1).

The proof now breaks up into two cases analogous to Case 1 and Case 2, with n_2 in place of n_1.

Continuing in this fashion we either get $\aleph_1$ pairwise nonisomorphic components at some finite level beneath the b_i's, or:

Case ω. There are numbers $0=n_0<n_1<n_2<\ldots<\omega$ and components C_k, $k<\omega$, such that for all $k<\omega$ there are elements $a^k_0,\ldots,a^k_k\in C_k$ satisfying

(i) $f^{n_0}(a^k_0)=f^{n_1}(a^k_1)=\ldots=f^{n_k}(a^k_k)=b_k$ (so $a^k_0=b_k$).

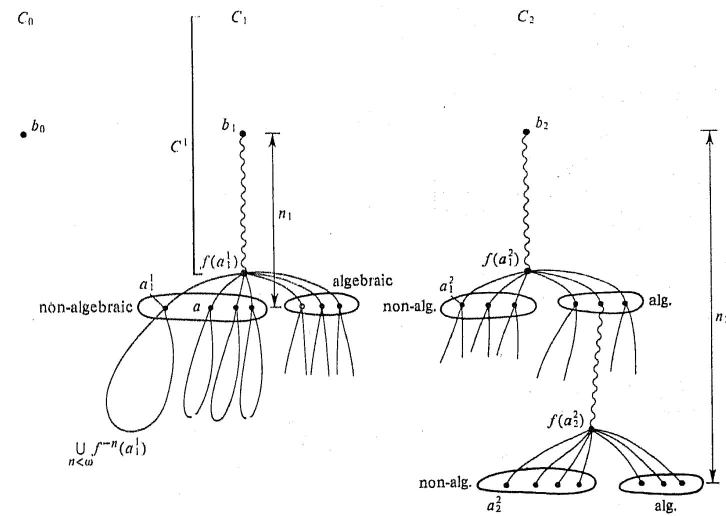
(ii) a^k_j is not algebraic over $f(a^k_j)$, for all $1\leqslant j\leqslant k$.

(iii) $a^k_{j_1}\notin\bigcup_{n<\omega} f^{-n}(a^k_{j_2})$ for all $j_1>j_2>0$.

(iv) The part of C_k above and including $f(a^k_k)$ is elementarily isomorphic to the part of C_l above and including $f(a^l_k)$, for all $k<l$.

(v) $|\{x\in C_k\colon \mathrm{tp}(x,f(a^k_k))=\mathrm{tp}(a^k_k,f(a^k_k))\}|\neq|\{x\in C_{k+1}\colon \mathrm{tp}(x,f(a^{k+1}_k))=\mathrm{tp}(a^k_k,f(a^k_k))\}|$.

See the figure.



Now we define a component $C = \bigcup_{m<\omega} C^m$, $C \equiv C_k$ for all k; $C^m \subset C^{m+1}$. Because of (v) there is $\sigma(k) \in \{k, k+1\}$ such that

$$\{x \in C_{\sigma(k)}\colon \operatorname{tp}(x, f(a_k^{\sigma(k)})) = \operatorname{tp}(a_k^k, f(a_k^k))\}$$

is finite. Let $C^0 = C_1 - \bigcup_{1 \leqslant n<\omega} f^{-n}(f(a_1^1))$. Thus $C^0 \cong C_2 - \bigcup_{1 \leqslant n<\omega} f^{-n}(f(a_1^2))$. Consider C^0 as being $C_{\sigma(1)} - \bigcup_{1 \leqslant n<\omega} f^{-n}(f(a_1^{\sigma(1)}))$. Now we describe what to attach to the bottom of C^0 (i.e., to $f(a_1^{\sigma(1)})$) in order to obtain C^1:

a) For every $x \in C_{\sigma(1)}$ such that $f(x) = f({}_1^{\sigma(1)})$, x is not algebraic over $f(a_1^{\sigma(1)})$, and $\operatorname{tp}(x, f(a_1^{\sigma(1)})) \neq \operatorname{tp}(a_1^{\sigma(1)}, f(a_1^{\sigma(1)}))$, attach $\bigcup_{n<\omega} f^{-n}(x)$.

b) For every $x \in C_2$ such that $f(x) = f(a_1^2)$, x is algebraic over $f(a_1^2)$, and $f(a_2^2) \notin \bigcup_{n<\omega} f^{-n}(x)$, attach $\bigcup_{n<\omega} f^{-n}(x)$.

c) For the unique x such that $f(x) = f(a_1^2)$, x is algebraic over $f(a_1^2)$, and $f(a_2^2) \in \bigcup_{n<\omega} f^{-n}(x)$, attach $\bigcup_{n<\omega} f^{-n}(x) - \bigcup_{n<\omega} f^{-n}(a_2^2)$.

In general, to obtain C^m go down to $f(a_m^{\sigma(m)})$ and discard the finite number of its predecessors which realize over it the same type as $a_m^{\sigma(m)}$. Then continue down C_{m+1}. (Of course it could be in (v) that one of the sets was empty. In that case we may have nothing to discard.)

Take $C = \bigcup_{m<\omega} C^m$.

From considerations like Lemma 4 follows that $C \equiv C_k$ for all k.

Now for each $I \subseteq \omega$ we define $C_I \succ C$:

If $m \in I$ attach directly below $f(a_m^{\sigma(m)})$ in C exactly one copy of the non-algebraic tail we discarded above.

If $k \notin I$ we do not add anything below $f(a_k^{\sigma(k)})$. By Lemma 1.2 $C_I \succ C$, and there are $2^{\aleph_0}$ of them which are not isomorphic.

References

[1] S. Burris, *Scott Sentences and a problem of Vaught for mono-unary algebras*, Fund. Math. 80 (1973), pp. 111–115.

[2] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fund. Math. 49 (1961), pp. 129–141.

[3] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, 1973.

[4] L. Marcus, *Minimal models of theories of one function symbol*, Israel J. Math. 18 (1974), pp. 117–131.

[5] A. Miller, *Vaught's conjecture for one unary operation*, preprint, July 1976.

[6] M. Rubin, *Theories of linear order*, Israel J. Math. 17 (1974), pp. 392–443.

[7] Yu. Shishmarev, *Categorical theories of a function*, Math. Notes 11 (1972), pp. 58–63.

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