In fact, for every $p \in X$, every sufficiently small neighbourhood of p has a countable, regular base of regions (see [4], p. 231, Theorem 8). Hence there exists a decreasing sequence of continua $K_j \subset X$ such that

$$(p) = \bigcap_{j=1}^{\infty} K_j \quad \text{and} \quad p \in \operatorname{Int} K_j \quad \text{for all } j .$$

Then $F(p) = \bigcap_{j=1}^{\infty} F(K_j)$ by virtue of (VIII). Therefore by (IX), for every open set $V \subset Y$ such that $F(p) \subset V$ there exists a j' such that $F(K_{j'}) \subset V$. (VI) follows.

PROBLEM. Can the c-function in Theorem 6 be replaced by a single-valued function?

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Products of perfectly normal spaces

by

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Abstract. Answering a question raised by R. W. Heath, we construct, assuming Continuum Hypothesis, for every natural n a separable and first countable space X such that

(a) X^n is perfectly normal;

(b) X^{n+1} is normal but X^{n+1} is not perfect.

The space X (and X^{n+1}) has cardinality ω_1 and can be made either Lindelöf or locally compact and locally countable.

We show that the existence of such spaces is independent of the axioms of set theory.

§ 1. **Introduction.** In 1969 R. W. Heath [7] raised a question whether for $n \geqslant 2$ there exist spaces X such that X^n is perfect but X^{n+1} is not. This question has also been repeated by D. Burke and D. Lutzer in [2] and has been brought to the author's attention by Eric van Douwen.

In this paper we give a positive answer to this question, constructing, under the assumption of the Continuum Hypothesis (CH), the following two examples.

EXAMPLE 1. (CH) For every $n<\omega$ there exists a first countable, locally compact, locally countable space X of cardinality ω_1 such that:

(a) X^n is perfectly normal and hereditarily separable;

(b) X^{n+1} is normal but X^{n+1} is not hereditarily normal [1].

EXAMPLE 2. (CH) For every $n<\omega$ there exists a first countable space X of cardinality ω_1 such that:

(a) X^n is hereditarily Lindelöf and hereditarily separable;

(b) X^{n+1} is Lindelöf but X^{n+1} is not hereditarily Lindelöf.

This paper is closely related to our paper [16] where, under the assumption of Martin's Axiom, positive results concerning the preservation of perfectness and perfect normality in product spaces are given.

* This paper was oryginated while the author was a Visiting Assistant Professor at the University of Pittsburgh in 1976/77.

(1) Let us recall that a perfectly normal space is hereditarily normal and that a Lindelöf space is perfect if and only if it is hereditarily Lindelöf.

We were unable to construct a "real" example answering Heath's question, although it is natural to conjecture that such "real" examples exist. Nevertheless, the following theorem, which is based on results obtained in [16] and other results already existing in the literature, shows that our Examples 1 and 2 are independent of the axioms of set theory (²).

THEOREM 1. *If* $n \geqslant 2$ *and* X *is a space such that* X^n *is perfectly normal but* X^{n+1} *is not perfect, then* X *cannot belong to any of the following classes of spaces*:

(a) *θ-refinable p-spaces*,

(b) *strict p-spaces*,

(c) *M-spaces*,

(d) *locally connected and locally compact spaces*,

(e) *linearly ordered spaces*,

(f) *monotonically normal p-spaces*,

(g) *pseudocompact spaces.*

Moreover, if Martin's Axiom plus the negation of the Continuum Hypothesis (MA + ¬CH) *is assumed and the cardinality of* X *is* ω_1, *then* X *cannot belong to any of the following classes of spaces*:

(h) *θ-refinable spaces*;

(i) *separable spaces*;

(j) *complete* ccc *spaces*;

(k) *generalized ordered spaces.*

It can be shown (see Section 2) that the assumption of $n \geqslant 2$ is essential in all the above statements (a) through (k), with the possible exception of (j). This should explain the motivation behind Heath's question.

Spaces described in Examples 1 and 2 cannot be both paracompact and locally compact (see [18] or Theorem 1, (a), (b) or (c)). One easily sees that they also cannot be both paracompact and locally countable.

The construction of Examples 1 and 2 relies heavily on the techniques oryginated by K. Kunen (see [9] and [11]) and E. van Douwen [4]. It also exploits the notion of n-cardinality (see [14]) and methods developed in [15].

Theorem 1 is proved in Section 2 and Examples 1 and 2 are constructed in Section 3.

§ 2. Proof of Theorem 1.

Proof of Theorem 1. (a) Kullman [12] proved that a θ-refinable p-space with a G_δ-diagonal is a Moore space; (b) Kullman also proved that a strict p-space with a regular G_δ-diagonal is a Moore space; (c) Chaber showed [3] that M-spaces with G_δ-diagonal are metrizable; (d) Reed and Zenor [17] proved that a locally connected and locally compact space whose square is perfectly normal is metrizable; (e) Lutzer [13] showed that linearly ordered spaces with G_δ-diagonals are metrizable; (f) monotonically normal p-spaces with G_δ-diagonals are metrizable [8]; (g) normal pseudocompact spaces are countably paracompact and thus M-spaces. (h) Since perfect θ-refinable spaces are subparacompact [1], this follows from Corollary 3 in [16]; (i) this is implied by Corollary 7 in [16]; (j) since under (MA + ¬CH) first countable complete ccc spaces are separable [6], (i) applies; (k) see [16]. ■

(²) For undefined notions and symbols the reader is referred to [5] and [2].

Let us now show that the assumption of $n \geqslant 2$ is essential in all statements (a) through (k) of Theorem 1, with the possible exception of (j).

The familiar example of the "double arrow" (see [5]; Exercise 3.10.C) due to Alexandroff and Urysohn is a compact, hereditarily Lindelöf, hereditarily separable, linearly ordered space without a G_δ-diagonal (compact spaces with G_δ-diagonals are metrizable [18]; see also Theorem 1, (a), (b) or (c)). This shows that $n \geqslant 2$ is essential in (a), (b), (c), (e), (f) and (g).

The Souslin line, whose existence is consistent with the axioms of set theory, is a locally connected, compact, perfectly normal, non-separable linearly ordered space, which does not have a G_δ-diagonal. This implies the necessity of $n \geqslant 2$ in (d).

If we build the "double arrow" starting with a subset of the unit interval of cardinality ω_1, then we will get a hereditarily Lindelöf and hereditarily separable linearly ordered space which, as one easily checks, does not have a G_δ-diagonal. Thus, $n \geqslant 2$ is necessary in (h), (i) and (k).

§ 3. Construction of Examples 1 and 2.

Let us first recall the definition of n-*cardinality*.

DEFINITION. [14] For a subset A of M^n, where M is an arbitrary set, we define the n-*cardinality* $|A|_n$ *of* A by

$$|A|_n = \max\{|B|:\ B \subset A \text{ and } p_i \neq q_i, \text{ for } i = 1, 2, \ldots, n \text{ and any two distinct points } p = (p_1, \ldots, p_n) \text{ and } q = (q_1, \ldots, q_n) \text{ of } B\}.$$

We say that A in n-*finite* (n-*countable*, n-*uncountable*) if its n-cardinality is finite (countable, uncountable). ■

The following proposition has been proved in [14].

PROPOSITION 2. *If* A *is not* n-*finite, then*

$$|A|_n = \min\{|Z|:\ Z \subset M \quad \text{and} \quad A \subset \bigcup_{i=1}^{n} (M^{i-1} \times Z \times M^{n-i})\}. ■$$

Construction of Example 2. For the sake of simplicity, we shall give a detailed construction of Example 2 only in case of $n = 2$. The proof of the general case is quite analogous.

Denote by M the space of irrationals. By Theorem 2 in [14] there exists a subset S of M such that

(1) $S^3 \cap F \neq \emptyset \neq (M \setminus S)^3 \cap F$, for every 3-uncountable closed subset F of M^3.

Let us fix a one-to-one correspondence between points of M and ordinals $\alpha<\omega_1 = 2^{\omega}$. From now on we shall identify M with $\omega_1 = \{\alpha\colon \alpha<\omega_1\}$. Denote by ε the topology on $M = \omega_1$ generated by the standard metric ϱ on the real line and let $\{A_\gamma\colon \gamma<\omega_1\}$ be the family of all countable subsets of $M^2 = \omega_1^2$. Clearly, we can assume that $\omega = \{\alpha\colon \alpha<\omega\}$ is dense in $\omega_1 = M$, $S\cap\omega = \emptyset$ and

(2) $$A_\gamma\subset\gamma^2 \quad \text{for every } \gamma<\omega_1 .$$

For every $\alpha<\omega_1$, $n<\omega$ and $i = 1, 2, 3$ we shall define subsets $B_i(\alpha, n)$ of $\omega_1 = M$ so that the following conditions are satisfied:

(3) For every $i = 1, 2, 3$ the collection $\{\mathcal{B}_i(\alpha)\}_{\alpha<\omega_1}$ of families $\mathcal{B}_i(\alpha) = \{B_i(\alpha, n)\}_{n<\omega}$ satisfies the axioms of a neighbourhood system and therefore generates a topology τ_i on $\omega_1 = M$ in which families $\mathcal{B}_i(\alpha)$ form bases of neighbourhoods of points $\alpha\in M$ (see [5]; Proposition 1.2.3);

(4) For every $i = 1, 2, 3$ the topology τ_i is finer than ε;

(5) For every $i = 1, 2, 3$, $n<\omega$ and $\alpha<\omega_1$ the sets $B_i(\alpha, n)$ are ε-closed and if $\alpha\in S$ then they are also ε-open;

(6) For every $\alpha\in M\setminus S$ we have $[B_1(\alpha, 0)\times B_2(\alpha, 0)\times B_3(\alpha, 0)]\cap\Delta = \{(\alpha, \alpha, \alpha)\}$, where $\Delta = \{(\beta, \beta, \beta)\colon \beta<\omega_1\}$ is the diagonal of M^3;

(7) For every $i, j = 1, 2, 3$, $\alpha<\omega_1$, $\beta<\alpha$ and $\gamma\leqslant\alpha$:
(a) $(\alpha, \alpha)\in \mathrm{Cl}_{\tau_i\times\tau_j}A_\gamma$, if $(\alpha, \alpha)\in \mathrm{Cl}_{\varepsilon\times\varepsilon}A_\gamma$;
(b) $(\alpha, \beta)\in \mathrm{Cl}_{\tau_i\times\tau_j}A_\gamma$, if $(\alpha, \beta)\in \mathrm{Cl}_{\varepsilon\times\tau_j}A_\gamma$;
(c) $(\beta, \alpha)\in \mathrm{Cl}_{\tau_i\times\tau_j}A_\gamma$, if $(\beta, \alpha)\in \mathrm{Cl}_{\tau_i\times\varepsilon}A_\gamma$.

Define X to be the topological sum of spaces $X_i = (M, \tau_i)$, i.e.

(8) $$X = \bigoplus_{i=1}^{3} (M, \tau_i) .$$

We shall show first that the space X has the desired properties.

From (3) and (4) it follows that X is first countable and Hausdorff. Conditions (5) and (4) imply that X is zero-dimensional, hence completely regular.

I. X^3 is Lindelöf. Suppose the contrary and let m be the smallest natural number $\leqslant 3$ such that X^m is not Lindelöf. Consequently, there exist $i_1, \ldots, i_m = 1, 2, 3$ such that the space $Y = X_{i_1}\times\ldots\times X_{i_m} = (M^m, \tau_{i_1}\times\ldots\times\tau_{i_m})$ is not Lindelöf. Let $\mathcal{U}$ be an open covering of Y. By (5) there exists a countable family $\mathcal{V}$ of euclidean-open subsets of M^m which covers S^m and refines $\mathcal{U}$. By (1) the euclidean-closed subset $F = (M^m\setminus\bigcup\mathcal{V})\times M^{3-m}$ of M^3 must be 3-countable, because $F\cap S^3 = \emptyset$. It follows that the subset $K = M^m\setminus\bigcup\mathcal{V}$ of M^m is m-countable and there exists a countable subset Z of M such that

$$K\subset L = \bigcup_{j=1}^{m} (M^{j-1}\times Z\times M^{m-j}) .$$

Since X^{m-1} is Lindelöf, the subspace L of Y is also Lindelöf and can be covered by a countable subfamily $\mathcal{W}$ of $\mathcal{U}$. The countable open covering $\mathcal{G} = \mathcal{V}\cup\mathcal{W}$ refines $\mathcal{U}$ and this contradicts our assumption that X^3 is not Lindelöf.

II. X^3 is not hereditarily Lindelöf. By (1) the set $M\setminus S$ must be uncountable, but by (6) the subspace

$$T = \{(\alpha, \alpha, \alpha)\in M^3\colon \alpha\in M\setminus S\}$$

of $X_1\times X_2\times X_3 = (M^3, \tau_1\times\tau_2\times\tau_3)$ is discrete-in-itself and thus T is not Lindelöf.

III. X^2 is hereditarily Lindelöf and hereditarily separable. The following four lemmas are essentially the same as those proved in [11].

LEMMA 1. *For every $k = 1, 2, 3$ and every subset A of M there exists a $\gamma<\omega_1$ such that if $\alpha\geqslant\gamma$ and $\alpha\in\mathrm{Cl}_\varepsilon A$, then $\alpha\in\mathrm{Cl}_{\tau_k}A$.*

Proof of Lemma 1. Let B be a countable euclidean-dense subset of A. There exists a $\gamma<\omega_1$ such that $B\times B = A_\gamma$. Let $\alpha\in\mathrm{Cl}_\varepsilon A$ and $\alpha\geqslant\gamma$. Then $\alpha\in\mathrm{Cl}_\varepsilon B$ and therefore $(\alpha, \alpha)\in\mathrm{Cl}_\varepsilon B\times\mathrm{Cl}_\varepsilon B = \mathrm{Cl}_{\varepsilon\times\varepsilon}B\times B = \mathrm{Cl}_{\varepsilon\times\varepsilon}A_\gamma$. Consequently, by (7), $(\alpha, \alpha)\in\mathrm{Cl}_{\tau_k\times\tau_k}A_\gamma = \mathrm{Cl}_{\tau_k\times\tau_k}B\times B = \mathrm{Cl}_{\tau_k}B\times\mathrm{Cl}_{\tau_k}B$ and $\alpha\in\mathrm{Cl}_{\tau_k}B\subset\mathrm{Cl}_{\tau_k}A$. ∎

LEMMA 2. *For every $k = 1, 2, 3$ the space X_k is perfect and hereditarily separable.*

Proof of Lemma 2. Let U be an open subset of X_k. By Lemma 1 the set $U\setminus\mathrm{Int}_\varepsilon U$ is countable, which easily implies that U is an F_σ-subset of X_k. Let B be an arbitrary subspace of X_k and let A be a euclidean-dense countable subset of B. By Lemma 1, there exists a $\gamma<\omega_1$ such that the countable set $A\cup\gamma$ is τ_k-dense in B. ∎

LEMMA 3. *For every $k = 1, 2, 3$ and every $A\subset M^2$ there exists a $\gamma<\omega_1$ such that if $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A$ and $\alpha\geqslant\gamma$, then $(\alpha, \beta)\in\mathrm{Cl}_{\tau_k\times\varepsilon}A$.*

Proof of Lemma 3. Let $\{V_m\}_{m<\omega}$ be a countable base for ε and let $A_m = \pi(A\cap(M\times V_m))\subset X_k$, where $\pi\colon X_k\times M\to X_k$ is the projection. By Lemma 1, for every $m<\omega$ there exists a $\gamma_m<\omega_1$ such that if $\alpha\geqslant\gamma_m$ and $\alpha\in\mathrm{Cl}_\varepsilon A_m$, then $\alpha\in\mathrm{Cl}_{\tau_k}A_m$. Let $\gamma = \sup\{\gamma_m\}_{m<\omega}$, $\alpha>\gamma$ and assume that $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A$. Take a neighbourhood $U\times V$ of (α, β) in $X_k\times M$, where $V = V_m$, for some m. Since $\alpha\in\mathrm{Cl}_\varepsilon\pi(A\cap(M\times V)) = \mathrm{Cl}_\varepsilon A_m$ and $\alpha\geqslant\gamma_m$, we have $\alpha\in\mathrm{Cl}_{\tau_k}A_m$. Therefore, there exists $\delta\in A_m\cap U$ and $\eta\in V$ such that $(\delta, \eta)\in A$. Consequently, $(\delta, \eta)\in A\cap(U\times V)$ and $(\alpha, \beta)\in\mathrm{Cl}_{\tau_k\times\varepsilon}A$. ∎

LEMMA 4. *For every $i, j = 1, 2, 3$ and every $A\subset M^2$ there exists a γ such that if $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A$ and $\alpha, \beta\geqslant\gamma$, then $(\alpha, \beta)\in\mathrm{Cl}_{\tau_i\times\tau_j}A$.*

Proof of Lemma 4. Let B be a euclidean-dense countable subset of A. There exists $\eta<\omega_1$ such that $B = A_\eta$. Applying Lemma 3 twice we find a $\gamma<\omega_1$, $\gamma\geqslant\eta$ such that if $\alpha, \beta\geqslant\gamma$ and $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A_\eta$, then $(\alpha, \beta)\in\mathrm{Cl}_{\tau_i\times\varepsilon}A_\eta\cap\mathrm{Cl}_{\varepsilon\times\tau_j}A_\eta$. If $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A$ and $\alpha, \beta\geqslant\gamma$, then $(\alpha, \beta)\in\mathrm{Cl}_{\varepsilon\times\varepsilon}A_\eta$, $(\alpha, \beta)\in\mathrm{Cl}_{\tau_i\times\varepsilon}A_\eta\cap\mathrm{Cl}_{\varepsilon\times\tau_j}A_\eta$, $\eta\leqslant\alpha$ and $\eta\leqslant\beta$. Depending on whether $\alpha = \beta$, $\alpha<\beta$ or $\alpha>\beta$ we use one of the conditions (7) (a), (b) or (c) to show that $(\alpha, \beta)\in\mathrm{Cl}_{\tau_i\times\tau_j}A_\eta\subset\mathrm{Cl}_{\tau_i\times\tau_j}A$. ∎

Since X^2 is Lindelöf it remains to show that X^2 is perfect and hereditarily separable. To this end, it is enough to prove that for arbitrary $i, j = 1, 2, 3$ the space $Y = X_i \times X_j$ has these properties.

Let U be an open subset of Z. By Lemma 4 there exists a $\gamma < \omega_1$ such that

$$U \setminus \mathrm{Int}_{\varepsilon\times\varepsilon} U \subset T_\gamma = \bigcup_{\alpha<\gamma} [\{\alpha\} \times X_j \cup X_i \times \{\alpha\}] .$$

By Lemma 2, T_γ is a perfect F_σ-subset of Y. Therefore,

$$U = \mathrm{Int}_{\varepsilon\times\varepsilon} U \cup (T_\gamma \cap U),$$

as a union of two F_σ-subsets, is an F_σ-subset of Y.

Let B be an arbitrary subspace of Y and let A be a countable euclidean-dense subset of A. By Lemma 4, there exists $\gamma < \omega_1$ such that $B \setminus \mathrm{Cl}_{\tau_i \times \tau_j} A \subset T_\gamma$. Since Lemma 2 implies that T is hereditarily separable, the space B is separable in Y.

This completes the proof of the properties of X. It remains to construct sets $B_i(\alpha, n)$ satisfying properties (3)–(7). We shall conduct the construction by induction on $\alpha < \omega_1$.

For every $\alpha < \omega$, $n < \omega$ and $i = 1, 2, 3$ define $B_i(\alpha, n) = \{\alpha\}$ and assume that $\alpha < \omega_1$ and that sets $B_i(\beta, n)$ have been constructed for $\beta < \alpha$, $i = 1, 2, 3$ and $n < \omega$. We will define sets $B_i(\alpha, n)$, for $i = 1, 2, 3$ and $n < \omega$.

If $\alpha \in S$, then $B_i(\alpha, n)$ will be an arbitrary ε-clopen subset of M of ϱ-diameter $< 1/(n+1)$ containing α.

Suppose $\alpha \in M \setminus S$. For every $\gamma \leqslant \alpha$ such that $(\alpha, \alpha) \in \mathrm{Cl}_{\varepsilon\times\varepsilon} A_\gamma$ there exists a sequence $\{(x_n, y_n)\}_{n<\omega}$ of points of A_γ such that $(x_n, y_n) \underset{\varepsilon\times\varepsilon}{\to} (\alpha, \alpha)$. By (2), for every n the set $P_n(\gamma) = \{x_n, y_n\}$ is contained in α.

For every $\gamma \leqslant \alpha$, $\beta < \alpha$ and $j = 1, 2, 3$, if $(\alpha, \beta) \in \mathrm{Cl}_{\varepsilon \times \tau_j|\alpha} A_\gamma$ then there exists a sequence $\{(x_n, y_n)\}_{n<\omega}$ of points of A_γ such that $(x_n, y_n) \xrightarrow[\varepsilon \times \tau_j|\alpha]{} (\alpha, \beta)$. We put $Q_n(\gamma, \beta, j) = \{x_n\}$.

Similarly, for every $\gamma \leqslant \alpha$, $\beta < \alpha$ and $i = 1, 2, 3$, if $(\beta, \alpha) \in \mathrm{Cl}_{\tau_i|\alpha \times \varepsilon} A_\gamma$, then there exists a sequence $\{(x_n, y_n)\}_{n<\omega}$ of points of A_γ such that $(x_n, y_n) \xrightarrow[\tau_i|\alpha \times \varepsilon]{} (\beta, \alpha)$. We put $R_n(\gamma, \beta, i) = \{y_n\}$.

Let us notice, that in the above described manner we have defined only countably many sets. It is easy to show that there exists a sequence $T = \{\alpha_m\}_{m<\omega}$ of different points $\alpha_m < \alpha$ which ε-converges to α and has the property

(9) for every $\gamma \leqslant \alpha$, $\beta < \alpha$ and $i, j = 1, 2, 3$ the sets $\{n < \omega\colon P_n(\gamma) \subset T\}$, $\{n < \omega\colon Q_n(\gamma, \beta, j) \subset T\}$ and $\{n < \omega\colon R_n(\gamma, \beta, i) \subset T\}$ are infinite, provided that the corresponding sets $P_n(\gamma)$, $Q(\gamma, \beta, j)$ and $R_n(\gamma, \beta, i)$ are defined.

It is clear, that we can decompose ω into three disjoint sets L_1, L_2, L_3 so that each of the sequences $T_k = \{\alpha_m\colon m \in L_k\}$ has the property (9), with T replaced by T_k.

Let us find disjoint ε-open sets U_m such that

(10) $\alpha_m \in U_m$, $\alpha \notin U_m$ and the ϱ-diameter of U_m is $< 1/(m+1)$.

By (4), for every $i = 1, 2, 3$ and $m < \omega$ we can find an $l < \omega$ such that $\alpha_m \in B_i(\alpha_m, l) \subset U_m$. Put $B_i(m) = B_i(\alpha_m, l)$ and define

(11) $$B_i(\alpha, n) = \{\alpha\} \cup \bigcup \{B_i(m)\colon m \geqslant n \text{ and } m \in \omega \setminus L_i\} .$$

Let us briefly check that the above defined family of sets $B_i(\alpha, n)$ satisfies conditions (3)–(7). One easily sees that conditions (3)–(6) are fulfilled. To illustrate the proof that (7) holds, let us show for instance that (7a) takes place.

Let $\alpha < \omega$, $\gamma \leqslant \alpha$, $i, j = 1, 2, 3$ and $(\alpha, \alpha) \in \mathrm{Cl}_{\varepsilon\times\varepsilon} A_\gamma$. It suffices to show that for every n

(12) $$B_i(\alpha, n) \times B_j(\alpha, n) \cap A_\gamma \neq \emptyset .$$

Take k different from i and j and notice that the intersection $B_i(\alpha, n) \cap B_j(\alpha, n)$ contains almost all elements of the sequence T_k. Therefore, by (9) there exists an n such that

$$P_n(\gamma) \subset B_i(\alpha, n) \cap B_j(\alpha, n) ,$$

which implies that

$$(x_n, y_n) \in B_i(\alpha, n) \times B_j(\alpha, n) \cap A_\gamma .$$

This proves (12) and completes the construction of Example 2. ■

Construction of Example 1. The construction of Example 1, although similar to the construction of Example 2, is nevertheless more complicated. The reason for it is that this time in order to ensure the normality of X^{n+1} we additionally use a technique introduced by van Douwen in [4] and later improved by the author in [15].

We shall only briefly sketch the construction in case of $n = 2$.

For every $\alpha < \omega_1$, $n < \omega$ and $i = 1, 2, 3$ we construct again sets $B_i(\alpha, n) \subset M = \omega_1$ so that conditions (3), (4) and (7), together with conditions (13), (14), (15) and (16) below, are satisfied.

(13) For every $i = 1, 2, 3$, $n < \omega$ and $\alpha < \omega_1$ the sets $B_i(\alpha, n)$ are ε-compact subsets of $\alpha + 1 = \{\beta\colon \beta \leqslant \alpha\}$;

(14) For every $\alpha \in M \setminus S$ we have

$$[B_1(\alpha, 0) \times B_2(\alpha, 0) \times B_3(\alpha, 0)] \cap \Delta \subset \{(\beta, \beta, \beta)\colon \beta = \alpha \text{ or } \beta < \omega\};$$

(15) $\{(\beta, \beta, \beta)\colon \beta < \omega\}$ is $(\tau_1 \times \tau_2 \times \tau_3)$-dense in Δ;

(16) For every pair A, B of countable subsets of M^3 if $\mathrm{Cl}_{\varepsilon\times\varepsilon\times\varepsilon} A \cap \mathrm{Cl}_{\varepsilon\times\varepsilon\times\varepsilon} B$ is 3-uncountable, then $\mathrm{Cl}_{\tau_1\times\tau_2\times\tau_3} A \cap \mathrm{Cl}_{\tau_1\times\tau_2\times\tau_3} B \neq \emptyset$.

We define X as in (8). First countability, complete regularity, local compactness and local countability of X easily follow from (3), (4) and (13). The normality of X^3 can be derived from (16) (cf. [4], [15]). The subspace $(M \setminus S)^3 \cap \Delta$ of $X_1 \times X_2 \times X_3$ is not normal, because by (15) it is separable and by (14) it contains a closed discrete subset $(M \setminus S \setminus \omega)^3 \cap \Delta$ of cardinality $\omega_1 = 2^\omega$ (use Jones' Lemma). The proof that X^2 is perfect and hereditarily separable is the same as in Example 2. ■

PROBLEMS

1. Does there exist a "real" example of a space X such that X^2 is perfect (or, better, perfectly normal) but X^3 is not? Can it have cardinality ω_1?

2. Does there exist a consistent example of a Moore space X such that X^2 is normal but X^3 is not?

3. Suppose X is generalized ordered and X^2 is perfect. Is X^ω perfect?

4. Suppose X^2 is perfectly normal. Is X submetrizable?

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Prolongationally stable discrete flows

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Abstract. Our objective in this paper is to obtain properties for certain homeomorphisms of Hausdorff spaces. In particular we extend the conclusions obtained by Knight in [7] for continuous flows of characteristic 0 on Hausdorff phase spaces to discrete flows.

1. Introduction and preliminaries. Flows of characteristic 0 from an important subclass of closed flows satisfying certain bilateral stability criteria. Ahmad introduced continuous flows of characteristics 0^+, 0^-, and $0^\pm$ in [1] and Knight introduced continuous flows of characteristic 0 in [5]. Classifications and characterizations of such flows were given in [1], [2], [5], [6], [7], and [8]. Lam made some observations about discrete and continuous flows containing subflows of characteristic $0^\pm$ in [9].

In this paper we extend the results of [7] to discrete flows. Much of the reasoning used in [7] yields similar discrete flow properties, however, several of the techniques employed either do not apply or directly apply here. Virtually all continuous flow results of [7] extend to discrete flows. In Section 2 we analyze discrete flows of characteristic 0 and in Section 3 we give characterizations of such flows.

The standard terminology, notations, and definitions used in the references cited above are used in this paper. The sets of integers, nonnegative integers, and nonpositive integers are denoted by Z, Z^+, and Z^-, respectively. By $C(x)$, $K(x)$, $L(x)$, $D(x)$, and $J(x)$ we mean the orbit, orbit closure, limit set, prolongation, and prolongational limit set of x, respectively. The corresponding positive and negative concepts carry the appropriate $+$ or $-$ superscript. The statements $y \in J^+(x)$ implies $x \in J^-(y)$, $y \in L^+(x)$ implies $x \in J^+(y)$, and $y \in K(x)$ implies $J^+(x) \subset J^+(y)$ are well known tools for continuous flows which are easily demonstrated for discrete flows.

A discrete or continuous flow (X, π) is said to be of *characteristic* 0^+ (0^-) if $D^+(x) = K^+(x)$ ($D^-(x) = K^-(x)$) for each point x in X, or equivalently, if $J^+(x) = L^+(x)$ ($J^-(x) = L^-(x)$) for each point x in X. A flow of characteristics 0^+ and 0^- is of characteristic $0^\pm$. We say that (X, π) is of *characteristic* 0 provided $D(x) = K(x)$ for each point x in X.