A. H. Lachlan, Singular properties of Morley rank
J. Siegel, Czech extensions and localization of homotopy functors
J. Marcus, The number of countable models of a theory of one unary function
A. Baüstreich, The theory of abelian $p$-groups with the quantifier $J$ is decidable
G. V. Cox, Luzin properties in the product space $S^{k}$
K. Nagami, Dimension of free $L$-spaces

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Singular properties of Morley rank

by

A. H. Lachlan (Burnaby, B. C.)

Abstract. Morley rank gives a measure of complexity of definable subsets of the universe of a given first-order structure. Let $M$ be an $\mathfrak{A}$-saturated structure with universe $|M|$. Let $A \subset |M| \times |M|$ be definable, $A_{a} = \{ b \in |M| : \langle a, b \rangle \in A \}$, and $A_{a}^{*} = \{ a \in |M| : \langle a, b \rangle \in A$ for some $b \in |M| \}$. A bound is computed for rank $\lambda$, given bounds for rank $\lambda^{0}$, and rank $\lambda_{a}$, $a \in |M|$, and the bound is shown to be best possible.

Related problems are investigated and Morley rank is compared with other ranks in particular with that of Lascar.

In this paper is investigated to what extent the rank of a formula $\psi(x, y)$ may be bounded given bounds on the rank of $\exists y \psi(x, y)$ and $\psi(a, y)$ for every element $a$ of the universe. We suppose that the model we are dealing with is appropriately saturated. The rank in question is that introduced in Morley [5] and for any formula $\phi(x)$ its Morley rank and degree are denoted $r(\phi(x))$ and $d(\phi(x))$ respectively. Intuitively the rank of $\phi(a)$ measures the complexity of the subset $A$ of the universe defined by $\phi(a)$. Thus we are investigating how the complexity of a definable binary relation $B$ is bounded given bounds on the complexity of $A$ and $B$, where

$$B = \{ (a, b) : a \in A \land b \in B(a) \}.$$

For countable ordinals $\alpha, \beta$ with $\beta \geq 1$ let $\phi(a, \beta)$ be the least $\gamma$ such that for every $\mathfrak{A}$-saturated structure $M$ and every formula $\psi(x, y)$ in $M$

$$r(\exists \psi(x, y)) = r(a, \beta) \land (\forall a \in M) [r(\psi(x, y)) < \beta] \rightarrow r(\psi(x, y)) < \gamma.$$

In § 1 we characterize $\phi$ by the equations:

$$\phi(0, \beta) = \beta$$

and for $\alpha \geq 1$

$$\phi(a, \beta) = \max(a + \beta, \sup_{\sigma < a} \phi(\sigma, \beta)),$$

where $\beta$ is $\beta - 1$ for $\beta < \omega$ and $\beta$ otherwise. For $m, n < \omega$ and $n \geq 1$ we have

$$\phi(m + 1, n + 1) = (m + 2) \cdot n + 1.$$

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1 — FUNDAMENTA MATHEMATICAE CVIII

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A closely related question asks how far the rank of \( \varphi(x) \land \psi(y) \) can be bounded given bounds on the ranks of \( \varphi(x) \) and \( \psi(y) \). Thus we are asking to what extent the complexity of the cartesian product of two definable sets is bounded by the complexity of those sets. For any ordinals \( \alpha, \beta \) we define \( \pi(\alpha, \beta) \) to be the least \( \gamma \) such that for every \( \kappa \)-saturated structure \( M \) and all formulas \( \varphi(x) \) and \( \psi(y) \) in \( M \)

\[
\pi(\alpha, \beta) = \alpha \& \pi(\varphi(x)) = \beta \Rightarrow \pi(\varphi(x) \land \psi(y)) < \gamma.
\]

It is immediate that \( \pi(\alpha, \beta) \leq \pi(\alpha, \beta + 1) \). In \( \S \ 2 \) we prove that if \( m \geq n \)

\[
\pi(m + 1, n + 1) \geq m + 1 + \frac{1}{m + 1 - n}
\]

and we establish an upper bound for \( \pi(\alpha, \beta + 1) \) which is sharper than \( \pi(\alpha, \beta + 2) \).

However, we are not able to characterize \( \pi \).

In \( \S \ 3 \) we characterize the functions \( \varphi \) and \( \pi \) in the context of categorical theories. Here \( \pi(\alpha, \beta) = \alpha(+)\beta(+)1 \) and \( \varphi(\alpha, \beta) = \sup \{ \alpha(+)\beta(+)1 \} \) where \( \alpha(+) \beta \) denotes the natural sum of \( \alpha, \beta \).

In \( \S \ 4 \) we compare our results with the corresponding properties obtained by Lascar [4, \( \S \ 5 \)] for a different kind of rank denoted here by \( \beta \). While Morley's rank may be regarded as being defined primarily on formulas, Lascar's rank is defined primarily on types complete over a set. It is defined, i.e. takes a value \( < \omega \), for all complete types when the theory being considered is superstable whereas Morley's rank is defined only for formulas which are \( \kappa \)-stable. Our conclusion here is that Lascar's rank is much better behaved than Morley's because for Lascar's rank the functions corresponding to \( \xi \) and \( \eta \) behave for all theories exactly as \( \pi \) and \( \varphi \) behave for categorical theories. Finally we examine the corresponding properties for the notion of degree introduced by Shahal [7] which like Morley's rank is defined primarily on formulas.

Our notation and terminology are mostly standard. Formulas are denoted by Greek letters \( \phi, \psi, \chi \), etc. We normally display the free variables which occur. If \( A \) is a subset of the universe of a structure defined by \( \varphi(x) \) we sometimes write \( r(A) \) instead of \( r(\varphi(x)) \) and \( x \in A \) for \( \varphi(x) \). Let \( \mathcal{F} \) be a family of subsets of \( A \) then \( \mathcal{F} \subset \mathcal{F} \) is said to be a basis for \( \mathcal{F} \) if the closure of \( \mathcal{F} \) under the Boolean operations includes \( \mathcal{F} \). If \( \mathcal{F} \) is a family of relations on \( A \) then \( \mathcal{F} \subset \mathcal{F} \) is a basis for \( \mathcal{F} \) if \( \mathcal{F} \) is included in the closure of \( \mathcal{F} \) under the Boolean operations, cartesian product, and permutation of arguments, where \( \mathcal{F} \) consists of \( \mathcal{F} \) together with all the relations on \( A \) definable in the pure theory of equality.

If a rank function is defined on formulas one can extend it to types by taking as the least value it takes on any finite conjunction of formulas in the type. Conversely, a rank function defined originally on complete types can be extended to formulas or types by taking as the least upper bound of its values on complete types containing the formula or extending the type respectively.

In a typescript of an earlier paper I made the false claim that \( \pi(\alpha, \beta) = \alpha(+)\beta(+)1 \). I am grateful to Daniel Lascar for pointing out my error and stimulating me to make the investigations recorded here. Lascar independently obtained some results about the function \( \xi \) — the fact that \( \pi(\alpha, \beta) \leq \pi(\alpha + 1, \beta + 1) \) for example.

1. Characterization of the function \( \varphi \). In this section we shall prove the inequalities which fix the values of the function \( \varphi \). First notice that in considering what may be the maximum rank of \( \psi(x, y) \) we may suppose that

\[
\forall x_1, \forall x_2, \forall y [\psi(x_1, y) \land \psi(x_2, y) \Rightarrow x_1 \neq x_2] .
\]

Such formulas will be called regular. It suffices to look at regular formulas, because we may form a new theory in which pairing functions are definable for the field of the formula \( \psi(x, y) \) and then \( \psi(x, y) \) may be replaced by

\[
\exists z [\psi(x, z) \land y = \langle x, z \rangle] .
\]

It is easy to prove:

**Lemma 0.** \( \varphi(\alpha, 1) = \alpha + 1 \).

**Applying this to** \( \psi(x, y) = \varphi(x, y) \) **we see that if** \( \varphi(x, y) = \varphi(\alpha, \beta) \).

**Lower bounds for the values of** \( \varphi \) **are given by:**

**Lemma 1.** \( \varphi(\alpha, \beta) \geq \alpha + \beta \).

**Lemma 2.** Let \( \leq \alpha \leq \beta \leq \alpha \), \( \leq \beta \leq \beta \leq 3 \leq \beta \), then

\[
\varphi(\alpha, \beta) \geq \sup \{ \varphi(\alpha i, \beta) \} + \beta .
\]

**Proof of Lemma 2.** We construct a structure \( M \) as follows. Choose \( \gamma < \beta \), \( \gamma \geq 2 \).

**Let** \( U \) **be a unary relation such that** \( U^M = \omega \) **and let** \( |M| - \omega \) **be infinite. Let** \( \{ A_i : i < \omega \} \) **be a partition of** \( |M| - \omega \) **into disjoint infinite sets.**

**Let** \( \Sigma \) **denote the set of all finite sequences**

\[
\langle \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma \rangle
\]

**such that** \( i > 0, \gamma = \gamma_0 > \gamma_1 > \gamma_2 > \gamma_3 > \gamma_4 > \gamma_5 > 0 \) **and** \( n_0, \ldots, n_i < \omega \). Choose infinite sets \( B_{\sigma} \)

**such that**

(i) \( |B_{\sigma}| = |M| - \omega \),

(ii) for all \( \sigma = \langle \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_i, \gamma \rangle \in \Sigma \), \( \gamma > 1 \)

\( \{ R_{\beta} : \sigma \in \Sigma & \leq l(\gamma) = 1(\sigma, \gamma) + 2 \}

**is a partition of** \( B_{\sigma} \),

(iii) for all \( \sigma \in \Sigma \) and all \( \beta \in [R_{\beta}, \leq \beta] = K_0 \),

When we complete the definition of \( M \) all the sets \( A_i, B_{\sigma} \) will become definable subsets of the universe. The sets \( B_{\sigma} \) ensure that the sets \( A_i \) have rank \( \gamma \) in a uniform way. Of course if we were to adjoin only unary relations corresponding to the sets \( A_i, B_{\sigma} \) then the sets \( A_i \) would have rank exactly \( \gamma \).
Let $R^M = \{ \langle i, \phi \rangle : \phi \in A_i \& i<\alpha \}$ and let $\psi(x, y)$ be the formula $R(x, y)$. For each $\sigma \in \Sigma$ let $U^M_\sigma$ be a unary relation symbol and let $U^M_{\Sigma} = B_\Sigma$. Define $\Sigma \subseteq \Sigma^*$ by:

$$\Sigma^* = \{ \sigma : \sigma = \langle \gamma_0, n_1, \gamma_1, \ldots, n_i, \gamma_i \rangle \in \Sigma \& \gamma_i = 1 \}.$$ 

Notice that the sets $B_\Sigma, \sigma \in \Sigma^*$, are just those in the family $\{ B_\sigma : \sigma \in \Sigma \}$ which are minimal with respect to inclusion. Let $\{ C_i : i<\omega \}$ and $\{ D_i : \sigma \in \Sigma^* \}$ be partitions of $\omega$ into disjoint infinite sets. Let $r^M = C_1$. Choose $\delta_i < \psi(a_i, \beta)$ for each $i < \omega$. For each $j \in \omega$ we choose a possibly infinite sequence $B^M_j$ of relations on $|M|$ as follows. Let $j \in D_\omega$ and choose $B^M_j$ such that if

$$M_j = \langle |M|, R^M \upharpoonright |M|, B^M_j \upharpoonright |M| \rangle$$

and

$$|M_j| = C_j \cup \bigcup \{ A_k \cap B_k : k \in C_j \}$$

then in every elementary extension of $M_j$:

$$r(\exists \psi(x, y)) < a_j, r(\psi(x, a)) < \beta$$

for every $a$, and $r(\psi(x, y)) \geq \delta_j$. Further we choose $B^M_j$ such that $B^M_j = B^M_j \upharpoonright |M|$ and such that every relation on $|M_j|$ definable in $M_j$ occurs in the sequence $B^M_j$. Such choice of $B^M_j$ is possible because $\delta_j < \psi(a_i, \beta)$. Our structure $M$ is now complete; we have

$$M = \langle |M|, R^M, U^M_{\Sigma^*}, V^M_{\Sigma^*}, B^M_j \rangle_{j \in \omega, i \in \omega, \sigma \in \Sigma}.$$ 

Without difficulty one can check that the relations displayed in the definition are a basis for the set of all relations on $|M|$ definable in $M$.

Consider $k<\omega$. There exists unique $j$ such that $k \in C_j$ and unique $\sigma \in \Sigma^*$ such that $j \in D_\sigma$. The relations on $A_k \cap B_k$ definable in $M$ are exactly the same as those definable in $M_j$. In $M_j, A_k \cap B_k$ is $< \beta$ whence the same is true in $M$. Next observe that $\{ A_k \cap B_k : \sigma \in \Sigma \}$ is a basis for the family of relations on $A_k \cap B_k$ definable in $M$. From this it follows that $A_k \cap B_k$ has rank $\beta$. Thus $A_k$ has rank $< \beta$.

Now consider $\omega$ as a subset of $|M|$. Let $S_j$ be the family of subsets of $C_j$ definable in $M_j$, then

$$\{ C_i : i<\omega \} \cup \bigcup (S_i : j < \omega)$$

is a basis for the family of subsets of $C_j$ definable in $M$. Reasoning as for $A_k \cap B_k$ above we see that in $M, \phi(C_j) < a_j$. Hence $r(\phi) < a_j$.

Let $\sigma \in \Sigma^*$ and consider the rank of the formula $\psi(x, y) \land x \in B_\Sigma$ in $M$. For each $j \in D_\sigma$, $\psi(x, y)$ has rank $\geq \delta_j$ in $M_j$ and hence also in $M$ the formula $\psi(x, y) \land x \in C_j \land y \in B_\Sigma$ has rank $\geq \delta_j$. Since $\Sigma$ is infinite we have

$$r(\psi(x, y) \land y \in B_\Sigma) \geq \sup \{ \delta_j + 1 : j \in D_\sigma \}.$$ 

Without loss of generality we may suppose that $\phi(a_i, \beta) : i < \omega$ is either a constant sequence or strictly increasing. Taking the first case we choose $\delta_i$ such that

$$\sup \{ \delta_i + 1 : i \in \omega \} = \sup \{ \phi(a_i, \beta) : i < \omega \}.$$ 

We just need to consider whether $\psi(a_i, \beta)$ is a limit ordinal or not. When $\phi(a_i, \beta) : i < \omega$ is strictly increasing, for each $i$ we choose $\delta_{i+1} = \phi(a_i, \beta)$. Thus we can choose the $\delta_i$ such that

$$\psi(x, y) \land y \in B_\Sigma \supset \{ \psi(a_i, \beta) : i < \omega \}$$

for each $\sigma \in \Sigma^*$. Now we prove by induction on $\gamma_i$ that for all $\sigma = \langle \gamma_0, n_1, \gamma_1, \ldots, n_i, \gamma_i \rangle \in M$

$$r(\psi(x, y) \land y \in B_\Sigma) \supset \{ \psi(a_i, \beta) + \gamma_i \}$$

Thus in $M$ we have $r(\psi(x, y)) \supset \{ \psi(a_i, \beta), \gamma \}$. Since $\gamma$ was chosen arbitrary $\gamma \psi(a_i, \beta)$ as required.

We now obtain a suitable upper bound for $\phi(a, \beta)$.

**Lemma 3.** Let $M$ be an $\kappa_0$-saturated structure and $\psi(x, y)$ be a formula in $M$ such that $r(\exists \psi(x, y)) \subseteq \beta$ and $r(\psi(x, a)) < \beta$ for all $a \in M$. Let $\alpha \geq 1, \beta \geq 2$ and

$$r(\psi(x, y)) \geq \sup \{ \psi(a, \beta) \}. $$

Then $\exists \psi(x, y)$ is regular, whence $r(\exists \psi(x, y)) = r(\psi(x, y))$. Further, we may suppose that $d(\exists \psi(x, y)) = 1$. Let $\Sigma$ be the set of finite sequences of the form

$$\langle a_0, a_1, \ldots, a_n, \beta \rangle$$

where $\beta = \beta_0 > \beta_1 > \ldots > \beta_n > 1$ and $n \geq 0$. Let $\Sigma$ denote the set of all $\sigma$ in $\Sigma$ maximal with respect to $\subseteq$. Since $r(\exists \psi(x, y)) < \beta$ we can find sets $B_\sigma, \sigma \in \Sigma$, definable in $M$ such that

(i) $B_\sigma$ is the solution set of $\exists \psi(x, y)$.

(ii) for all $\sigma = \langle a_0, a_1, \ldots, a_n, \beta \rangle$ with $\beta > 1$

$$\langle \beta ; \sigma \in \Sigma \& t(\sigma) = t(\sigma + 2) \rangle$$

is a family of disjoint subsets of $B_\sigma$.

(iii) for all $\sigma = \langle a_0, a_1, \beta_1, \ldots, a_n, \beta_n \rangle, r(B_\sigma) \supset \gamma + (\gamma, \beta)$. Let

$$M' = M \cup \langle B_\sigma \rangle$$

be such that $M' \models \exists \psi(x, y)$ and $r(\psi(a, \langle M' \rangle)) = r(\exists \psi(x, y))$. For each $A \subseteq |M|$ definable in $M$ let $A'$ denote the corresponding subset of $|M'|$ definable in $M'$. Let $A_\sigma$ denote the solution set of $\psi(a, \beta)$, and for all $\sigma \in \Sigma$ by then by induction on $\beta_i$ for all $\sigma = \langle a_0, a_1, \ldots, a_n, \beta \rangle \in \Sigma$, $r(B_\sigma \cap A') \supset \beta_i$. Taking $\sigma < \beta$ we have $r(A_\sigma) \supset \beta$, contradiction. Now fix $\sigma \in \Sigma$ and $n < \omega$ such that $|B_\sigma \cap A_\sigma| = n$. Let

$$C = \{ b \in M : |B_\sigma \cap A_\sigma| = n \}.$$
By choice of $tp(a, |M|)$, $r(C) = r(\exists y \psi(x, y))$. Let

$$\alpha' = r(\exists y \psi(x, y) \land x \notin C).$$

Since $d(\exists y \psi(x, y)) = 1$ by assumption, $\alpha' < a$. Let $D$ be the solution set in $M$ of $\exists y (\psi(x, y) \land x \in C \land y \in B_a) \Rightarrow D = B_a$.

As remarked above, when $\varphi(x, y)$ is regular, $r(\exists y \varphi(x, y)) = r(\varphi(x, y))$. Applying this to the formula defining $D$,

$$r(D) = r(\psi(x, y) \land x \in C \land y \in B_a).$$

By Lemma 0

$$r(\psi(x, y) \land x \in C \land y \in B_a) = r(\exists y (\psi(x, y) \land x \in C \land y \in B_a)) = r(C).$$

Combining the equations established above we have

$$r(D) = r(\exists y \psi(x, y)) \leq a.$$

Now $B_a - D$ is included in the solution set of $\exists y (\psi(x, y) \land y \notin C)$ whence

$$r(B_a - D) \leq r(\exists y (\psi(x, y) \land y \notin C)) = r(\gamma)(a', \beta).$$

Clearly $r(B_a)$ is equal to one of $r(D)$ and $r(B_a - D)$ and $r(B_a) \geq \gamma$. Thus we have the conclusion in case $\beta < a$.

Now let $\beta \geq a$ then $-\beta = \beta$. If $B_a' \cap A_\beta'$ is nonempty for all $\sigma \in V$ then by induction on $\beta'$ for all $\sigma = \langle 0, n_1, 1, \ldots, n_n, \beta' \rangle \in V$, $r(B_a \cap A_\beta') \geq \beta'$. Taking $\sigma = \langle 0, \beta \rangle$ we have a contradiction since $-\beta = \beta$. The rest of the argument runs as for the case $\beta < a$ except that we take $n = 0$ which makes $D = \emptyset$. Thus in this case the only possibility is $\gamma \leq a'$.

The function $q$ may now be characterized by:

**Theorem 1.** $q(0, \beta) = \beta$ and for $\alpha > 1$

$$q(a, \beta) = \max(a + 1, \sup \{q(a', \beta) + \langle \gamma \rangle \}.$$ 

Proof. It is obvious that $q(0, \beta) = \beta$. Let $\alpha > 1, \beta > 2$ and suppose that $q(a, \beta)$ exceeds the given expression. Let $\gamma = \max(a + 1, \sup \{q(a', \beta)\})$ if $\beta < a$ and be $\sup \{q(a', \beta)\}$ otherwise. Let $\psi(x, y)$ be a formula witnessing that $q(a, \beta) >$ the given expression then $r(\psi(x, y)) \geq \gamma (\beta)$. From Lemma 3 we have a contradiction, whence $q(a, \beta)$ is less than or equal to the given expression when $a > 1$ and $\beta > 2$. From Lemmas 0 and 2, $q(a, 1) = a + 1$ and it is easy to check directly that $q(a, 1) \geq a + 2$. From Lemmas 1 and 2, $q(a, \beta)$ gives the given expression when $a > 1$ and $\beta > 2$. Since the given expression takes the values $a + 1$, $a + 2$ for $\beta = 1, 2$ respectively the proof is complete.

2. The function $\pi$. Here our information is incomplete. Clearly $\pi(a, \beta) \leq q(a, \beta + 1)$ but in general the inequality is strict. We now give an example which shows that $\pi(a, \beta) \leq a(+) \beta(+) + 1$ is not true in general. This contradicts a previous claim of the author [2, Lemma 1, p. 162]. Fortunately the only use made of that lemma in [2] was to show that if $\varphi(x)$ and $\psi(y)$ both have finite rank then so also does $\varphi(x) \land \psi(y)$. The truth of this assertion is clear from Theorem 1.

Let $A$ denote the set of algebraic numbers. We furnish $A^2$ with the following functions: the projections $p_0, p_1$ onto the diagonal and $+$, $-$ on the diagonal. The theory of the diagonal is the theory of algebraically closed fields which is well-known to be strongly minimal, that is to say, the rank and degree of the universe are both 1. It is easy to see that $A^2$ has rank 2 and degree 1. We consider elements $(x, y)$ of $B = A^2 \times A^2$. For $n < a$ let $B_n \subset B$ denote the solution set of

$$p_0(x) + p_1(x) + p_0(y) + p_1(y) = (n, n)$$

and for $k < a$ let $B_n \subset B$ denote the intersection of $B_n$ with the solution set of

$$p_0(x) + 2p_1(x) + 2p_0(y) + p_1(y) = (k, k).$$

There is a unique $F_n, k: A^2 \to A^2$ such that $(x, y) \in B_n$ if and only if

$$F_n, k(p_0(x), p_1(x)) = (p_0(y), p_1(y)).$$

Now $F_n, k$ being 1-1 onto and definable takes definable subsets onto definable subsets of the same rank and degree; similarly for $F_{n, i}$. Since the number of definable subsets is $n_0$, we can choose $a_n, i_1, \ldots, i_{n_0}$ such that they are all distinct and

$$C = \{a_n, i: n, k < a \} \cup \{F(a_n, i): n, k, i < a\}$$

has finite intersection with each definable subset of $A^2$ of rank 1. Now we define a structure

$$M = \langle |M|, U^M, p_0^M, p_1^M, +^M, \cdot^M, R^M, R^M \rangle, i < a.$$ 

First let $|M| = A^2 \cup (A^2 \times a \times a), U^M = A^2$. Next $p_0^M, p_1^M, +^M, \cdot^M$ are defined as before on $A^2$ and on the rest of the universe are to be trivial. Let

$$R^M = \{\langle a, i, j \rangle: a \in A^2, i, j < a\}$$

and finally for $c \in C, i < a$ let

$$R^M = \{\langle c, i, j \rangle: j < a\}.$$ 

Let $\varphi(x)$ be $\neg U(x)$. Clearly

$$R^M = \{\langle c, i, j \rangle: j < a\}.$$ 

is a basis for the family of subsets of $A^2$ definable in $M$. If $S \subset A^2$ has rank $\leq 1$ in $A^2$ then $S \cap C < a_0$ whence $S \times a \times a$ has rank 2 in $|M|$. It follows easily that $\varphi(x)$ has rank 3 and degree 1 in $M$. Now let

$$B_n = \{\langle x, i, j \rangle, \langle y, k \rangle, \langle z, l \rangle: x, y, z \in B_n \land i, j, k, l < a\}$$
and $B_{a,b}$ be defined similarly from $B_{a,b}$. For any $c = a_{n,k}$ it is clear that

$$(c \times a_0 \times a_0) \cup ((F_n,c) \times a_0 \times a_0)$$

is a subset of $B_{a,b}$ of rank 4 in $M$. Since the $a_{n,k}$ are different for different $i$, $r(B_{a,b}) > 5$. Thus $r(B_{a,b}) \geq 6$ whence $r(p(x) \land q(y)) \geq 7$. Thus this example shows that $\pi(3, 3) > 7$.

The example can be generalized as follows. Let $m \geq n \geq 2$. Form a model $M$ with universe

$$A_0 \cup A_0' \cup (A_0' \times a_0 \times a_0) \cup (A_0' \times a_0 \times a_0)$$

where $A_0, A_1$ are disjoint copies of the algebraic numbers. We include the projection functions on $A_0', A_1'$ and also the functions $+$ and $\cdot$, all these functions being trivial elsewhere. Let $U_i = A_i$ for $i < 2$ and

$$R = \{ \langle a, b, c \rangle : a \in A_0, b \in A_0', c \leq a \} .$$

As in the case $m = n = 2$ we can easily construct nested definable subsets of $A_0' \times A_1'$ of depth $n+1$ such that the innermost ones have the form $(c) = (c_0, c_1)$ where $c_0 \in A_1'$, and such that if $C_i$ denotes the set of all $c_i$ then $C_i \cap S$ is finite for each $S \subseteq A_1'$ which is definable and of rank $\geq n-1$. Now for each $c = (c_0, c_1)$ we adjoint to $M$ relations on $((c_0) \times a_0 \times a_0) \cup ((c_1) \times c_0 \times c_0)$ such that this set has rank $\pi(m, n) = 4$ and $\psi(x)$ has an $\omega$-saturated model $M$.

LEMMA 4. If $m > n \geq 2$ then $\pi(m, n) = \pi(m, n) + 3 + 4(m + n)$.

The best result we have in the other direction is:

LEMMA 5. For $a, \beta > 1$

$$\pi(a, \beta + 1) \leq \max \{ \pi(a, \beta + 1), \sup \{ \pi(a, \beta + 1) + 1 \} \}$$

Proof. Let $\psi(x), \varphi(x)$ be formulas in a structure $M$ having ranks $a, \beta + 1$ respectively and both of degree 1. Let $\psi(x), \varphi(x)$ be formulas in $M$ which partition $\theta(x) \land \psi(x)$.

LEMMA 6. Let $\varphi(x)$ be a formula in a model $M$ of an $\omega$-categorical theory $\phi$.

(i) If $\theta(x)$ is a formula in $M$ for all $n < \omega$,

(ii) If $\theta(x)$ is a formula in $M$ for all $n < \omega$ and $\theta(x)$ is a formula in $M$ for all $n < \omega$,

(iii) If $\theta(x)$ is a formula in $M$ for all $n < \omega$ and $\theta(x)$ is a formula in $M$ for all $n < \omega$,

(iv) If $\theta(x)$ is a formula in $M$ for all $n < \omega$ and $\theta(x)$ is a formula in $M$ for all $n < \omega$,

(v) If $\theta(x)$ is a formula in $M$ for all $n < \omega$ and $\theta(x)$ is a formula in $M$ for all $n < \omega$,

where $\theta(x)$ is a formula in $M$ for all $n < \omega$ and $\theta(x)$ is a formula in $M$ for all $n < \omega$.

This lemma is not stated explicitly but follows easily from the facts assembled to prove the main theorem of [1].
Lemma 8. In a categorical theory all $\alpha, \beta \geq 1$

$$\varphi(\alpha, \beta) \leq \max \left( \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}, \sup_{\gamma \geq \alpha} \{ \varphi(\alpha, \gamma) + 1 \} \right).$$

Proof. We follow the same line as in the proof of Lemma 5. Thus let $\psi(x, y)$ be a formula in a model $M$ of a categorical theory satisfying:

$$r(\exists y \psi(x, y)) \leq \alpha$$

and $r(\psi(x, a)) \leq \beta$

for all $a$ in every elementary extension of $M$. Without loss of generality let $d(\exists y \psi(x, y)) = 1$. Fix $a$ in an elementary extension $M'$ of $M$ such that $r(\exists y \psi(x, a)) \leq \alpha$ and $r(\psi(x, a)) \leq \beta$.

Now let $\psi'(x, y)$ denote $\theta(x) \land \psi(x, y)$ and $\varphi'(x, y)$ denote $\neg \theta(x) \land \psi(x, y)$. For all $a'$ we have $r(\psi'(a', x)) \leq \beta$ whence there exists $\gamma \leq \beta$ such that for all $a'$, $r(\psi'(a', x)) \leq \gamma$ either because rank is finite in the case of $\kappa_0$-category or because there are only finitely many types over a finite set in the case of $\kappa_0$-category. Also since $\langle tp(a, M) \rangle = \kappa_0 \Rightarrow \exists y \psi(y, y)$ and $d(\exists y \psi(x, y)) = 1$ we have $r(\exists y \psi'(x, y)) \leq \alpha$. Since $\psi(x, y)$ is equivalent to $\psi'(x, y) \land \psi''(x, y)$, $r(\psi'(x, y)) = r(\psi''(x, y))$. Thus in case $r(\psi'(x, a)) \leq \beta$ then

$$r(\psi'(x, y)) \leq \max \left( \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}, \sup_{\gamma \geq \alpha} \{ \varphi(\alpha, \gamma) + 1 \} \right).$$

Now suppose $r(\psi(x, a)) = \beta$ and $d(\psi(x, a)) = \kappa_0$. Let $\psi(x, y), \gamma \leq \kappa_0$ be formulas in $M$ constituting a partition of $\psi(x, y)$. We can choose $\gamma \leq \kappa_0$ such that $r(\psi(x, a)) \leq \beta$. Reasoning exactly as for $\psi(x, y)$ above we have

$$r(\psi(x, y)) \leq \max \left( \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}, \sup_{\gamma \geq \alpha} \{ \varphi(\alpha, \beta) + 1 \} \right)$$

which gives

$$r(\psi(x, y)) \leq \max \left( \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}, \sup_{\gamma \geq \alpha} \{ \varphi(\alpha, \gamma) + 1 \} \right).$$

Thus in either case we have the desired conclusion.

Theorem 2. For a categorical theory

$$\varphi(\alpha, \beta) + 1 = \pi(\alpha, \beta) = \alpha(+) \beta(+) 1$$

and $\varphi(\alpha, \beta) = \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}$.

Proof. It is immediate that $\varphi(\alpha, \beta) \leq \varphi(\alpha, \beta) + 1$. By a straightforward induction $\varphi(\alpha, \beta) + 1 \leq \pi(\alpha, \beta)$, see Wierzejewski [10]. From Lemma 8 we see by induction that $\varphi(\alpha, \beta) \leq \alpha(+) \beta$. This proves the first part. Note that $\varphi(\alpha, \beta) = \alpha(+) \beta$. Let $\varphi(x, y)$ be a formula such that $r(\exists y \psi(x, y)) = \alpha$ and $r(\psi(x, a)) \leq \gamma$ for all $a$ in some $\kappa_0$-saturated model. Since

$$r(\exists y \psi(x, y)) \leq \varphi(\alpha, \beta) \leq \alpha(+) \beta$$

we have $\varphi(\alpha, \beta) \leq \sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}$. On the other hand if there are formulas $\chi(x)$ and $\theta(x)$ with ranks $\alpha, \gamma$ respectively then $r(\chi(x) \land \theta(x)) = \alpha(+) \gamma$ — this is the same observation of Wierzejewski just mentioned. Thus $\sup_{\gamma \geq \alpha} \{ \varphi(\gamma, \beta) + 1 \}$ which completes the proof of the second part.

4. Comparison with other ranks. In [4, §5] Lascar employed a rank defined on types complete over some subset of a model. From now on we suppose that the theory under consideration is superstable. For all $\pi \leq \alpha$, for all $A$, and all $p \in S(A)$ we define $u(p) \in On$. The key feature of the definition is that if $p \in S(A)$, $q \in S(B)$, $A \subseteq B$, and $p \subseteq q$ then

$$u(q) < u(p) = q \text{forks over } A.$$
From [3, Propositions 2, 3, pp. 56, 57] it follows that this notion is unchanged for an $\kappa_0$-stable theory if we write $a$ for $\alpha$. From Theorem 3 Lascar deduced that for any $\xi$ there exists $m<\alpha$ depending only on $tp(\xi, B)$ such that $A$ is independent over $B$ and for each $a \in A$, $\{a\}$ is not independent over $B \cup Rn a$, then $|A| < \aleph_\xi$. This answer a question raised in [2, p. 166], because it shows that the upper dimension of a model is finite if its lower dimension is finite. As a corollary Lascar obtained an elegant proof that a countable superstable theory has $\geq \kappa_0$ countable models if it is not $\kappa_0$-parametrizable. Thus even for the investigation of $\kappa_0$-stable theories Lascar’s rank is more powerful than Morley’s.

The closest relative to Lascar’s rank amongst ranks defined on formulas is Shelah’s notion of degree, not to be confused with degree in Morley’s sense. The degree of a formula $\varphi$ is denoted $s(\varphi)$, see [7] or [8] for a precise definition. The key clause says that $s(\varphi(x)) \geq \beta + 1$ if and only if there is a formula $\psi(x, \bar{y})$ and sequences $\bar{z}_i, i \in \omega$, such that:

1. $s(\varphi(x)) \leq \beta$ for all $i$
2. The $\psi(x, \bar{y})$’s are almost contradictory.

The relationship between $s$ and $u$ is given by:

**Lemma 9.** Let the parameters of $\varphi(x)$ be $A$. Then $s(\varphi(x)) \geq n$ if and only if there exists $p \in S^*(A)$ such that $u(p) \geq n$ and $\varphi(x) \in p$. The “if” part holds for arbitrary $A \in On$ in place of $n$.

**Proof.** For induction suppose the result holds for $n$. First let $s(\varphi(x)) \geq n + 1$. As Shelah noticed in [7], in the above definition we can suppose that $\{\bar{z}_i, i < \omega\}$ is an indiscernible set, and further that it is indiscernible over $A$. Thus in the present case we have $s(\varphi(x) \land \psi(x, \bar{y})) \geq n$. By the induction hypothesis there exist $B \supseteq A$ and $q \in S^*(B)$ such that $u(q) \geq n$ and $\varphi(x) \land \psi(x, \bar{y}) \in q$. Clearly $q$ forks over $A$, whence $u(q) > n$ and we may set $p = q$.

For the other direction suppose that $p \in S^*(A)$, $u(p) \geq n + 1$, and $\varphi(x) \in p$. Then there exists $B \supseteq A$ and $q \in S^*(B)$ such that $q \supseteq p$, $q$ forks over $A$, and $u(q) \geq n$. Further using [3, Theorem 10, p. 41] we may suppose that there is a formula $\psi(x, \bar{y})$, and a set $\{\bar{z}_i, i < \omega\}$ indiscernible over $A$ such that $\psi(x, \bar{y}) \in q$ and the formulas $\varphi(x) \land \psi(x, \bar{y})$ are almost contradictory. Now $s(\varphi(x) \land \psi(x, \bar{y})) \geq n$ by the induction hypothesis whence $s(\varphi(x)) \geq n + 1$.

This lemma is best possible in the following sense. For any $\alpha$, there exists a countable superstable theory for which $s(\varphi(x)) = \alpha$ and yet for every $A$ and $p \in S^*(A)$, $u(p) \leq \alpha$. Suppose $T$ is the theory for $\alpha$ and $R$ be a new binary relation symbol. Let the axioms of $T$ say that $R$ is an equivalence relation with infinitely many equivalence classes, that $T$ restricted to any one of the equivalence classes is $T$, and that no relationships hold between elements of different equivalence classes. Then $T'$ is good for $\alpha + 1$. Further, if $T_0, T_1, \ldots$ are theories corresponding to $\alpha_0, \alpha_1, \ldots$, then their disjoint union corresponds to $\sup \alpha_i$.

References


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