Corollary. *If a dendroid X contains a Q-point p such that* (18) *holds, then for every continuous selection σ: $C(X) \to X$ we have $\sigma(K) = p$ (here K denotes the limit continuum mentioned in the definition of a Q-point).*

The author does not know if condition (18) is essential in the corollary, i.e., if there exists a selectible dendroid containing a Q-point p for which (18) fails and with $\sigma(K) \neq p$. Recently Mr. S. T. Czuba has found a dendroid (even a fan) with a Q-point p for which (18) does not hold, but this example is not selectible.

Consider now the dendroid D_0 described in [5], p. 305. Let X be a continuum obtained from D_0 by shrinking the horizontal straight line segment of D_0 to which the points $p_1, p_2, \ldots$ belong (see the picture of C_0, Fig. 1 on p. 305 of [5]) to a point p. It is evident that X is a countable plane fan with a Q-point p. As it was recently shown by Dr. T. Maćkowiak, the fan X is selectible. Thus the existence of a Q-point in a countable plane fan X does not imply that X is not selectible.

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Fixed point theorems for λ-dendroids

by

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Abstract. Fixed point theorems are proved for functions whose values and the images as well as the inverse-images of continua are continua.

§ 1. Introduction. Throughout this paper X will denote an arbitrary *λ-dendroid*, i.e. a hereditarily decomposable and hereditarily unicoherent metric continuum. We shall consider, under the name *c-functions*, functions having continua $F(p) \subset X$ as values, non-empty for all $p \in X$. If $p \in F(p)$, then the point p will be called a *fixed point* of F.

In [5] I proved that if F is upper semi-continuous, then

(I) *there exists a fixed point of F.*

With the aid of papers [5] and [6] we prove here that the fixed point theorem (I) holds under the same remaining assumptions, even without the upper semi-continuity of the function F.

First, a stronger fixed point theorem (§ 3, Theorem 1) is proved under the following two conditions ([5], p. 113, (I) and (II)):

(II) *for every continuum $K \subset X$ the image $F(K) \subset X$ is a continuum,*

(III) *the property $K \cap F(K) \neq \emptyset$ is inductive for continua $K \subset X$,* where the *image* $F(K)$ means the union $\bigcup \{F(p): p \in K\}$, and a property is called *inductive* provided that for every decreasing sequence of sets having this property their common part also has this property (see [4], p. 54). Then we prove a theorem stating exactly that (I) follows from (II) and (III) (§ 3, Theorem 2), which is next applied to considering the following condition on the sets $F^{-1}(K) = \{p \in X: F(p) \cap K \neq \emptyset\}$:

(IV) *for every continuum $K \subset F(K)$ the set $F^{-1}(K)$ is a continuum.*

Namely, fixed point theorems are proved for c-functions satisfying (IV) or (II) (§ 4, Theorems 3 and 4), which imply a common generalization of the fixed point theorems by Gray [1], by the present author [6] and by Smithson [9], [10] (§ 4, Corollary).

Finally the following property of a non-degenerated subcontinuum E of X will be considered:

(V) *for any continuum* $K\subset X$ *the inequalities* $K\cap E\neq\emptyset\neq K-E$ *imply the inclusion* $E\subset K$.

It will be stated under assumptions (V), (III) and (II) that if $E\cap F(E)\neq\emptyset$, then E contains a fixed point of F (§ 5, Theorem 5). Simultaneously (V) ensures (§ 6, Theorem 6) the existence of non semi-continuous c-functions satisfying (II), (III) and (IV).

The remainder of the paper is devoted to examples proving the essentiality of the assumptions (ab denote the unique continuum irreducible between points a and b).

§ 2. Preliminaries on irreducible subcontinua of X. Recall that the assumption of *hereditary unicoherence* (see [4], p. 162; see also [5], p. 106) means that the common part of every two subcontinua of X is a subcontinuum of X, and thus that for every two points $a, b\in X$ a subcontinuum *irreducible* between them (see [4], p. 190 and 193, Theorem 1) is unique in X. Therefore the hereditary unicoherence of X means that for every subcontinuum $K\subset X$

(2.1) $$a\in K \text{ and } b\in K \quad\text{imply}\quad ab\subset K.$$

By a theorem of Kuratowski (see [4], p. 193, Theorem 4; see also [7]), if $a\in K\subset X$ and the continuum K cannot be decomposed into a union of two proper subcontinua containing a, then the point a is a *point irreducibility* of K, i.e. $K=ab$ for some point $b\in X$. By the same theorem, the *set of irreducibility* Ab, i.e. the set of all points $p\in X$ such that $pb=ab$ (see [5], p. 106; see also [3], p. 230), is a continuum by virtue of the *hereditary decomposability* of X (see [7], p. 52). Moreover it follows that for every irreducible subcontinuum $ab\subset X$

(2.2) $$Ab \text{ is a continuum,}$$

(2.3) $$a\in Ab\subset ab,$$

(2.4) $$Ab=ab \quad\text{implies}\quad a=b,$$

in view of the definition of Ab. According to (2.1), for every continuum $K\subset X$

(2.5) $$a\in X-K \text{ and } b\in K \quad\text{imply}\quad Ab\subset X-K.$$

For every point $a\in X$ we define an equivalence relation $\multimap$, called *association* (see [5], p. 106), between irreducible subcontinua of X which contain the point a as a point of irreducibility, such that

(2.6) $$ab\multimap ac \quad\text{implies}\quad Ab=Ac.$$

Finally, I call a continuum E an *end continuum* of X [1] provided that E forms a set of irreducibility of every irreducible subcontinuum of X which meets both E and $X-E$.

[1] This notion is a generalization of the notion of E-set by Miller [8], p. 184.

§ 3. Fixed point theorems for c-functions satisfying (II) and (III). By virtue of (III), there exists a continuum $K\subset X$ such that

(3.1) $$K\cap F(K)\neq\emptyset,$$

(3.2) the continuum $K\subset X$ is irreducible with respect to (3.1)

(see [4], p. 54). Relations between such subcontinua K of X lead to the existence of a fixed point of F (see [5], p. 120, proof of the corollary). For them, the following statement, valid for any set X, points to the possibility of using relations between irreducible subcontinua:

LEMMA 1. *For an arbitrary set-valued function F mapping X into itself and for every continuum $K\subset X$ the condition* (3.2) *implies*

(3.3) $$K=ab \quad\text{and}\quad b\in F(a) \quad\text{for some } a, b\in X.$$

Proof. For every point $b\in F(K)\cap K$ there exists a point $a\in K$ such that $b\in F(a)$ by the definition of the image $F(X)$, and simultaneously $b\in K$. Hence any irreducible subcontinuum $ab\subset K$ (see [4], p. 192, Theorem 1; here (2.1) for the λ-dendroid X) satisfies (3.1). (3.2) implies (3.3).

Lemma 1 will be used in the sequel. Now, we will apply Kuratowski's theorem on the point of irreducibility to the proof of the following

LEMMA 2. *Let $a\in X$, $b\in F(a)$ and $Ab\cap F(Ab)\neq\emptyset$ for a c-function F satisfying* (II) *and* (III). *Then there exists a subcontinuum $K\subset X$ such that*

(3.4) $$a\in K\subset Ab,$$

(3.5) *the continuum K is irreducible with respect to* (3.1) *and* (3.4),

and then there exists a point $c\in X$ such that

(3.6) $$ac=K,$$

(3.7) $$ab\subset F(ac),$$

(3.8) $$Ab\cap\big(F(ac-Ca)\big)=\emptyset.$$

Proof. Since Ab is a continuum and $a\in Ab$ in view of (2.2) and (2.3), and Ab satisfies (3.1) by assumption, there exists a continuum K satisfying (3.5) by virtue of (III).

To prove (3.6) for some point $c\in X$, suppose the contrary. Then by the theorem of Kuratowski, there exist two continua $K_i\subset X$ such that

(3.9) $$K_1\cup K_2=K,$$

(3.10) $$a\in K_i \quad\text{and}\quad K_i\neq K.$$

Since $b\in F(a)$ by assumption, it follows that $b\in F(K_i)$, and by (3.5) $K_i\cap F(K_i)=\emptyset$, whence $a\notin F(K_i)$. Since $F(K_i)$ is a continuum by (II), it follows according to (2.5) that $Ab\cap F(K_i)=\emptyset$. Therefore $Ab\cap\big(F(K_1)\cup F(K_2)\big)=\emptyset$. Since $F(K_1)\cup F(K_2)=F(K_1\cup K_2)$ by the definition of the images, we have $Ab\cap F(K)=\emptyset$ by (3.9), which contradicts (3.1) or (3.4).

Thus (3.6) holds for some $c \in X$.

Properties (3.1), (3.4) and (3.6) of the continuum K imply that $Ab \cap F(ac) \neq \emptyset$. Since $b \in F(ac)$ by assumption, we have $a \in F(ac)$ according to (2.5) and (II). (2.1) implies (3.7).

Finally, the set $ac - Ca$ being the *composant* of a in ac (see [4], p. 208 for this concept), we have

(3.11) $$ac - Ca = \bigcup_j K_j,$$

where the continua K_j all satisfy (3.10) in view of (3.6). Then $Ab \cap F(K_j) = \emptyset$ for all j, as above by (2.5) and (II). Since $\bigcup_j F(K_j) = F(\bigcup_j K_j)$ by the definition of images, it follows by (3.11) that $Ab \cap F(ac - Ca) = \emptyset$.

THEOREM 1. *For every c-function F satisfying* (II) *and* (III) *and for every two points* $a, b \in X$ *satisfying* $b \in F(a)$ *and* $Ab \cap F(Ab) = \emptyset$, *there exists a fixed point c of F such that* $ab \prec ac$.

Proof. It follows from Lemma 2 that Theorem II of [5] holds under asumptions (II) and (III) (see [5], p. 118–119, (15)–(17)). The two remaining auxiliary theorems, I and III, of [5] are both derived there from the two properties (II) and (III). Hence (see [6], p. 762–763) the theorem follows.

THEOREM 2. *Every c-function F satisfying* (II) *and* (III) *has a fixed point* (*property* (I)).

Proof. Let $K \subset X$ be a continuum having property (3.2) and therefore satisfying (3.3) by Lemma 1. Then a is a fixed point of F when $a = b$. In the opposite case, $Ab \neq ab$ in view of (2.4). Since Ab is a subcontinuum of ab by (2.2) and (2.3), it follows by (3.2) and (3.3) that $Ab \cap F(Ab) = \emptyset$ and $b \in F(a)$. Thus, by Theorem 1, there exists a fixed point of F.

Remark 1. The theorem of Kuratowski remains true without any axiom of separability for connected and compact topological spaces in which every closed and connected set with a non-empty interior is decomposable (see [7], p. 52). Then also the above fixed point theorems hold for generalized non-metric λ-dendroids where sequences are replaced by transfinite sequences (see [5], p. 109, Remark 2, for the remaining statement which is needed in the fixed point theorems).

§ 4. Fixed point theorems for *c*-functions satisfying (II) **or** (IV). In the fixed point theorem below we will consider two cases, $F(X) \subset X$ and $X \subset F(X)$, instead of considering a c-function F mapping X into itself or *onto* itself, i.e. such that $F(X) = X$.

Recall first that, for an arbitrary set-valued function F mapping X onto a space $F(X)$, we have for every $q \in F(X)$

(4.1) $$p \in F^{-1}(q) \quad \text{iff} \quad q \in F(p),$$

by the definition of the set $F^{-1}(q)$. Hence by the definition of images $F(K)$ for any $K \subset X$

(4.2) $$K \cap F^{-1}(q) \neq \emptyset \quad \text{iff} \quad q \in F(K),$$

whence it follows that $F^{-1}(q) \neq \emptyset$ iff $q \in F(X)$. Thus a set-valued function F^{-1} mapping $F(X)$ onto X is defined such that $(F^{-1})^{-1} = F$ by (4.1), i.e. that

(4.3) $$(F^{-1})^{-1}(p) = F(p) \quad \text{for all } p \in X.$$

THEOREM 3. *If* $F(X) \subset X$ *where F is a c-function satisfying* (II), *and for every* $q \in F(X)$ *the set* $F^{-1}(q)$ *is compact, then there exists a fixed point of F.*

Proof. We have to verify that F satisfies (III).

Let a sequence of continua $K_j \subset X$ be decreasing and let $K_j \cap F(K_j) \neq \emptyset$ for $j = 1, 2, \ldots$ Then by (II) the non-empty sets $K_j \cap F(K_j)$ are compact, and their sequence is decreasing by the definition of the images $F(K_j)$. It follows by a theorem of Cantor (see [4], p. 2) that $\bigcap_{j=1}^{\infty} (K_j \cap F(K_j)) \neq \emptyset$, i.e. that $(\bigcap_{j=1}^{\infty} K_j) \cap \bigcap_{j=1}^{\infty} F(K_j) \neq \emptyset$.

Now it suffices to verify that $\bigcap_{j=1}^{\infty} F(K_j) \subset F(\bigcap_{j=1}^{\infty} K_j)$.

Let $q \in \bigcap_{j=1}^{\infty} F(K_j)$, i.e. $q \in F(K_j)$ for all j. Then $K_j \cap F^{-1}(q) \neq \emptyset$ by (4.2). Since the sequence $K_j \cap F^{-1}(q)$ is decreasing and $F^{-1}(q)$ is compact by assumption, it follows by the theorem of Cantor that $\bigcap_{j=1}^{\infty} (K_j \cap F^{-1}(q)) \neq \emptyset$. Hence $(\bigcap_{j=1}^{\infty} K_j) \cap F^{-1}(q) \neq \emptyset$, and therefore by (4.2), $q \in F(\bigcap_{j=1}^{\infty} K_j)$.

Thus F satisfies (III) and, (II) being satisfied by assumption, Theorem 2 holds.

Remark 2. The general statement is true for any set-valued function F mapping a topological space X into a set Y: *the following two conditions are equivalent*

1^0 $F^{-1}(q)$ *is countably compact for every* $q \in F(X)$,

2^0 $F(\bigcap_{j=1}^{\infty} A_j) = \bigcap_{j=1}^{\infty} F(A_j)$ *for every decreasing sequence of closed sets* $A_j \subset X$.

This equivalence remains true replacing 1^0 *"compact" by "closed" and in* 2^0 *"closed" by "countably compact".*

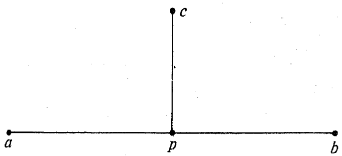
THEOREM 4. *If* $X \subset F(X)$ *where F is a c-function satisfying* (IV) *mapping X onto a space* $F(X)$, *then there exists a fixed point of F.*

Proof. $F^{-1}(X) \subset X$ by the definition of the set $F^{-1}(X)$, and in view of (4.3) the c-function F^{-1} satisfies (II) by the assumed property (IV) of F. Also by (4.3), $(F^{-1})^{-1}(p)$ is compact for every $p \in F^{-1}(X)$, $F(p)$ being a continuum by the assumption that F is a c-function. Thus, by applying Theorem 3 to the c-function F^{-1}, we find that there exists a fixed point of F^{-1} which is a fixed point of F in view of (4.1).

COROLLARY. *If* $F(X) \subset X$ *or* $X \subset F(X)$ *for a c-function F satisfying* (II) *and* (IV), *then there exists a fixed point of F.*

The above corollary is a generalization of [9] (see also [10], p. 599) to λ-dendroids in the domain of continua, some generalizations being possible also for semi-continua (see [4], p. 188 for this concept). Simultaneously, it extends an analogous theorem of [6] (p. 765, Theorem 3) to the case where the c-function is non-semi-continuous (the main Theorem 2 of [6] being derived from the above properties (II) and (III) only). Here observe that Theorem 4 above cannot be proved for c-functions satisfying (II) and (III) only. This is shown by the following simple

EXAMPLE 1. Let X be a segment ab in a Euclidean plane with end points a and b and let p be an interior point of ab. Now let ac denote an arbitrary arc lying in the plane such that $ac \cap ab = ap$ and $p \neq c$.



Then no function f mapping ap onto bc and pb onto ca has a fixed point if $f(p) = c$, because $ap \cap bc = (p) = pb \cap ca$. However, f is continuous if both the images $f(ap)$ and $f(pb)$ are continuous.

In particular, for continuous monotone functions the above corollary is in fact proved in a paper by Gray (see [1], p. 503). The question arises whether the method of his proof can be extended to set-valued functions.

§ 5. End continua and fixed points. Every end continuum E of X has by definition the property that for every irreducible continuum $ab \subset X$

(5.1) $$ab \cap E \neq \emptyset \neq ab - E \quad \text{imply} \quad E = Ab \text{ or } E = Ba\,,$$

and thus

(5.2) $$a \in E \text{ and } b \in X - E \quad \text{imply} \quad E = Ab\,,$$

which implies (V) directly by (2.1) and (2.3). The properties (5.1), (5.2) and (V) are of course not equivalent (see § 6, Examples 2 and 3). Now observe that (V) means exactly that E is nowhere dense in every other subcontinuum of X which contains E; namely, the following lemma holds:

LEMMA 3. *A subcontinuum E of X has property* (V) *if and only if*

(5.3) $$a \in E \text{ and } b \in X - E \quad \text{imply} \quad E \subset Ab\,.$$

Proof. Assume (V) to be satisfied and let

(5.4) $$a \in E \quad \text{and} \quad b \in X - E\,.$$

Then by (V),

(5.5) $$E \subset ab\,,$$

and now if we set

(5.6) $$K = \overline{ab - E}\,,$$

condition (V) will be applied to the continuum K (see [4], p. 193, Theorem 3). Since $E \cup K = ab$ by (5.5) and (5.6), and ab is connected, we have $K \cap E \neq \emptyset$. Simultaneously $K - E \neq \emptyset$, being $b \in ab - E$ by (5.4); thus $E \subset K$.

In this way it is proved that $E - K = \emptyset$, i.e. in view of (5.6), that E is a nowhere dense subcontinuum of ab containing the point a by (5.4). Since Ab is a subcontinuum of ab in view of (2.2) and (2.3), and thus a maximal subcontinuum of ab containing a and nowhere dense in ab (see [3], p. 243, Corollary 2), it follows that $E \subset Ab$, i.e. (5.3).

The converse statement follows directly from (2.1) and (2.3).

THEOREM 5. *For every c-function F satisfying* (II) *and* (III) *mapping X into itself, and for every subcontinuum E of X such that $E \cap F(E) \neq \emptyset$ and* (V), *there exists a fixed point belonging to E.*

Proof. It follows from the assumed inequality (3.1) for the continuum E that E contains a subcontinuum

(5.7) $$K \subset E$$

satisfying (3.1) and (3.2), and thus also (3.3) by Lemma 1.

If $a \neq b$, then $Ab \neq ab$ in view of (2.4). Hence, according to (3.3),

(5.8) $$Ab \subset K \quad \text{and} \quad Ab \neq K\,,$$

and hence $Ab \cap F(Ab) = \emptyset$ by (3.2). Since $b \in F(a)$ by (3.3), it follows by Theorem 1 that there exists a fixed point c of F such that $ab \multimap ac$. Hence $Ac = Ab$ by (2.6). It follows by (5.7) and (5.8) that $E - Ac \neq \emptyset$. Since $a \in E$ by (5.7) and (3.3), we have $c \in E$ by (5.3) in Lemma 3.

Theorem 5 above yields a partial answer to the problem raised in [6] (p. 766).

§ 6. Essentiality of assumptions. The essentiality of the assumptions in Theorems 1–5 will first be proved for an essential part of them, namely for the following two implications:

$$(\text{II}) \wedge (\text{IV}) \Rightarrow (\text{III}) \wedge (\text{II}) \Rightarrow (\text{I})\,.$$

The essentiality of (II) is shown by the function f_1 defined on the unit interval $I = \{t\colon 0 \leqslant t \leqslant 1\}$ of the real line by the formula $f_1(t) = (t+\frac{1}{2}) \bmod 1$, i.e.

(6.1) $$f_1(t) = \begin{cases} t+\frac{1}{2} & \text{for} \quad t<\frac{1}{2}\,, \\ t-\frac{1}{2} & \text{for} \quad \frac{1}{2} \leqslant t\,. \end{cases}$$

The function f_1 does not have property (I), and (III) is satisfied.

Indeed, let a sequence of continua $K_j \subset I$ be decreasing and let $K_j \cap f_1(K_j) \neq \emptyset$ for $j = 1, 2, \ldots$ Suppose first that $1 \notin K_j$ for all j (greater than some integer), so that the inverse images $f_1^{-1}(K_j)$ are compact by (6.1). Hence $\bigcap_{j=1}^{\infty} (K_j \cap f_1^{-1}(K_j)) \neq \emptyset$ by the Cantor theorem, and therefore $(\bigcap_{j=1}^{\infty} K_j) \cap f_1^{-1}(\bigcap_{j=1}^{\infty} K_j) \neq \emptyset$. It follows that $(\bigcap_{j=1}^{\infty} K_j) \cap f_1(\bigcap_{j=1}^{\infty} K_j) \neq \emptyset$ in view of (4.2). This inequality holds also in the case where $1 \in K_j$ for all j, directly by (6.1).

The essentiality of (III) is proved by the function defined on the same unit interval I by the formula

$$(6.2)\qquad f_2(t) = \begin{cases} \omega(t) & \text{for} \quad \omega(t) \neq t, \\ f_1(\omega(t)) & \text{for} \quad \omega(t) = t, \end{cases}$$

which actually does not have property (I), and satisfies (II) if ω denotes the function of Cesáro (see [4], p. 131) defined on the closed interval. Indeed, ω attains every value between 0 and 1 in every interval (ibidem); thus f_2 also has this property (see also [2], p. 20 for another function with this property).

The essentiality of (IV) can be stated by the function defined on the closed interval $J = \{t\colon\ -1 \leqslant t \leqslant 1\}$ by the formula

$$(6.3)\qquad f_3(t) = \begin{cases} \sin \dfrac{1}{t} & \text{for } t \neq 0, \\ s & \text{for } t = 0, \end{cases}$$

where s denotes an arbitrary number from J.

The function f_3 satisfies (II), in view of $f_3([-\varepsilon, 0]) = J = f_3([0, \varepsilon])$ for every $\varepsilon > 0$, and hence whenever $s \neq 0$, f_3 does not have property (III) either.

The essentiality of (IV), (III) and (II) will now be considered with respect to the following conditions:

(VI) *for every closed set $B \subset F(X)$ the set $F^{-1}(B)$ is closed,*

(VII) *for every $q \in F(X)$ the set $F^{-1}(q)$ is closed,*

(VIII) *if a sequence of continua $K_j \subset X$ is decreasing, then* $F(\bigcap_{j=1}^{\infty} K_j) = \bigcap_{j=1}^{\infty} F(K_j)$.

Property (VI) means of course the upper semi-continuity of F, which for (single-valued) functions simply becomes continuity, and the following relations hold in view of Remark 2:

$$(\mathrm{VI}) \Rightarrow (\mathrm{VII}) \Rightarrow (\mathrm{VIII}).$$

These two implications are essential, is shown, in particular, by the following two examples of functions.

Example 2. Let X be the closure in a Euclidean plane of the diagram of the function f_3, and let E denote the segment of condensation of X, i.e. the segment of the y-axis with end points -1 and 1. Then the formula

$$(6.4)\qquad f_4(p) = \begin{cases} p & \text{for} \quad p \in X - E, \\ -p & \text{for} \quad p \in E \end{cases}$$

defines a non-continuous function mapping X onto itself and satisfying (II) and (IV).

Indeed, if a continuum $K \subset X$ is contained in E or in $X - E$, then the image $f_4(K)$ is a continuum because f_4 is continuous in each of these two sets E and $X - E$ separately. If $K \cap E \neq \emptyset \neq K - E$, then $E \subset K$ (property (V)), and then $f_4(K) = K$ by the definition of f_4. Thus f_4 satisfies (I), and hence also (IV) is satisfied, because $f_4^{-1} = f_4$ by (6.4).

Example 3. For X and E as above, let g denote the orthogonal projection of X onto E. Then the function

$$(6.5)\qquad f_5(p) = g(f_4(p)) \quad \text{for every } p \in X$$

does not have property (VII). Namely, if we take for q the point 1 of the y-axis, the inverse-image $f_5^{-1}(q)$ is not closed because $f_5^{-1}(q) = f_4^{-1}(g^{-1}(q))$. To prove (VIII) for the function f_5 it suffices to proceed in the same way as in the preceding example, considering images $f_5(K_j)$ for a decreasing sequence of continua $K_j \subset X$.

The function f_5 satisfies (II) as a composition of two functions satisfying (II).

Property (V) of the subcontinuum E of X is essential in Examples 2 and 3; it follows here by (5.2), of course. Approximating this plane continuum X by a topological ray (as in [3], p. 266, Lemma 1) we define a continuum by adding this ray to X, where the segment E does not have property (5.2) and satisfies (V) in view of Lemma 3. However, these two examples can be extended to such a large continuum.

In general, the following holds:

Theorem 6. *If X contains a subcontinuum E which satisfies* (V) *nontrivially, then there exists a non semi-continuous c-function F, mapping X onto itself, with the properties* (II) *and* (IV).

Proof. E being non-trivial, it is a proper subset of X and it contains more than one point. Setting for an arbitrary $a \in E$

$$F(p) = \begin{cases} (p) & \text{for} \quad p \in X - E, \\ (a) & \text{for} \quad p \in E - (a), \\ E & \text{for} \quad p = a \end{cases}$$

we define a c-function F mapping X into itself.

F is non semi-continuous. Since E is nowhere dense in X by Lemma 3, there exists a convergent sequence $p_j \in X - E$ with $\lim_j p_j \in E - (a)$. Then $F(\lim_j p_j) = (a)$ and $F(p_j) = (p_j)$ by the definition of F, and hence $\mathrm{Lt}\, F(p_j) = \lim p_j$. It follows that $\mathrm{Lt}\, F(p_j) \cap F(\lim p_j) = \emptyset$, which proves the non semi-continuity of F (see [4], p. 61–62, Theorems 1, 2).

F satisfies (II). If a continuum $K \subset X$ satisfies $K \subset X - E$ or $K \subset E$, then the image $F(K)$ is a continuum: K, E or (a) by the definition of F. If $K \cap E \neq \emptyset \neq K - E$, then $E \subset K$ by (V), and hence $F(K) = K$.

F satisfies IV. It suffices to verify, regarding (4.1), that $F^{-1} = F$.

Remark 3. Such examples as 2 and 3 above are impossible among c-functions defined on the real line, and similarly Theorem 6 is not true for X locally connected. Indeed, for every set-valued function F mapping a locally connected and locally compact metric space X into a topological space Y such that

(IX) *for every continuum $K \subset X$ the image $F(K)$ is compact* the properties (VI), (VII) and (VIII) are equivalent, i.e.

$$(\mathrm{VIII}) \Rightarrow (\mathrm{VI}).$$

In fact, for every $p \in X$, every sufficiently small neighbourhood of p has a countable, regular base of regions (see [4], p. 231, Theorem 8). Hence there exists a decreasing sequence of continua $K_j \subset X$ such that

$$(p) = \bigcap_{j=1}^{\infty} K_j \quad \text{and} \quad p \in \operatorname{Int} K_j \quad \text{for all } j .$$

Then $F(p) = \bigcap_{j=1}^{\infty} F(K_j)$ by virtue of (VIII). Therefore by (IX), for every open set $V \subset Y$ such that $F(p) \subset V$ there exists a j' such that $F(K_{j'}) \subset V$. (VI) follows.

PROBLEM. Can the c-function in Theorem 6 be replaced by a single-valued function?

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Products of perfectly normal spaces

by

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Abstract. Answering a question raised by R. W. Heath, we construct, assuming Continuum Hypothesis, for every natural n a separable and first countable space X such that

(a) X^n is perfectly normal;

(b) X^{n+1} is normal but X^{n+1} is not perfect.

The space X (and X^{n+1}) has cardinality ω_1 and can be made either Lindelöf or locally compact and locally countable.

We show that the existence of such spaces is independent of the axioms of set theory.

§ 1. Introduction. In 1969 R. W. Heath [7] raised a question whether for $n \geqslant 2$ there exist spaces X such that X^n is perfect but X^{n+1} is not. This question has also been repeated by D. Burke and D. Lutzer in [2] and has been brought to the author's attention by Eric van Douwen.

In this paper we give a positive answer to this question, constructing, under the assumption of the Continuum Hypothesis (CH), the following two examples.

EXAMPLE 1. (CH) For every $n<\omega$ there exists a first countable, locally compact, locally countable space X of cardinality ω_1 such that:

(a) X^n is perfectly normal and hereditarily separable;

(b) X^{n+1} is normal but X^{n+1} is not hereditarily normal (1).

EXAMPLE 2. (CH) For every $n<\omega$ there exists a first countable space X of cardinality ω_1 such that:

(a) X^n is hereditarily Lindelöf and hereditarily separable;

(b) X^{n+1} is Lindelöf but X^{n+1} is not hereditarily Lindelöf.

This paper is closely related to our paper [16] where, under the assumption of Martin's Axiom, positive results concerning the preservation of perfectness and perfect normality in product spaces are given.

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(1) Let us recall that a perfectly normal space is hereditarily normal and that a Lindelöf space is perfect if and only if it is hereditarily Lindelöf.