COROLLARY. If a dendroid \( X \) contains a \( Q \)-point \( p \) such that (18) holds, then for every continuous selection \( \alpha : \mathcal{C}(X) \to X \) we have \( \sigma(K) = p \) (here \( K \) denotes the limit continuum mentioned in the definition of a \( Q \)-point).

The author does not know if condition (18) is essential in the corollary, i.e., if there exists a selectable dendroid containing a \( Q \)-point \( p \) for which (18) fails and with \( \sigma(K) \neq p \). Recently Mr. S. T. Czaub has found a dendroid (even a fan) with a \( Q \)-point \( p \) for which (18) does not hold, but this example is not selectable.

Consider now the dendroid \( D_2 \), described in [5], p. 305. Let \( X \) be a continuum obtained from \( D_2 \) by shrinking the horizontal straight line segment of \( D_2 \) to which the points \( p_1, p_2, \ldots \) belong (see the picture of \( C_\rho \), Fig. 1 on p. 305 of [5]) to a point \( p \). It is evident that \( X \) is a countable plane fan with a \( Q \)-point \( p \). As it was recently shown by Dr. T. Mackowiak, the fan \( X \) is selectable. Thus the existence of a \( Q \)-point in a countable plane fan \( X \) does not imply that \( X \) is not selectable.

References

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Fixed point theorems for \( \lambda \)-dendroids

by

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Abstract. Fixed point theorems are proved for functions whose values and the images as well as the inverse-images of continua are continua.

§ 1. Introduction. Throughout this paper \( X \) will denote an arbitrary \( \lambda \)-dendroid, i.e. a hereditarily decomposable and hereditarily unicoherent metric continuum. We shall consider, under the name \( \epsilon \)-functions, functions having continua \( F(p) \subseteq X \) as values, non-empty for all \( p \in X \). If \( p \in F(p) \), then the point \( p \) will be called a fixed point of \( F \).

In [5] I proved that if \( F \) is upper semi-continuous, then

(I) there exists a fixed point of \( F \).

With the aid of papers [5] and [6] we prove here that the fixed point theorem (I) holds under the same remaining assumptions, even without the upper semi-continuity of the function \( F \).

First, a stronger fixed point theorem (§ 3, Theorem 1) is proved under the following two conditions ([5], p. 113, (I) and (II)):

(II) for every continuum \( K \subseteq X \) the image \( F(K) \subseteq X \) is a continuum,

(III) the property \( K \cap F(K) \neq \emptyset \) is inductive for continua \( K \subseteq X \), where the image \( F(K) \) means the union \( \bigcup \{ F(p) : p \in K \} \), and a property is called inductive provided that for every decreasing sequence of sets having this property their common part also has this property (see [6], p. 54). Then we prove a theorem stating exactly that (I) follows from (II) and (III) (§ 3, Theorem 2), which is next applied to considering the following condition on the sets \( F^{-1}(K) = \{ p \in X : F(p) \cap K \neq \emptyset \} \): (IV) for every continuum \( K \subseteq F(K) \) the set \( F^{-1}(K) \) is a continuum.

Namely, fixed point theorems are proved for \( \epsilon \)-functions satisfying (IV) or (II) (§ 4, Theorems 3 and 4), which imply a common generalization of the fixed point theorems by Gray [1], by the present author [6] and by Smithson [9], [10] (§ 4, Corollary).

Finally the following property of a non-degenerated subcontinuum \( E \subseteq X \) will be considered:
§ 3. Fixed point theorems for c-functions satisfying (II) and (III). By virtue of (III), there exists a continuum $K \subset X$ such that

\[(1.1) \quad K \cap F(K) \neq \emptyset.\]

(3.2) the continuum $K \subset X$ is irreducible with respect to (3.1)

(see [4], p. 54). Relations between such subcontinua $K$ of $X$ lead to the existence of a fixed point of $F$ (see [5], p. 120, proof of the corollary). For the, following statement, valid for any set X, points to the possibility of using relations between irreducible subcontinua:

Lemma 1. For an arbitrary set-valued function $F$ mapping $X$ into itself and for every continuum $K \subset X$ the condition (3.2) implies

\[(3.3) \quad K = ab \quad \text{and} \quad b \in F(a) \quad \text{for some} \ a, b \in X.\]

Proof. For every point $b \in F(K) \cap K$ there exists a point $a \in K$ such that $b \in F(a)$ by the definition of the image $F(X)$, and simultaneously $b \in K$. Hence any irreducible subcontinuum $ab \subset K$ (see [4], p. 192, Theorem 1, here (2.1) for the $\lambda$-dendroid $X$) satisfies (3.1). (3.2) implies (3.3).

Lemma 1 will be used in the sequel. Now, we will apply Kuratowski’s theorem on the point of irreducibility to the proof of the following

Lemma 2. Let $a \in X$, $b \in F(a)$ and $Ab \cap F(\lambda b) \neq \emptyset$ for a c-function $F$ satisfying (II) and (III). Then there exists a subcontinuum $K \subset X$ such that

\[(3.4) \quad a \in K \subset Ab,\]

(3.5) the continuum $K$ is irreducible with respect to (3.1) and (3.4),

and then there exists a point $c \in X$ such that

\[(3.6) \quad ac = K,\]

\[(3.7) \quad ab \subset F(ac),\]

\[(3.8) \quad Ab \cap (F(ac - Ca)) = \emptyset.\]

Proof. Since $Ab$ is a continuum and $a \in Ab$ in view of (2.2) and (2.3), and $Ab$ satisfies (3.1) by assumption, there exists a continuum $K$ satisfying (3.5) by virtue of (III).

To prove (3.6) for some point $c \in X$, suppose the contrary. Then by the theorem of Kuratowski, there exist two continua $K_1 \subset X$ such that

\[(3.9) \quad K_1 \cup K_2 = K,\]

\[(3.10) \quad a \in K_1 \quad \text{and} \quad K_1 \neq K.\]

Since $b \in F(a)$ by assumption, it follows that $b \in F(K_1)$, and by (3.5) $K_1 \cap F(K_1) = \emptyset$, whence $a \notin F(K_1)$. Since $F(K_1)$ is a continuum by (II), it follows according to (2.5) that $Ab \cap F(K_1) = \emptyset$. Therefore $Ab \cap F(K_2) = F(K_1) \cup F(K_2)$ by the definition of the images, we have $Ab \cap F(K) = \emptyset$ by (3.9), which contradicts (3.1) or (3.4).
Thus (3.6) holds for some $c \in X$.

Properties (3.1), (3.4) and (3.6) of the continuum imply that $A \cap F(ac) \neq \emptyset$.

Since $b \in F(ac)$ by assumption, we have $a \in F(ac)$ according to (2.5) and (II). (2.1) implies (3.7).

Finally, the set $ac - Ca$ being the component of $a$ in $ac$ (see [4], p. 208 for this concept), we have

\[ ac - Ca = \bigcup_{j=1}^{n} K_{j}, \]

where the continua $K_j$ all satisfy (3.10), in view of (3.6). Then $A \cap F(K_j) = \emptyset$ for all $j$, as above by (2.5) and (II). Since $\bigcup_{j} F(K_j) = F(\bigcup_{j} K_j)$ by the definition of images, it follows by (3.11) that $A \cap F(ac - Ca) = \emptyset$.

**Theorem 1.** For every $c$-function $F$ satisfying (II) and (III), and for every two points $a, b \in X$ satisfying $b \in F(a)$ and $A \cap F(ab) = \emptyset$, there exists a fixed point $c$ of $F$ such that $ab = ac$.

**Proof.** It follows from Lemma 2 that Theorem II of [5] holds under assumptions (II) and (III) (see [5], p. 188–190, 151–171). The two remaining auxiliary theorems, I and III, of [5] are both derived from the two properties (II) and (III). Hence (see [6], p. 762–763) the theorem follows.

**Theorem 2.** Every $c$-function $F$ satisfying (II) and (III) has a fixed point (property (I)).

**Proof.** Let $K = X$ be a continuum having property (3.2) and therefore satisfying (3.3) by Lemma 1. Then $a$ is a fixed point of $F$ when $a = b$. In the opposite case, $A \neq ab$ in view of (2.4). Since $ab$ is a subcontinuum of $ab$ by (2.2) and (2.3), it follows by (3.2) and (3.3) that $A \cap F(ab) = \emptyset$ and $b \in F(a)$. Thus, by Theorem 1, there exists a fixed point of $F$.

**Remark 1.** The theorem of Kuratowski remains true without any axiom of separability for connected and compact topological spaces in which every closed and connected set with a non-empty interior is decomposable (see [7], p. 53). Then also the above fixed point theorems hold for generalized non-metric $\lambda$-dendroids where sequences are replaced by transfinite sequences (see [5], p. 109, Remark 2, for the remaining statement which is needed in the fixed point theorems).

**§ 4. Fixed point theorems for $c$-functions satisfying (II) or (IV).** In the fixed point theorem below, we will consider two cases, $F(X) = X$ and $X \subset F(X)$, instead of considering a $c$-function $F$ mapping $X$ onto itself or onto itself, i.e., such that $F(X) = X$.

Recall that, for an arbitrary $c$-function $F$ mapping $X$ onto a space $F(X)$, we have for every $x \in F(X)$

\[ p \in F^{-1}(x) \iff x \in F(p), \]

by the definition of the $F^{-1}(x)$. Hence by the definition of images $F(K)$ for any $K \subset X$

\[ \forall x \in X \quad \forall p \in F^{-1}(x) \iff x \in F(p), \]

whence it follows that $F^{-1}(x) \neq \emptyset$ if $x \in F(X)$, i.e., that a set-valued function $F^{-1}$ mapping $F(X)$ onto $X$ is defined such that $(F^{-1})^{-1} = F$ by (4.1), i.e., that $F^{-1}$ is a function $F^{-1}(p) = \{p \in F^{-1}(x) \mid x \in F(p)\}$ for all $p \in F(X)$.

**Theorem 3.** If $F(X) = X$ where $F$ is a $c$-function satisfying (II), and for every $q \in F(X)$, the set $F^{-1}(q)$ is compact, then there exists a fixed point of $F$.

**Proof.** We have to verify that $F$ satisfies (II).

Let a sequence of continua $K_j \subset X$ be decreasing and let $K_j \cap F(K_j) \neq \emptyset$ for $j = 1, 2, \ldots$. Then by (II) the non-empty sets $K_j \cap F(K_j)$ are compact, and their sequence is decreasing by the definition of the images $F(K_j)$. It follows by a theorem of Cantor (see [4], p. 208) that \( \bigcap_{j=1}^{\infty} (K_j \cap F(K_j)) \neq \emptyset \), i.e., that \( (\bigcap_{j=1}^{\infty} K_j) \cap F(K_j) \neq \emptyset \).

Now it suffices to verify that \( \bigcap_{j=1}^{\infty} F(K_j) = F(\bigcap_{j=1}^{\infty} K_j) \).

Let $q \in F(K_j)$, i.e., $q \in F(K_j)$ for all $j$. Then $K_j \cap F^{-1}(q) \neq \emptyset$ by (4.2). Since the sequence $K_j \cap F^{-1}(q)$ is decreasing and $F^{-1}(q)$ is compact by assumption, it follows by the theorem of Cantor that \( \bigcap_{j=1}^{\infty} (K_j \cap F^{-1}(q)) \neq \emptyset \). Hence $F^{-1}(q) = \emptyset$, and therefore by (4.2), $q \in F(\bigcap_{j=1}^{\infty} K_j)$.

Thus $F$ satisfies (II) and, (II), being satisfied by assumption, Theorem 2 holds.

**Remark 2.** The general statement is true for any set-valued function $F$ mapping a topological space $X$ into a set $Y$: the following two conditions are equivalent:

\[ F^{-1}(y) \] is countably compact for every $y \in F(X)$,

\[ 2^{F(X)} \] is compact by “closed” and in $2^{F(X)}$ “closed” by “countably compact”.

**Theorem 4.** If $X \subset F(X)$ where $F$ is a $c$-function satisfying (IV) mapping $X$ onto a space $F(X)$, then there exists a fixed point of $F$. 

**Proof of $F^{-1}(X) \subset X$ by the definition of the set $F^{-1}(X)$, and in view of (4.3) the $c$-function $F^{-1}$ satisfies (II) by the assumed property (IV) of $F$. Also by (4.3), $(F^{-1})^{-1}(p)$ is compact for every $p \in F^{-1}(X)$, $F(p)$ being a continuum by the assumption that $F$ is a $c$-function. Thus, by applying Theorem 3 to the $c$-function $F^{-1}$, we find that there exists a fixed point of $F^{-1}$ which is a fixed point of $F$ in view of (4.1).

**Corollary.** If $F(X) \subset X$ or $X \subset F(X)$ for a $c$-function $F$ satisfying (II) and (IV), then there exists a fixed point of $F$.

The above corollary is a generalization of [9] (see also [10], p. 599) to $\lambda$-dendroids in the domain of continua, some generalizations being possible also for semi-continua (see [4], p. 188 for this concept). Simultaneously, it extends an analogous
theorem of [6] (p. 765, Theorem 3) to the case where the e-function is non-semi-continuous (the main Theorem 2 of [6] being derived from the above properties (II) and (III) only). Here observe that Theorem 4 above cannot be proved for e-functions satisfying (II) and (III) only. This is shown by the following simple

**Example 1.** Let $X$ be a segment $ab$ in a Euclidean plane with end points $a$ and $b$ and let $p$ be an interior point of $ab$. Now let $ac$ denote an arbitrary arc lying in the plane such that $ac \cap ab = ap$ and $p \neq c$.

Then no function $f$ mapping $ap$ onto $bc$ and $pb$ onto $ca$ has a fixed point if $f(p) = c$, because $ap \cap bc = (p) = pb \cap ca$. However, $f$ is continuous if both the images $f(ap)$ and $f(pb)$ are continuous.

In particular, for continuous monotone functions the above corollary is in fact proved in a paper by Gray (see [1], p. 450). The question arises whether the method of his proof can be extended to set-valued functions.

**§ 5. End continua and fixed points.** Every end continuum $E$ of $X$ has by definition the property that for every irreducible continuum $ab \subset X$

\[(5.1)\]  
$ab \cap E \neq \emptyset \neq ab - E$ imply $E = Ab$ or $E = Ba$,

and thus

\[(5.2)\]  
$a \in E$ and $b \in X - E$ imply $E = Ab$,

which implies (V) directly by (2.1) and (2.5). The properties (5.1), (5.2) and (V) are of course not equivalent (see § 6, Examples 2 and 3). Now observe that (V) means exactly that $E$ is nowhere dense in every other subcontinuum of $X$ which contains $E$; namely, the following lemma holds:

**Lemma 3.** A subcontinuum $E$ of $X$ has property (V) if and only if

\[(5.3)\]  
$a \in E$ and $b \in X - E$ imply $E \subset Ab$.

**Proof.** Assume (V) to be satisfied and let

\[(5.4)\]  
$a \in E$ and $b \in X - E$.

Then by (V),

\[(5.5)\]  
$E \subset ab$,

and now if we set

\[(5.6)\]  
$K = \overline{ab - E}$,

condition (V) will be applied to the continuum $K$ (see [4], p. 435, Theorem 3). Since $E \cup K = ab$ by (5.5) and (5.6), and $ab$ is connected, we have $K \cap E = K \neq \emptyset$. Simultaneously $K - E \neq \emptyset$, being $b \in ab - E$ by (5.4); thus $E \subset K$.

In this way it is proved that $E \cap F(E) = \emptyset$, i.e. in view of (5.6), that $E$ is a nowhere dense subcontinuum of $ab$ containing the point a by (5.4). Since $ab$ is a subcontinuum of $ab$ in view of (2.1) and (2.3), and thus a maximal subcontinuum of $ab$ containing a and nowhere dense in $ab$ (see [3], p. 453, Corollary 2), it follows that $E = Ab$, i.e. (5.3).

The converse statement follows directly from (2.1) and (2.3).

**Theorem 5.** For every e-function $E$ satisfying (II) and (III) mapping $X$ into itself, and for every subcontinuum $E$ of $X$ such that $E \cap F(E) = \emptyset$ and (V), there exists a fixed point belonging to $E$.

**Proof.** It follows from the assumed inequality (3.1) for the continuum $E$ that $E$ contains a subcontinuum $K \subset E$

satisfying (3.1) and (3.2), and thus also (3.3) by Lemma 1.

If $a \neq b$, then $E \neq F(E)$ in view of (2.4). Hence, according to (3.3),

\[(5.8)\]  
$E = K$ and $F(E) = \emptyset$,

and hence $E \cap F(E) = \emptyset$ by (3.2). Since $b \in F(E)$ by (3.3), it follows by Theorem 1 that there exists a fixed point $c$ of $F$ such that $ab = ac$. Hence $Ac = Ab$ by (2.6). It follows by (5.7) and (5.8) that $E = Ac \neq \emptyset$. Since $a \in E$ by (5.7) and (3.3), we have $c \in E$ by (3.3) in Lemma 3.

Theorem 5 above yields a partial answer to the problem raised in [6] (p. 766).

**§ 6. Essentiality of assumptions.** The essentiality of the assumptions in Theorems 1–5 will first be proved for an essential part of them, namely for the following two implications:

\[(II) \land (IV) \implies (III) \land (II) = (I)\]

The essentiality of (II) is shown by the function $f_1$ defined on the unit interval $I = \{ t \mid 0 \leq t < 1 \}$ of the real line by the formula $f_1(t) = (t^{+}+1) \bmod 1$, i.e.

\[(6.1)\]  
$f_1(t) = \begin{cases} t^{+}+1 & \text{for } t < \frac{1}{2} \\ t & \text{for } t \geq \frac{1}{2} \end{cases}$

The function $f_1$ does not have property (I), and (III) is satisfied.

Indeed, let a sequence of continua $K_j = I$ be decreasing and let $K_j \cap f_1(K_j) \neq \emptyset$ for $j = 1, 2, 3, \ldots$ Suppose first that $1 \notin K_j$ for all $j$ (greater than some index), so that the inverse images $f_1^{-1}(K_j)$ are compact by (6.1). Hence

\[(6.2)\]  
$\bigcap_{j=1}^{\infty} f_1^{-1}(K_j) = \emptyset$.

by the Cantor theorem, and therefore

\[(6.3)\]  
$\bigcap_{j=1}^{\infty} f_1^{-1}(K_j) = \emptyset$.

It follows that

\[(6.4)\]  
$\bigcap_{j=1}^{\infty} f_1^{-1}(K_j) = \emptyset$.

This inequality holds also in the case where $1 \in K_j$ for all $j$, directly by (6.1).
The essentiality of (III) is proved by the function defined on the same unit interval $I$ by the formula

$$f_3(t) = \begin{cases} \sin \frac{t}{t} & \text{for } t \neq 1, \\ \frac{1}{t} & \text{for } t = 1, \end{cases}$$

which actually does not have property (I), and satisfies (II) if $\omega$ denotes the function of Cesàro (see [4], p. 131) defined on the closed interval. Indeed, $\omega$ attains every value between 0 and 1 in every interval (hence); thus $f_3$ also has this property (see also [2], p. 20 for another function with this property).

The essentiality of (IV) can be stated by the function defined on the closed interval $J = \{r: -1 \leq r \leq 1\}$ by the formula

$$f_4(t) = \begin{cases} \sin \frac{t}{t} & \text{for } t \neq 1, \\ \frac{1}{t} & \text{for } t = 1, \end{cases}$$

where $s$ denotes an arbitrary number from $J$.

The function $f_4$ satisfies (II), in view of $f_4([-e, 0]) = J = f_3(0, e)$ for every $e > 0$, and hence whenever $s \neq 0$, $f_4$ does not have property (III) either.

The essentiality of (IV), (III) and (II) will now be considered with respect to the following conditions:

(VI) for every closed set $B \subseteq X$ the set $F^{-1}(B)$ is closed,

(VII) for every $q \in X$ the set $F^{-1}(q)$ is closed,

(VIII) if a sequence of continua $K_j \subseteq X$ is decreasing, then $F(\cap K_j) = \cap F(K_j)$.

Property (VI) means of course the upper semi-continuity of $F$, which for (single-valued) functions simply becomes continuity, and the following relations hold in view of Remark 2:

$$\text{VI} \Rightarrow \text{VII} \Rightarrow \text{VIII}.$$

These two implications are essential, is shown, in particular, by the following two examples of functions.

**Example 2.** Let $X$ be the closure in a Euclidean plane of the diagonal of the function $f$, and let $E$ denote the segment of condensation of $X$, i.e. the segment of the $y$-axis with end points $-1$ and 1. Then the formula

$$f_X(p) = \begin{cases} p & \text{for } p \in X, \\ -p & \text{for } p \not\in X, \end{cases}$$

defines a non-continuous function mapping $X$ onto itself and satisfying (II) and (IV).

Indeed, if a continuum $K \subseteq X$ is contained in $E$ or is in $X - E$, then the image $f_3(K)$ is a continuum because $f_3$ is continuous in each of these two sets $E$ and $X - E$ separately. If $K \subseteq E \not\subseteq K - E$, then $E \cup K$ (property (V)), and then $f_3(K) = K$ by the definition of $f_3$. Thus $f_3$ satisfies (I), and hence also (IV) is satisfied, because $f_3^{-1} = f_3$ by (6.4).
In fact, for every $p \in X$, every sufficiently small neighbourhood of $p$ has a countable, regular base of regions (see [4], p. 231, Theorem 8). Hence there exists a decreasing sequence of continua $K_j \subset X$ such that

$$ (p) = \bigcap_{j=1}^{\infty} K_j \quad \text{and} \quad p \in \text{Int} K_j \quad \text{for all} \quad j. $$

Then $F(p) = \bigcap_{j=1}^{\infty} F(K_j)$ by virtue of (VIII). Therefore by (IX), for every open set $V \subset Y$ such that $F(p) \subset V$ there exists a $j'$ such that $F(K_{j'}) \subset V$. (VI) follows.

**Problem.** Can the $c$-function in Theorem 6 be replaced by a single-valued function?

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**Products of perfectly normal spaces**

by

**Teodor C. Przymusiński (Warszawa)**

**Abstract.** Answering a question raised by R. W. Heath, we construct, assuming Continuum Hypothesis, for every natural $n$ a separable and first countable space $X$ such that

(a) $X^n$ is perfectly normal;
(b) $X^{n+1}$ is normal but $X^{n+1}$ is not perfect.

The space $X$ and $X^{n+1}$ has cardinality $\omega$, and can be made either Lindelöf or locally compact and locally countable.

We show that the existence of such spaces is independent of the axioms of set theory.

**§ 1. Introduction.** In 1969 R. W. Heath [7] raised a question whether for $n \geq 2$ there exist spaces $X$ such that $X^n$ is perfect but $X^{n+1}$ is not. This question has also been repeated by D. Burke and D. Lutzer in [2] and has been brought to the author’s attention by Eric van Douwen.

In this paper we give a positive answer to this question, constructing, under the assumption of the Continuum Hypothesis (CH), the following two examples.

**Example 1. (CH)** For every $n < \omega$ there exists a first countable, locally compact, locally countable space $X$ of cardinality $\omega$, such that:

(a) $X^n$ is perfectly normal and hereditarily separable;
(b) $X^{n+1}$ is normal but $X^{n+1}$ is not hereditarily normal (\textsuperscript{*}).

**Example 2. (CH)** For every $n < \omega$ there exists a first countable space $X$ of cardinality $\omega$, such that:

(a) $X^n$ is hereditarily Lindelöf and hereditarily separable;
(b) $X^{n+1}$ is Lindelöf but $X^{n+1}$ is not hereditarily Lindelöf.

This paper is closely related to our paper [16] where, under the assumption of Martin’s Axiom, positive results concerning the preservation of perfectness and perfect normality in product spaces are given.

\textsuperscript{*} This paper was originated while the author was a Visiting Assistant Professor at the University of Pittsburgh in 1976/77.

(\textsuperscript{*}) Let us recall that a perfectly normal space is hereditarily normal and that a Lindelöf space is perfect if and only if it is hereditarily Lindelöf.