Singular properties of Morley rank *

by

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Abstract. Morley rank gives a measure of complexity of definable subsets of the universe of a given first-order structure. Let M be an $\aleph_0$-saturated structure with universe $|M|$. Let $A \subset |M| \times |M|$ be definable, $A_a = \{b \in |M|: \langle a, b\rangle \in A\}$, and $A^0 = \{a \in |M|: \langle a, b\rangle \in A$ for some $b \in |M|\}$. A bound is computed for rank (A) given bounds for rank (A^0) and rank (A_a), $a \in |M|$, and the bound is shown to be best possible.

Related problems are investigated and Morley rank is compared with other ranks in particular with that of Lascar.

In this paper is investigated to what extent the rank of a formula $\psi(x, y)$ may be bounded given bounds on the rank of $\exists y \psi(x, y)$ and $\psi(a, x)$ for every element a of the universe. We suppose that the model we are dealing with is appropriately saturated. The rank in question is that introduced in Morley [5] and for any formula $\varphi(\bar{x})$ its Morley rank and degree are denoted $r(\varphi(\bar{x}))$ and $d(\varphi(\bar{x}))$ respectively. Intuitively the rank of $\varphi(x)$ measures the complexity of the subset A of the universe defined by $\varphi(x)$. Thus we are investigating how the complexity of a definable binary relation B is bounded given bounds on the complexity of A and B_a where $B = \{(a, b): a \in A \;\&\; b \in B_a\}$.

For countable ordinals α, β with $\beta \geqslant 1$ let $\varrho(\alpha, \beta)$ be the least γ such that for every $\aleph_0$-saturated structure M and every formula $\psi(x, y)$ in M

$$r(\exists y \psi(x, y)) = \alpha \;\&\; (\forall a \in M)[r(\psi(a, x)) < \beta]. \rightarrow r(\psi(x, y)) < \gamma .$$

In § 1 we characterize ϱ by the equations:

$$\varrho(0, \beta) = \beta$$

and for $\alpha \geqslant 1$

$$\varrho(\alpha, \beta) = \max(\alpha+\beta, \sup_{\alpha' < \alpha} \{\varrho(\alpha', \beta)\} + (^{-}\beta))$$

where $^{-}\beta$ is $\beta - 1$ for $\beta < \omega$ and β otherwise. For $m, n < \omega$ and $n \geqslant 1$ we have

$$\varrho(m+1, n+1) = (m+2)n+1 .$$

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A closely related question asks how far the rank of $\varphi(x)\wedge\psi(y)$ can be bounded given bounds on the ranks of $\varphi(x)$ and $\psi(x)$. Thus we are asking to what extent the complexity of the cartesian product of two definable sets is bounded by the complexity of those sets. For any ordinals α, β we define $\pi(\alpha, \beta)$ to be the least γ such that for every $\aleph_0$-saturated structure M and all formulas $\varphi(x)$ and $\psi(y)$ in M

$$r(\varphi(x)) = \alpha \;\&\; r(\varphi(y)) = \beta\,. \rightarrow r(\varphi(x)\wedge\psi(y))<\gamma\,.$$

It is immediate that $\pi(\alpha, \beta)\leqslant\varrho(\alpha, \beta+1)$. In § 2 we prove that if $m\geqslant n$,

$$\pi(m+1, n+1)\geqslant m+3+\tfrac{1}{2}(n^2+n)$$

and we establish an upper bound for $\pi(\alpha, \beta+1)$ which is sharper than $\varrho(\alpha, \beta+2)$. However, we are not able to characterize π.

In § 3 we characterize the functions ϱ and π in the context of categorical theories. Here $\pi(\alpha, \beta) = \alpha(+)\beta(+)1$ and $\varrho(\alpha, \beta) = \sup\{\alpha(+)\gamma(+)1\}$ where $\alpha(+)\beta$ denotes the natural sum of α, β.

In § 4 we compare our results with the corresponding properties obtained by Lascar [4, § 5] for a different kind of rank denoted here by u. While Morley's rank may be regarded as being defined primarily on formulas, Lascar's rank is defined primarily on types complete over a set. It is defined, i.e. takes a value $<\infty$, for all complete types when the theory being considered is superstable whereas Morley's rank is defined only for formulas which are $\aleph_0$-stable. Our conclusion here is that Lascar's rank is much better behaved than Morley's because for Lascar's rank the functions corresponding to π and ϱ behave for all theories exactly as π and ϱ behave for categorical theories. Finally we examine the corresponding properties for the notion of degree introduced by Shelah [7] which like Morley's rank is defined primarily on formulas.

Our notation and terminology are mostly standard. Formulas are denoted by Greek letters φ, ψ, χ, etc. We normally display the free variables which occur. If A is a subset of the universe of a structure defined by $\varphi(x)$ we sometimes write $r(A)$ instead of $r(\varphi(x))$ and $x\in A$ for $\varphi(x)$ Let $\mathscr{F}$ be a family of subsets of A then $\mathscr{B}\subset\mathscr{F}$ is said to be a *basis for* $\mathscr{F}$ if the closure of $\mathscr{B}$ under the Boolean operations includes $\mathscr{F}$. If $\mathscr{F}$ is a family of relations on A then $\mathscr{B}\subset\mathscr{F}$ is a *basis for* $\mathscr{F}$ if $\mathscr{F}$ is included in the closure of $\mathscr{B}'$ under the Boolean operations, cartesian product, and permutation of arguments, where $\mathscr{B}'$ consists of $\mathscr{B}$ together with all the relations on A definable in the pure theory of equality.

If a rank function is defined on formulas one can extend it to types by taking as value the least value it takes on any finite conjunction of formulas in the type. Conversely, a rank function defined originally on complete types can be extended to formulas or types by taking as value the least upper bound of its values on complete types containing the formula or extending the type respectively.

In a typescript of an earlier paper I made the false claim that $\pi(\alpha, \beta) = \alpha(+)\beta(+)1$. I am grateful to Daniel Lascar for pointing out my error and

stimulating me to make the investigations recorded here. Lascar independently obtained some results about the function π — the fact that $\pi(\alpha, \beta)\leqslant(\alpha+1)\beta+1$ for example.

1. Characterization of the function ϱ. In this section we shall prove the inequalities which fix the values of the function ϱ. First notice that in considering what may be the maximum rank of $\psi(x, y)$ we may suppose that

$$\models \forall x_1\forall x_2\forall y[\psi(x_1, y)\wedge\psi(x_2, y).\rightarrow x_1 = x_2 \neq y]\,.$$

Such formulas will be called *regular*. It suffices to look at regular formulas, because we may form a new theory in which pairing functions are definable for the field of the formula $\psi(x, y)$ and then $\psi(x, y)$ may be replaced by

$$\exists z[\psi(x, z)\wedge y = \langle x, z\rangle]\,.$$

It is easy to prove:

LEMMA 0. $\varrho(\alpha, 1) = \alpha+1$.

Applying this to $\psi(x, y) = \varphi(y, x)$ we see that if $\varphi(x, y)$ is regular then $r(\varphi(x, y)) = r(\exists y\varphi(y, x))$.

Lower bounds for the values of ϱ are given by:

LEMMA 1. $\varrho(\alpha, \beta)\geqslant\alpha+\beta$.

LEMMA 2. *Let* $1\leqslant a_0\leqslant\alpha_1\leqslant\ldots<\alpha$ *and* $3\leqslant\beta$, *then*

$$\varrho(\alpha, \beta)\geqslant \sup_{i<\omega}\{\varrho(\alpha_i, \beta)\}+^{-}\beta\,.$$

Proof of Lemma 2. We construct a structure M as follows. Choose $\gamma<\beta$, $\gamma\geqslant 2$. Let U be a unary relation such that $U^M = \omega$ and let $|M|-\omega$ be infinite. Let $\{A_i\colon i<\omega\}$ be a partition of $|M|-\omega$ into disjoint infinite sets.

Let Σ denote the set of all finite sequences

$$\langle\gamma_0, n_1, \gamma_1, \ldots, n_i, \gamma_i\rangle$$

such that $i\geqslant 0$, $\gamma = \gamma_0>\gamma_1>\ldots>\gamma_i>0$ and $n_1, \ldots, n_i<\omega$. Choose infinite sets B_σ, $\sigma\in\Sigma$, such that

(i) $B_{\langle\gamma\rangle} = |M|-\omega$,

(ii) for all $\sigma = \langle\gamma_0, n_1, \gamma_1, \ldots, n_i, \gamma_i\rangle\in\Sigma$, if $\gamma_i>1$

$$\{B_\tau\colon\ \sigma\subset\tau\in\Sigma \;\&\; l(\tau) = l(\sigma)+2\}$$

is a partition of B_σ,

(iii) for all $\sigma\in\Sigma$ and all i, $|B_\sigma\cap A_i| = \aleph_0$.

When we complete the definition of M all the sets A_i, B_σ will become definable subsets of the universe. The sets B_σ ensure that the sets A_i have rank $\geqslant\gamma$ in a uniform way. Of course if we were to adjoin only unary relations corresponding to the sets A_i, B_σ then the sets A_i would have rank exactly γ.

Let $R^M = \{\langle i, a\rangle: a \in A_i \& i<\omega\}$ and let $\psi(x, y)$ be the formula $R(x, y)$. For each $\sigma \in \Sigma$ let U_σ be a unary relation symbol and let $U_\sigma^M = B_\sigma$. Define $\Sigma' \subset \Sigma$ by:

$$\Sigma' = \{\sigma: \sigma = \langle\gamma_0, n_1, \gamma_1, \ldots, n_i, \gamma_i\rangle \in \Sigma \text{ and } \gamma_i = 1\}.$$

Notice that the sets B_σ, $\sigma \in \Sigma'$, are just those in the family $\{B_\sigma: \sigma \in \Sigma\}$ which are minimal with respect to inclusion. Let $\{C_i: i<\omega\}$ and $\{D_\sigma: \sigma \in \Sigma'\}$ be partitions of ω into disjoint infinite sets. Let $V_i^M = C_i$. Choose $\delta_i < \varrho(\alpha_i, \beta)$ for each $i<\omega$. For each $j \in \omega$ we choose a possibly infinite sequence $\underline{R}_j^M$ of relations on $|M|$ as follows. Let $j \in D_\sigma$ and choose $\underline{R}_j^M$ such that if

$$M_j = \langle |M_j|, R^M \upharpoonright |M_j|, \underline{R}_j^M \upharpoonright |M_j|\rangle$$

and

$$|M_j| = C_j \cup \bigcup \{A_k \cap B_\sigma: k \in C_j\}$$

then in every elementary extension of M_j: $r(\exists y \psi(x, y)) \leqslant \alpha_j$, $r(\psi(a, x)) < \beta$ for every a, and $r(\psi(x, y)) \geqslant \delta_j$. Further we choose $\underline{R}_j^M$ such that $\underline{R}_j^M = \underline{R}_j^M \upharpoonright |M_j|$ and such that every relation on $|M_j|$ definable in M_j occurs in the sequence $\underline{R}_j^M$. Such choice of $\underline{R}_j^M$ is possible because $\delta_j < \varrho(\alpha_j, \beta)$. Our structure M is now complete; we have

$$M = \langle |M|, R^M, U_\sigma^M, V_i^M, \underline{R}_j^M\rangle_{\sigma \in \Sigma, i<\omega, j<\omega}.$$

Without difficulty one can check that the relations displayed in the definition are a basis for the set of all relations on $|M|$ definable in M.

Consider $k<\omega$. There exists unique j such that $k \in C_j$ and unique $\sigma \in \Sigma'$ such that $j \in D_\sigma$. The relations on $A_k \cap B_\sigma$ definable in M are exactly the same as those definable in M_j. In M_j, $A_k \cap B_\sigma$ has rank $<\beta$ whence the same is true in M. Next observe that $\{(A_k - B_\sigma) \cap B_\tau: \tau \in \Sigma\}$ is a basis for the family of relations on $A_k - B_\sigma$ definable in M. From this it follows that $A_k - B_\sigma$ has rank γ. Thus A_k has rank $<\beta$.

Now consider ω as a subset of $|M|$. Let S_j be the family of subsets of C_j definable in M_j, then

$$\{C_i: i<\omega\} \cup \bigcup \{S_j: j<\omega\}$$

is a basis for the family of subsets of ω definable in M. Reasoning as for $A_k \cap B_\sigma$ above we see that in M, $\varrho(C_j) \leqslant \alpha_j$. Hence $r(\omega) \leqslant \alpha$.

Let $\sigma \in \Sigma'$ and consider the rank of the formula $\psi(x, y) \wedge y \in B_\sigma$ in M. For each j in D_σ, $\psi(x, y)$ has rank $\geqslant \delta_j$ in M_j and hence also in M the formula $\psi(x, y) \wedge x \in C_j \wedge y \in B_\sigma$ has rank $\geqslant \delta_j$. Since D_σ is infinite we have

$$r(\psi(x, y) \wedge y \in B_\sigma) \geqslant \sup\{\delta_j + 1: j \in D_\sigma\}.$$

Without loss of generality we may suppose that $\langle\varrho(\alpha_i, \beta): i<\omega\rangle$ is either a constant sequence or strictly increasing. Taking the first case we choose δ_i such that $\sup\{\delta_i + 1: i<\omega\} = \varrho(\alpha_0, \beta)$; we just need to consider whether $\varrho(\alpha_0, \beta)$ is a limit ordinal or not. When $\langle\varrho(\alpha_i, \beta): i<\omega\rangle$ is strictly increasing, for each i we choose $\delta_{i+1} = \varrho(\alpha_i, \beta)$. Thus we can choose the δ_i such that

$$r(\psi(x, y) \wedge y \in B_\sigma) \geqslant \sup\{\delta_i + 1: i<\omega\} = \sup_{i<\omega} \varrho(\alpha_i, \beta)$$

for each $\sigma \in \Sigma'$.

Now we prove by induction on γ_i that for all $\sigma = \langle\gamma, n_1, \gamma_1, \ldots, n_i, \gamma_i\rangle$ in M

$$r(\psi(x, y) \wedge y \in B_\sigma) \geqslant \sup_{j<\omega} \{\varrho(\alpha_j, \beta)\} + (^-\gamma_i).$$

Thus in M we have $r(\psi(x, y)) \geqslant \sup_{j<\omega} \{\varrho(\alpha_j, \beta)\} + (^-\gamma)$. Since γ was chosen arbitrary $<\beta$ we have

$$\varrho(\alpha, \beta) \geqslant \sup_{i<\omega} \{\varrho(\alpha_i, \beta)\} + (^-\beta)$$

as required.

We now obtain a suitable upper bound for $\varrho(\alpha, \beta)$.

Lemma 3. *Let M be an $\aleph_0$-saturated structure and $\psi(x, y)$ be a formula in M such that $r(\exists y \psi(x, y)) \leqslant \alpha$ and $r(\psi(a, x)) < \beta$ for all a in M. Let $\alpha \geqslant 1$, $\beta \geqslant 2$ and $r(\psi(x, y)) \geqslant \gamma + (^-\beta)$. There exists $\alpha' < \alpha$ such that if $\beta < \omega$ then either $\gamma \leqslant \alpha$ or $\gamma < \varrho(\alpha', \beta)$; and if $\beta \geqslant \omega$ then $\gamma < \varrho(\alpha', \beta)$.*

Proof. We may suppose that $\psi(x, y)$ is regular, whence $r(\exists y \psi(y, x)) = r(\psi(x, y))$. Further, we may suppose that $d(\exists y \psi(x, y)) = 1$. Let Σ be the set of finite sequences of the form

$$\langle\beta_0, n_1, \beta_1, \ldots, n_i, \beta_i\rangle$$

where $\beta = \beta_0 > \beta_1 > \ldots > \beta_i \geqslant 1$ and $n_1, \ldots, n_i < \omega$. Let Σ' denote the set of all σ in Σ maximal with respect to $\subset$. Since $r(\exists y \psi(y, x)) \geqslant \gamma + (^-\beta)$ we can find sets B_σ, $\sigma \in \Sigma$, definable in M such that

(i) $B_{\langle\beta\rangle}$ is the solution set of $\exists y \psi(y, x)$,

(ii) for all $\sigma = \langle\beta_0, n_1, \beta_1, \ldots, n_i, \beta_i\rangle$ with $\beta_i > 1$

$$\{B_\tau: \sigma \subset \tau \in \Sigma \,\&\, l(\tau) = l(\sigma) + 2\}$$

is a family of disjoint subsets of B_σ,

(iii) for all $\sigma = \langle\beta_0, n_1, \beta_1, \ldots, n_i, \beta_i\rangle$, $r(B_\sigma) \geqslant \gamma + (^-\beta_i)$.

Let $M' \succ M$ and $a \in |M'| - |M|$ be such that $M' \models \exists y \psi(a, y)$ and $r(\mathrm{tp}(a, |M|)) = r(\exists y \psi(x, y))$. For each $A \subset |M|$ definable in M let A' denote the corresponding subset of $|M'|$ definable in M'. Let A'_a denote the solution set of $\psi(a, x)$ and for all $b \in M$ let A_b denote the solution set of $\psi(b, x)$ in M.

Let $\beta < \omega$ then $^-\beta = \beta - 1$. If $B'_\sigma \cap A'_a$ is infinite for all $\sigma \in \Sigma'$ then by induction on β_i: for all $\sigma = \langle\beta_0, n_1, \beta_1, \ldots, n_i, \beta_i\rangle \in \Sigma$, $r(B'_\sigma \cap A'_a) \geqslant \beta_i$. Taking $\sigma = \langle\beta\rangle$ we have $r(A'_a) \geqslant \beta$, contradiction. Now fix $\sigma \in \Sigma'$ and $n<\omega$ such that $|B'_\sigma \cap A'_a| = n$. Let

$$C = \{b \in M: |B_\sigma \cap A_b| = n\}.$$

By choice of $tp(a, |M|)$, $r(C) = r(\exists y\psi(x, y))$. Let

$$\alpha' = r(\exists y\psi(x, y) \wedge x \notin C).$$

Since $d(\exists y\psi(x, y)) = 1$ by assumption, $\alpha' < \alpha$. Let D be the solution set in M of $\exists y(\psi(y, x) \wedge y \in C \wedge x \in B_\sigma)$ then $D \subset B_\sigma$.

As remarked above, when $\varphi(x, y)$ is regular, $r(\exists y\varphi(y, x)) = r(\varphi(x, y))$. Applying this to the formula defining D,

$$r(D) = r(\psi(x, y) \wedge x \in C \wedge y \in B_\sigma).$$

By Lemma 0

$$r(\psi(x, y) \wedge x \in C \wedge y \in B_\sigma) = r\big(\exists y(\psi(x, y) \wedge x \in C \wedge y \in B_\sigma)\big) = r(C).$$

Combining the equations established above we have

$$r(D) = r(\exists y\psi(x, y)) \leqslant \alpha.$$

Now $B_\sigma - D$ is included in the solution set of $\exists y(\psi(y, x) \wedge y \notin C)$ whence

$$r(B_\sigma - D) \leqslant r\big(\exists y(\psi(y, x) \wedge y \notin C)\big) < \varrho(\alpha', \beta).$$

Clearly $r(B_\sigma)$ is equal to one of $r(D)$ and $r(B_\sigma - D)$, and $r(B_\sigma) \geqslant \gamma$. Thus we have the conclusion in case $\beta < \omega$.

Now let $\beta \geqslant \omega$ then $^-\beta = \beta$. If $B'_\sigma \cap A'_\alpha$ is nonempty for all $\sigma \in \Sigma'$ then by induction on β_i: for all $\sigma = \langle \beta_0, n_1, \beta_1, \ldots, n_i, \beta_i \rangle \in \Sigma$, $r(B'_\sigma \cap A'_\alpha) \geqslant {}^-\beta_i$. Taking $\sigma = \langle \beta \rangle$ we have a contradiction since $^-\beta = \beta$. The rest of the argument runs as for the case $\beta < \omega$ except that we take $n = 0$ which makes $D = \emptyset$. Thus in this case the only possibility is $\gamma < \varrho(\alpha', \beta)$.

The function ϱ may now be characterized by:

THEOREM 1. *$\varrho(0, \beta) = \beta$ and for $\alpha \geqslant 1$*

$$\varrho(\alpha, \beta) = \max\big(\alpha + \beta, \sup_{\alpha' < \alpha}\{\varrho(\alpha', \beta)\} + (^-\beta)\big).$$

Proof. It is obvious that $\varrho(0, \beta) = \beta$. Let $\alpha \geqslant 1$, $\beta \geqslant 2$ and suppose that $\varrho(\alpha, \beta)$ exceeds the given expression. Let $\gamma = \max(\alpha + 1, \sup_{\alpha' < \alpha}\{\varrho(\alpha', \beta)\})$ if $\beta < \omega$ and be $\sup_{\alpha' < \alpha}\{\varrho(\alpha', \beta)\}$ otherwise. Let $\psi(x, y)$ be a formula witnessing that $\varrho(\alpha, \beta) >$ the given expression then $r(\psi(x, y)) \geqslant \gamma + (^-\beta)$. From Lemma 3 we have a contradiction, whence $\varrho(\alpha, \beta)$ is less than or equal the given expression when $\alpha \geqslant 1$ and $\beta \geqslant 2$. From Lemma 0, $\varrho(\alpha, 1) = \alpha + 1$ and it is easy to check directly that $\varrho(\alpha, 2) \geqslant \alpha + 2$. From Lemmas 1 and 2, $\varrho(\alpha, \beta) \geqslant$ the given expression when $\alpha \geqslant 1$ and $\beta \geqslant 3$. Since the given expression takes the values $\alpha + 1$, $\alpha + 2$ for $\beta = 1, 2$ respectively the proof is complete.

2. The function π. Here our information is incomplete. Clearly $\pi(\alpha, \beta) \leqslant \varrho(\alpha, \beta + 1)$ but in general the inequality is strict. We now give an example which shows that $\pi(\alpha, \beta) \leqslant \alpha(+)\beta(+)1$ is not true in general. This contradicts a previous

claim of the author [2, Lemma 1, 12, p. 162]. Fortunately the only use made of that lemma in [2] was to show that if $\varphi(x)$ and $\psi(y)$ both have finite rank then so also does $\varphi(x) \wedge \psi(y)$. The truth of this assertion is clear from Theorem 1.

Let A denote the set of algebraic numbers. We furnish A^2 with the following functions: the projections p_0, p_1 onto the diagonal and $+$ and $\cdot$ on the diagonal. The theory of the diagonal is the theory of algebraically closed fields which is well-known to be strongly minimal, that is to say, the rank and degree of the universe are both 1. It is easy to see that A^2 has rank 2 and degree 1. We consider elements (x, y) of $B = A^2 \times A^2$. For $n < \omega$ let $B_n \subset B$ denote the solution set of

$$p_0(x) + p_1(x) + p_0(y) + p_1(y) = (n, n)$$

and for $k < \omega$ let $B_{n,k}$ denote the intersection of B_n with the solution set of

$$p_0(x) + 2p_1(x) + 2p_0(y) + p_1(y) = (k, k).$$

There is a unique $F_{n,k}\colon A^2 \to A^2$ such that $(x, y) \in B_{n,k}$ if and only if

$$F_{n,k}(p_0(x), p_1(x)) = (p_0(y), p_1(y)).$$

Now $F_{n,k}$ being 1-1 onto and definable takes definable subsets onto definable subsets of the same rank and degree; similarly for $F_{n,k}^{-1}$. Since the number of definable subsets is $\aleph_0$ we can choose $a_{n,k,i}$ in A^2 for $n, k, i < \omega$ such that they are all distinct and

$$C = \{a_{n,k,i}\colon n, k, i < \omega\} \cup \{F(a_{n,k,i})\colon n, k, i < \omega\}$$

has finite intersection with each definable subset of A^2 of rank 1. Now we define a structure

$$M = \langle |M|, U^M, p_0^N, p_1^M, {+^M}, {\cdot^M}, R^M, R^M_{c,i} \rangle_{c \in C,\ i < \omega}.$$

First let $|M| = A^2 \cup (A^2 \times \omega \times \omega)$, $U^M = A^2$. Next $p_0^M, p_1^M, +^M, \cdot^M$ are defined as before on A^2 and on the rest of the universe are to be trivial. Let

$$R^M = \{\langle a, \langle a, i, j \rangle\rangle\colon a \in A^2, i, j < \omega\}$$

and finally for $c \in C$, $i < \omega$ let

$$R^M_{c,i} = \{\langle c, i, j \rangle\colon j < \omega\}.$$

Let $\varphi(x)$ be $\neg U(x)$. Clearly

$$\{R^M_{c,i}\colon c \in C, i < \omega\} \cup \{S \times \omega \times \omega\colon S \subset A^2, S \text{ definable in } A^2\}$$

is a basis for the family of subsets of A^2 definable in M. If $S \subset A^2$ has rank $\leqslant 1$ in A^2 then $|S \cap C| < \aleph_0$ whence $S \times \omega \times \omega$ has rank 2 in $|M|$. It follows easily that $\varphi(x)$ has rank 3 and degree 1 in M. Now let

$$B'_n = \{\langle\langle x, i, j \rangle, \langle y, k, l \rangle\rangle\colon \langle x, y \rangle \in B_n \text{ and } i, j, k, l < \omega\}$$

and $B'_{n,k}$ be defined similarly from $B_{n,k}$. For any $c = a_{n,k,i}$ it is clear that

$$(\{c\}\times\omega\times\omega)\times(\{F_{n,k}(c)\}\times\omega\times\omega)$$

is a subset of $B'_{n,k}$ of rank 4 in M. Since the $a_{n,k,i}$ are different for different i, $r(B'_{n,k})>5$. Thus $r(B'_n)\geqslant 6$ whence $r(\varphi(x)\wedge\varphi(y))\geqslant 7$. Thus this example shows that $\pi(3,3)>7$.

The example can be generalized as follows. Let $m\geqslant n\geqslant 2$. Form a model M with universe

$$A_0^n\cup A_1^n\cup(A_0^n\times\omega\times\omega)\cup(A_1^n\times\omega\times\omega)$$

where A_0, A_1 are disjoint copies of the algebraic numbers. We include the projection functions on A_0^n, A_1^n and also the functions $+$ and $\cdot$, all these functions being trivial elsewhere. Let $U_i^M = A_i^n$ for $i<2$ and

$$R^M = \{\langle a, i, j\rangle:\ a\in A_0^n\cup A_1^n,\ i,j<\omega\}.$$

As in the case $m = n = 2$ we can easily construct nested definable subsets of $A_0^n\times A_1^n$ of depth $n+1$ such that the innermost ones have the form $\{c\} = \{(c_0, c_1)\}$ where $c_i\in A_i^n$, and such that if C_i denotes the set of all c_i then $C_i\cap S$ is finite for each $S\subset A_i^n$ which is definable and of rank $\geqslant n-1$. Now for each $c = (c_0, c_1)$ we adjoin to M relations on $(\{c_0\}\times\omega\times\omega)\times(\{c_1\}\times\omega\times\omega)$ such that this set has rank $\pi(m,n)-1$ while $\{c_0\}\times\omega\times\omega$ and $\{c_1\}\times\omega\times\omega$ have ranks m, n respectively. Taking $\varphi_i(x)$, $i<2$, to be the formula whose solution set is $A_i^n\times\omega\times\omega$ we have $r(\varphi_0(x)) = m+1$, $r(\varphi_1(x)) = n+1$, and $r(\varphi_0(x)\wedge\varphi_1(x))\geqslant\pi(m,n)+n$. Thus we have:

LEMMA 4. *If $m\geqslant n>0$ then $\pi(m+1, n+1)>\pi(m,n)+n+1$.*

The best result we have in the other direction is:

LEMMA 5. *For $\alpha,\beta>1$*

$$\pi(\alpha,\beta+1)\leqslant\max\{\varrho(\alpha,\beta+1),\ \sup_{\alpha'<\alpha}\{\pi(\alpha',\beta+1)+1\}\}$$

Proof. Let $\varphi(x)$, $\psi(x)$ be formulas in a structure M having ranks α, $\beta+1$ respectively and both of degree 1. Let $\theta_0(x,y)$, $\theta_1(x,y)$ be formulas in M which partition $\varphi(x)\wedge\psi(y)$. Let $a\in|\overline{M}|-|M|$, $\models\varphi(a)$ and $r(\mathrm{tp}(a,|M|)) = \alpha$. Since $d(\psi(x)) = 1$ and θ_0, θ_1 are disjoint one of the formulas $\theta_0(a,x)$, $\theta_1(a,x)$ has rank $<\beta+1$. Denote this formula by $\theta(a,x)$. From [2, Lemma 1.8, p. 160] there is a formula $\chi(x)$ in M implying $\varphi(x)$ such that for all b in $\overline{M}$

$$[\models\chi(b)]\equiv[\models\varphi(b)\ \text{and}\ r(\theta(b,x))\leqslant\beta].$$

Since $\models\chi(a)$ we have $r(\chi(x)) = \alpha$ and $r(\varphi(x)\wedge\neg\chi(x))<\alpha$. Observe that

$$r(\chi(x)\wedge\theta(x,y))<\varrho(\alpha,\beta+1)$$

and

$$r(\varphi(x)\wedge\neg\chi(x)\wedge\theta(x,y))\leqslant\pi(\alpha',\beta+1)$$

for some $\alpha'<\alpha$. Since $\chi(x)\wedge\theta(x,y)$, $\varphi(x)\wedge\neg\chi(x)\wedge\theta(x,y)$ partition $\theta(x,y)$, for some $\alpha'<\alpha$ we have

$$r(\theta(x,y))<\max(\varrho(\alpha,\beta+1),\ \pi(\alpha',\beta+1)+1),$$

which proves the lemma.

From Lemma 4 we deduce by induction that

$$\pi(m+1, n+1)\geqslant m+3+\tfrac{1}{2}(n^2+n)$$

and conjecture that equality holds. From Lemma 5, in the opposite direction we have

$$\pi(m+1, n+1)\leqslant\varrho(m+1, n+1) = (m+2)n+1.$$

3. The functions π and ϱ for a categorical theory. Throughout this section we assume that the values of α and β are limited to whatever values are sensible for the kind of theory concerned. In the case of $\aleph_1$-categorical theories rank can take only finite values thus π is allowed only finite arguments and $\varrho(\alpha,\beta)$ is considered for $\alpha<\omega$ and $\beta\leqslant\omega$. In the case of $\aleph_0$-categorical theories $\pi(\alpha,\beta)$ is considered for $\alpha,\beta<\lambda$ and $\varrho(\alpha,\beta)$ for $\alpha<\lambda$ and $\beta\leqslant\lambda$, where λ is the least ordinal which is not the rank of a formula in any $\aleph_0$-categorical theory. Morley [5] conjectured that $\lambda = \aleph_0$ but the value is still unknown.

Let $\varrho'(\alpha,\beta)$ denote the least γ such that for every $\aleph_0$-saturated structure M and every formula $\psi(x,y)$ in M

$$r(\exists y\psi(c,y))\leqslant\alpha\ \&\ (\forall a\in M)[r(\psi(a,x))\leqslant\beta].\rightarrow r(\psi(x,y))\leqslant\gamma.$$

We shall show that for a categorical theory

$$\varrho'(\alpha,\beta)(+)1 = \pi(\alpha,\beta) = \alpha(+)\beta(+)1$$

and

$$\varrho(\alpha,\beta) = \sup_{\gamma<\beta}\{\alpha(+)\gamma(+)1\}.$$

LEMMA 6. *Let $\varphi(\bar{x},\bar{y})$ be a formula in a model M of an $\aleph_1$-categorical theory then*

(i) *$\{\langle r(\varphi(\bar{x},\bar{b})), d(\varphi(\bar{x},\bar{b}))\rangle:\ \bar{b}\in M\}$ is finite and $\subset\omega^2$,*

(ii) *for all $m,n<\omega$ there exists $\chi(\bar{y})$ in M such that for each $\bar{b}$ in any elementary extension of M*

$$[(\varphi(\bar{x},\bar{b})) = m\ \&\ d(\varphi(\bar{x},\bar{b})) = n]\equiv[\models\chi(\bar{b})],$$

(iii) *there exists a formula $\theta(\bar{x},\bar{y},\bar{z})$ in M and $k<\omega$ such that if $r(\varphi(\bar{x},\bar{b}))>0$ there exists an indiscernible sequence $\bar{c}_0, \bar{c}_1, \ldots$ such that the formulas $\theta(\bar{x},\bar{b},\bar{c}_0)$ $\theta(\bar{x},\bar{b},\bar{c}_1), \ldots$ are k-almost disjoint and*

$$r(\theta(\bar{x},\bar{b},\bar{c}_j)\wedge\varphi(\bar{x},\bar{b})) = r(\varphi(\bar{x},\bar{b}))-1.$$

This lemma is not stated explicitly but follows easily from the facts assembled to prove the main theorem of [1].

LEMMA 7. *Let $\psi(x, y)$ be a formula in a model M of a categorical theory. For each α there exists a formula $\theta(y)$ in M such that for all $b \in \overline{M}$*

$$r(\psi(x, b)) < \alpha \equiv \models \theta(b).$$

Proof. This is immediate from (ii) of the last lemma for $\aleph_1$-categorical theories. For $\aleph_0$-categorical theories the result is immediate from the fact that there are only a finite number of 1-types over the finite set of parameters occurring in $\psi(x, t)$.

LEMMA 8. *In a categorical theory all $\alpha, \beta \geqslant 1$*

$$\varrho'(\alpha, \beta) \leqslant \max\big(\sup_{\gamma<\alpha}\{\varrho'(\gamma, \beta)+1\}, \sup_{\gamma<\beta}\{\varrho'(\alpha, \gamma)+1\}\big).$$

Proof. We follow the same line as in the proof of Lemma 5. Thus let $\psi(x, y)$ be a formula in a model M of a categorical theory satisfying:

$$r(\exists y \psi(x, y)) \leqslant \alpha \quad \text{and} \quad r(\psi(a, x)) \leqslant \beta$$

for all a in every elementary extension of M. Without loss of generality let $d(\exists y \psi(x, y)) = 1$. Fix a in an elementary extension M' of M such that $r(\mathrm{tp}(a, |M|)) = \alpha$ and $\models \exists y \psi(a, y)$.

Suppose $r(\psi(a, x)) < \beta$ then from the last lemma there exists a formula θ in M such that for all a' in any elementary extension of M

$$r(\psi(\alpha', x)) < \beta \equiv \models \theta(a').$$

Now let $\psi'(x, y)$ denote $\theta(x) \wedge \psi(x, y)$ and $\psi''(x, y)$ denote $\neg\theta(x) \wedge \psi(x, y)$. For all a' we have $r(\psi'(a', x)) < \beta$ whence there exists $\gamma < \beta$ such that for all a', $r(\psi'(a', x)) \leqslant \gamma$ either because rank is finite in the case of $\aleph_1$-categoricity or because there are only finitely many 1-types over a finite set in the case of $\aleph_0$-categoricity. Also since $(\mathrm{tp}(a, |M|)) = \alpha$, $\models \exists y \psi(a, y)$, and $d(\exists y \psi(x, y)) = 1$ we have $r(\exists y \psi''(x, y)) < \alpha$. Since $\psi(x, y)$ is equivalent to $\psi'(x, y) \wedge \psi''(x, y)$, $r(\psi(x, y)) = \max\big(r(\psi'(x, y)), r(\psi''(x, y))\big)$. Thus in case $r(\psi(a, x)) < \beta$ then

$$r(\psi(x, y)) < \max\big(\sup_{\gamma<\beta}\{\varrho'(\alpha, \gamma)+1\}, \sup_{\gamma<\alpha}\{\varrho'(\gamma, \beta)+1\}\big).$$

Now suppose $r(\psi(a, x)) = \beta$ and $d(\psi(a, x)) = n$. Let $\psi_i(x, y)$, $i \leqslant n$, be formulas in M constituting a partition of $\psi(x, y)$. We can choose $j \leqslant n$ such that $r(\psi_j(a, x)) < \beta$. Reasoning exactly as for $\psi(x, y)$ above we have

$$r(\psi_j(x, y)) < \max\big(\sup_{\gamma<\beta}\{\varrho'(\alpha, \gamma)+1\}, \sup_{\gamma<\alpha}\{\varrho'(\gamma, \beta)+1\}\big)$$

which gives

$$r(\psi(x, y)) \leqslant \max\big(\sup_{\gamma<\beta}\{\varrho'(\alpha, \gamma)+1\}, \sup_{\gamma<\alpha}\{\varrho'(\gamma, \beta)+1\}\big).$$

Thus in either case we have the desired conclusion

THEOREM 2. *For a categorical theory*

$$\varrho'(\alpha, \beta)(+)1 = \pi(\alpha, \beta) = \alpha(+)\beta(+)1$$

and $\varrho(\alpha, \beta) = \sup_{\gamma<\beta}\{\varrho'(\alpha, \gamma)+1\}$.

Proof. It is immediate that $\pi(\alpha, \beta) \leqslant \varrho'(\alpha, \beta)(+)1$. By a straightforward induction $\alpha(+)\beta(+)1 \leqslant \pi(\alpha, \beta)$, see Wierzejewski [10]. From Lemma 8 we see by induction that $\varrho'(\alpha, \beta) \leqslant \alpha(+)\beta$. This proves the first part. Note that $\varrho'(\alpha, \beta) = \alpha(+)\beta$. Let $\psi(x, y)$ be a formula such that $r(\exists y \psi(x, y)) = \alpha$ and $r(\psi(a, x)) \leqslant \gamma$ for all a in some $\aleph_0$-saturated model. Since

$$r(\psi(x, y)) \leqslant \varrho'(\alpha, \gamma) < \alpha(+)\gamma(+)1$$

we have $\varrho(\alpha, \beta) \leqslant \sup_{\gamma<\beta}\{\alpha(+)\gamma(+)1\}$. On the other hand if there are formulas $\chi(x)$ and $\theta(x)$ with ranks α, γ respectively then $r(\chi(x) \wedge \theta(y)) = \alpha(+)\gamma$ — this is the same observation of Wierzejewski just mentioned. Thus $\rho(\alpha, \beta) \geqslant \sup_{\gamma<\beta}\{\alpha(+)\gamma(+)1\}$ which completes the proof of the second part.

4. Comparison with other ranks. In [4, § 5] Lascar employed a rank defined on types complete over some subset of a model. From now on we suppose that the theory under consideration is superstable. For all $n < \omega$, for all A, and all $p \in S_n(A)$ we define $u(p) \in On$. The key feature of the definition is that if $p \in S^n(A)$, $q \in S^n(B)$, $A \subset B$, and $p \subset q$ then

$$u(q) < u(p) \equiv q \text{ forks over } A.$$

Subject to this stipulation u is to be least possible. The notion of forking is due to Shelah and an extensive treatment of it will appear in [8]. Here we rely on the account given by Lascar [3].

In order to compare Lascar's rank with Morley's it is convenient to identify types with the corresponding infinite conjunctions. Thus when it is convenient we regard types as formulas in this extended sense. If $p(x, y)$ is a 2-type, by $\exists y p(x, y)$ we denote the 1-type

$$\{\exists y \textstyle\bigwedge p_0(x, y) \colon p_0 \subset p \;\&\; |p_0| < \aleph_0\}.$$

Note that if p is complete over A then so is $\exists y p(x, y)$. In [4, Thm. 8, p. 82] Lascar showed that the rank u is well behaved with respect to existential quantification.

THEOREM 3 (Lascar). *Let $p(x, y) \in S^2(A)$ be realized by $\langle a, b\rangle$. Let $u(\exists y p(x, y)) = \alpha$, $u(p(a, x)) = \beta$, then*

$$\beta + \alpha \leqslant u(p(x, y)) \leqslant \alpha(+)\beta.$$

Following Lascar let us call a subset A of a model *independent over B* if for all $a \in A$

$$u(\mathrm{tp}(a, B \cup (A - \{a\}))) = u(\mathrm{tp}(a, B)).$$

From [3, Propositions 2, 3, pp. 56, 57] it follows that this notion is unchanged for an $\aleph_0$-stable theory if we write r for u. From Theorem 3 Lascar deduced that for any $\bar{c}$ there exists $m<\omega$ depending only on $\mathrm{tp}(\bar{c}, B)$ such that if A is independent over B and for each $a \in A$, $\{a\}$ is not independent over $B \cup \mathrm{Rng}\,\bar{c}$, then $|A|<m$. This answers a question raised in [2, p. 166], because it shows that the upper dimension of a model is finite if its lower dimension is finite. As a corollary Lascar obtained an elegant proof that a countable superstable theory has $\geqslant \aleph_0$ countable models if it is not $\aleph_0$-categorical. Thus even for the investigation of $\aleph_0$-stable theories Lascar's rank is more powerful than Morley's.

The closest relative to Lascar's rank amongst ranks defined on formulas is Shelah's notion of degree, not to be confused with degree in Morley's sense. (This notion first appeared in [6] under the name rank and with a different definition.) The degree of a formula φ is denoted $s(\varphi)$, see [7] or [8] for a precise definition. The key clause says that $s(\varphi(x)) \geqslant \beta+1$ if and only if there is a formula $\psi(x, \bar{y})$ and sequences $\bar{c}_i$, $i<|T|^+$, such that:

(1) $s[\varphi(x) \wedge \psi(x, \bar{c}_i)] \geqslant \beta$ for all i,

(2) the $\psi(x, \bar{c}_i)$'s are almost contradictory.

The relationship between s and u is given by:

Lemma 9. *Let the parameters of $\varphi(x)$ be in A. Then $s(\varphi(x)) \geqslant n$ if and only if there exists $p \in S^1(A)$ such that $u(p) \geqslant n$ and $\varphi(x) \in p$. The "if" part holds for arbitrary $\alpha \in On$ in place of n.*

Proof. For induction suppose the result holds for n. First let $s(\varphi(x)) \geqslant n+1$. As Shelah noticed in [7], in the above definition we can suppose that $\{\bar{c}_i\colon i<|T|^+\}$ is an indiscernible set, and further that it is indiscernible over A. Thus in the present case we have $s[\varphi(x) \wedge \psi(x, \bar{c}_i)] \geqslant n$. By the induction hypothesis there exist $B \supset A$ and $q \in S^1(B)$ such that $u(q) \geqslant n$ and $\varphi(x) \wedge \psi(x, \bar{c}_0) \in q$. Clearly q forks over A, whence $u(q \restriction A) > u(q)$ and we may set $p = q \restriction A$.

For the other direction suppose that $p \in S^1(A)$, $u(p) \geqslant n+1$, and $\varphi(x) \in p$. Then there exists $B \supset A$ and $q \in S^1(B)$ such that $q \supset p$, q forks over A, and $u(q) \geqslant n$. Further using [3, Theorem 10, p. 41] we may suppose that there is a formula $\psi(x, \bar{y})$ and a set $\{\bar{c}_i\colon i<\omega\}$ indiscernible over A such that $\psi(x, c_0) \in q$ and the formulas $\psi(x, c_i)$ are almost contradictory. Now $s(\varphi(x) \wedge \psi(x, \bar{c}_i)) \geqslant n$ by the induction hypothesis whence $s(\varphi(x)) \geqslant n+1$.

This lemma is best possible in the following sense. For any $\alpha<\omega_1$ there exists a countable superstable theory for which $s(x=x)=\alpha$ and yet for every A and $p \in S^1(A)$, $u(p) \leqslant \omega$. Suppose T is the theory for α and R be a new binary relation symbol. Let the axioms of T' say that R is an equivalence relation with infinitely many equivalence classes, that T' restricted to any one of the equivalence classes is T, and that no relationships hold between elements of different equivalence classes. Then T' is good for $\alpha+1$. Further, if $T_0, T_1, \ldots$ are theories corresponding to $\alpha_0, \alpha_1, \ldots$ then their disjoint union corresponds to $\sup\limits_{i<\omega} \alpha_i$.

It turns out that if $s(\exists y \psi(x, y)) \leqslant \alpha$ and $s(\psi(a, x)) \leqslant n$ for all $a \in \overline{M}$ then $s(\psi(x, y)) \leqslant \alpha+n$. This follows immediately from Theorem 3 and Lemma 9 when $\alpha<\omega$. Otherwise one has to make a direct proof. If $s(\psi(a, x))$ is not bounded by a natural number then we can place no bound on $s(\psi(x, y))$ given bounds on $s(\exists y \psi(x, y))$ and $s(\psi(a, x))$. To see this we use the same idea as in the last paragraph.

In an $\aleph_1$-categorical theory the ranks r, u, and s are all the same as may be seen from Lemma 6. Lascar proved that r and u are the same for an $\aleph_0$-categorical theory, see [3, Corollary 10, p. 79]. It follows that for an $\aleph_0$-categorical theory r and s are the same. For suppose $r(\varphi(x)) = \alpha$ then there exist A and $p \in S^1(A)$ such that $r(p) = \alpha$ and $\varphi(x) \in p$. By Lascar's result $u(p) = \alpha$, whence $s(\varphi(x)) \geqslant \alpha$ by Lemma 9. Thus $s(\varphi(x)) \geqslant r(\varphi(x))$. On the other hand it is easy to show that $r(\varphi(x)) \geqslant s(\varphi(x))$ for every theory $\aleph_0$-categorical or not. Thus r, u, and s are all the same for any categorical theory.

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