Contractibility and continuous selections

by

J. J. Charatonik (Wrocław)

Abstract. Some relations are studied between the contractibility and the existence of a continuous selection on the hyperspace of all subcontinua of a dendroid. Also some necessary conditions are found for the existence of such a selection.

In what follows a continuum means a nonempty compact connected Hausdorff space. If X is a topological space, then 2^X denotes the space of all nonempty closed subsets of X with the Vietoris topology (see e.g. [11]; if X is a compact metric space, then the Vietoris topology agrees with the topology induced by Hausdorff metric). The subspace of closed and connected subsets of X is denoted $C(X)$. A continuous selection for a family $\mathscr{A} \subset 2^X$ is a continuous function $\sigma\colon \mathscr{A} \to X$ such that $\sigma(A) \in A$ for each $A \in \mathscr{A}$. If, in addition, the condition $\sigma(B) \in A \subset B$ implies $\sigma(A) = \sigma(B)$ for each $A, B \in \mathscr{A}$, then σ is called a rigid selection (see [14], p. 1041).

Kuratowski, Nadler and Young [10] have proved that if X is a metrizable continuum, then a continuous selection for 2^X exists if and only if X is an arc. If one seeks a continuous selection on $C(X)$, however, then such a simple characterization of those continua X is not known and it seems to be rather a hard problem. A very important approach to solve this question was made by Nadler and Ward [13] who proved that if a metrizable continuum X admits a continuous selection for $C(X)$, then X is a dendroid (recall that a dendroid means a metrizable continuum which is both arcwise connected and hereditarily unicoherent). In the same paper they shown that the class of metric continua X which admit a continuous selection on $C(X)$ is a proper subclass of the dendroids, but it is larger than the class of smooth dendroids (see [14] for details).

It is tempting — by examples showed in [13] — to conjecture that among the dendroids the existence of a continuous selection for the space of subcontinua is related in some way to the property of being contractible. Nadler asks [12] whether contractibility of dendroids implies the existence of a such selection. This paper does not answer the question, however, it is a contribution to the attempt to find some relations between discussed properties of dendroids.

For shortness we formulate the following definition. A continuum X is called *selectible* if the hyperspace $C(X)$ of its subcontinua admits a continuous selection.

Thus the above mentioned result of [13], Lemma 3, p. 370, can be reformulated as

Proposition 1 (Nadler and Ward). *Each metrizable selectible continuum is a dendroid.*

Proposition 2. *Each selectible dendroid is a continuous image of the Cantor fan, and therefore it is uniformly arcwise connected.*

Indeed, if X is a selectible dendroid, then it is a continuous image of the hyperspace $C(X)$ of its subcontinua under a selection $\sigma\colon C(X)\to X$. Further, $C(X)$ is in turn a continuous image of the Cantor fan F under a continuous mapping $f\colon F\to C(X)$ by the Kelley result ([8], Theorem 2.7, p. 25). The composite $f\sigma$ is the required mapping. The second part of the proposition follows from the Kuperberg result ([9], Theorem 3.5, p. 322).

Remark that Propositions 1 and 2 hold true if the term "a selectible continuum" is replaced by "a contractible curve" (here a curve means a continuum of dimension one) — see [4], Propositions 1, 4 and 5.

Proposition 3. *There exists a non-contractible and selectible plane dendroid having two ramification points.*

Proof. Let A_n be the line segment joining $(0,1)$ and $(2^{-n},0)$ in the plane, where $n=0,1,2,\dots$, and let T be the line segment joining $(0,1)$ and $(0,0)$. Let $D_1 = T\cup\bigcup\{A_n\colon n=0,1,2,\dots\}$ and put $D=D_1\cup D_2$, where D_2 is the reflection of D_1 about the origin (see Fig. 1). It is appeerent that D is a plane dendroid with

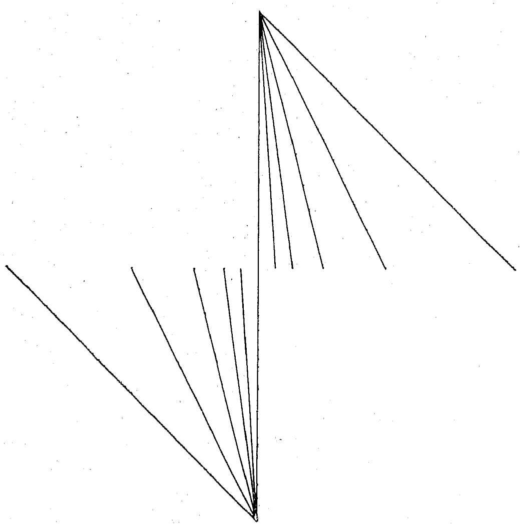

Fig. 1

two ramification points, and it follows that D is not contractible. In fact, either observe that the origin in an R-point (i.e. a degenerate R-arc) for D and apply Corollary 6 of [7], or apply the Bennett theorem ([1], Theorem 1, p. 47). We shall now define a continuous selection on $C(D)$. The idea comes from Nadler and Ward's proof of Theorem 3 of [13], and we include the description of the selection in matter in order to make the exposition reasonably self-contained. Note first that D_1 and D_2 are both smooth dendroids with initial points $(0,1)$ and $(0,-1)$ respectively and thus they admit closed partial orders Γ_1 and Γ_2. Further, it is known that each subcontinuum of D_i has a (unique) least element with respect to Γ_i (where $i=1$ or 2) (cf. [14], p. 1043) and hence according to Theorem 1 of [14], p. 1042, if $\sigma_i\colon C(D_i)\to D_i$ ($i=1$ or 2) is defined by $\sigma_i(A)=$ the least element of A (relative to Γ_i), then σ_i is a continuous selection. Moreover, by Lemma 4 of [13], p. 371, we have $\sigma_1(T)=(0,1)$ and $\sigma_2(T')=(0,-1)$, where T' is the reflection of T about the origin. For each $A\in C(T\cup T')$ let $\varphi_1(A)$ be the second coordinate of $\sigma_1(A\cap T)$ if $A\cap T\neq\emptyset$, and otherwise let $\varphi_1(A)=0$. Similarly, let $\varphi_2(A)$ be the second coordinate of $\sigma_2(A\cap T')$ if $A\cap T'\neq\emptyset$, and otherwise let $\varphi_2(A)=0$. Now define $\delta(A)=\big(0,\varphi_1(A)+\varphi_2(A)\big)$ for $A\in C(T\cup T')$ and let $\delta=\sigma_i$ for $A\in C(D_i)$. Then δ is a continuous selection on $C(D_1)\cup C(D_2)\cup C(T\cup T')$. The extension of δ to a continuous selection σ on $C(D)$ is now straightforward. If

$$A\in C(D)\setminus\big(C(D_1)\cup C(D_2)\cup C(T\cup T')\big),$$

define $\sigma(A)=\delta\big(A\cap(T\cup T')\big)$.

Let us recall that a dendroid with exactly one ramification point is called a *fan*. The ramification point is called the *top of the fan*. If, moreover, the dendroid has a countable set of its end points, then it is called a *countable fan*.

Proposition 4. *There exists a countable plane fan which is non-contractible and selectible.*

Proof. Let a point p be the pole (i.e. the origin) of the polar coordinate system in the Euclidean plane. Put in the polar coordinates (ϱ,φ)

$$p_0=(1,0),\quad p_n=(1,2^{1-n}),\quad q_n=(\tfrac{1}{2},\tfrac{3}{4}\cdot 2^{1-n})\quad\text{and}\quad r_n=(\tfrac{1}{2},2^{1-n})$$

for $n=1,2,\dots$ and take

$$X=\overline{pp_0}\cup\bigcup_{n=1}^{\infty}(\overline{pp_{2n}}\cup\overline{p_{2n}q_{2n}})\cup\bigcup_{n=1}^{\infty}\overline{pr_{2n-1}},$$

where $\overline{ab}$ stands for the straight line segment joining points a and b (see Fig. 2). So we see that X is a countable plane fan with the top p and with end points p_0, q_{2n}, r_{2n-1} for $n=1,2,\dots$ Let us observe that the midpoint $s=(\frac{1}{2},0)$ of $\overline{pp_0}$ is the common limit point of end points q_{2n} and r_{2n-1} if n tends to infinity, which implies that s is an R-point ([7], Definition 4) and therefore X is non-contractible ([7], Corollary 6).

Before proving that X is selectible observe that if a point (ϱ, φ) is in X, then $\varrho \in [0, 1]$. Given any subcontinuum Y of X we put

$$x = \min\{\varrho \in [0,1]\colon (\varrho, \varphi) \in Y\} \quad \text{and} \quad y = \max\{\varrho \in [0,1]\colon (\varrho, \varphi) \in Y\},$$

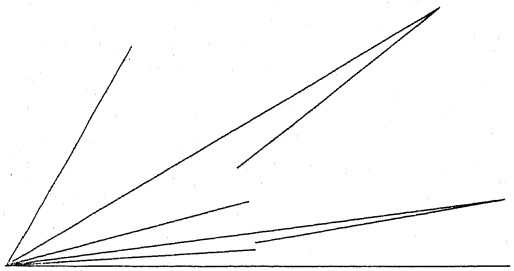

Fig. 2

and we denote by z the radius, i.e. the first coordinate, of the point $\sigma(Y) \in Y$, where $\sigma\colon C(X) \to X$ is the selection to be defined. To determine a point $\sigma(Y)$ we define first z as a function of two variables x and y. Since $\sigma(Y) \in Y$, we must have

(1) $$0 \leqslant x \leqslant z \leqslant y \leqslant 1,$$

and therefore, if $x = y$, that is, if Y is a point, say a, then $z = x = y$ and consequently $\sigma(Y) = a$.

(2) If $x = 0$, we admit $z = 0$.

It means that if $p \in Y$, then $\sigma(Y) = p$. If $x > 0$, then $p \notin Y$, that is $Y \subset X \setminus \{p\}$, and therefore Y is an arc.

(3) If $\frac{1}{2} \leqslant x \leqslant y = 1$, we define $z = 1$.

In this case either Y is a straight line segment contained in $\overline{sp_0}$ with $p_0 \in Y$, and we take $\sigma(Y) = p_0$ then, or Y is a subarc of $\overline{pp_{2n}} \cup \overline{p_{2n}q_{2n}}$ with $p_{2n} \in Y$ for a natural n, and we take $\sigma(Y) = p_{2n}$.

In the rest cases the arc Y is a straight line segment except for the case when, for some $n = 1, 2, \ldots$,

$$Y \cap (\overline{pp_{2n}} \setminus \{p_{2n}\}) \neq \emptyset \neq Y \cap (\overline{p_{2n}q_{2n}} \setminus \{p_{2n}\}).$$

This is the only case for which we can have two different points of Y with the same radius-coordinate. Of two points of Y with the same radius-coordinate z we let $\sigma(Y)$ to be lying in $\overline{pp_{2n}}$. This condition guarantees to us that $\sigma(Y)$ is defined provided its radius-coordinate z is. Further, the continuity of z (as a function of reals x and y) implies the continuity of σ, since x and y are, by their definitions, continuous functions of Y. So, to finish the proof, we ought to define a continuous

function z of two variables x and y satisfying conditions (1), (2) and (3). We can consider x, y and z as the rectangle Cartesian coordinates of a point belonging to the graph G of the function z to be defined. This function is defined in the triangle $0 \leqslant x \leqslant y \leqslant 1$, and, as it follows from (1), its graph G is situated in the unit cube $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$, above the mentioned triangle $0 \leqslant x \leqslant y \leqslant 1$, $z = 0$, above the plane $x = z$ and under the plane $y = z$. Probably the simplest way to guarantee conditions (2) and (3) is to take G composed of straight line segments joining points $(0, u, 0)$ with $(1 - \frac{1}{2}u, 1, 1)$, where u runs over the closed segment $[0, 1]$. Elementary calculations show that the function z is then defined by the formula

$$z = x - \tfrac{1}{2}y + 1 - \sqrt{(x - \tfrac{1}{2}y + 1)^2 - 2x}.$$

Therefore the continuous selection $\sigma\colon C(X) \to X$ is defined.

The next problem, which we consider now, concerns relations between hereditary contractibility and selectibility of dendroids. First of all remark that selectibility is a hereditary property, that is, if a continuum X admits a continuous selection on $C(X)$ and if Y is a subcontinuum of X, then Y also admits a continuous selection on $C(Y)$. Namely, if σ is a continuous selection on $C(X)$, then $\sigma | C(Y)$ is a continuous selection on $C(Y)$. In contrast to this, contractibility is not a hereditary property, even for countable plane fans (see [7], Proposition 12). For fans, however, we have the following.

Proposition 5. *A fan X is hereditarily contractible if and only if it admits a rigid selection on $C(X)$.*

Indeed, for fans hereditary contractibility is equivalent to smoothness ([7], Corollary 17), which in turn is equivalent to the existence of a rigid selection on $C(X)$ ([14], Theorem 2, p. 1043).

The hypothesis that the continuum X under consideration is a fan is superfluous in one direction. Namely it follows from Theorem 2 of [14], p. 1043 and from Proposition 14 of [7] that

Proposition 6. *If a dendroid X admits a rigid selection on $C(X)$, then it is hereditarily contractible.*

It is shown by Proposition 3 that the hypothesis of rigidity of the selection is essential in Proposition 6. Concerning the inverse to Proposition 6 let us note that the dendroid D constructed in the proof of Theorem 3 of [13], p. 372–374, is hereditarily contractible, selectible, but not smooth, and therefore it does not admit any rigid selection. An open question is, however, whether hereditarily contractible dendroids are always selectible, or, in other words, whether hereditary contractibility implies selectibility not only for fans — as it is stated in Proposition 5 — but for all dendroids.

Nadler and Ward proved ([13], Lemma 4, p. 371) that if K is the limit segment of the harmonic fan X with the top p and if σ is a continuous selection on $C(X)$, then $\sigma(K) = p$. We generalize this fact proving some necessary conditions of selectibility of dendroids. To show these conditions we need a definition and a lemma. The author is indebted to the referee for pointing out a way of making the proof of the lemma much shorter than it was done in the previous draft of this paper.

Given two disjoint subcontinua P and Q of a dendroid X, an arc pq such that $pq \cap P = \{p\}$ and $pq \cap Q = \{q\}$ is called the *irreducible junction between P and Q* and is denoted by $L(P, Q)$. It is known ([3], T20, p. 195) that such a junction always does exist and it is uniquely determined.

LEMMA. *Given two subcontinua Y_0 and Y_1 of a dendroid X, there exists an arc $\mathscr{B}$ in the hyperspace $C(X)$ of subcontinua of X having Y_0 and Y_1 as its end points and with the property that each member of $\mathscr{B}$ contains either Y_0 or Y_1 and is contained in $Y_0 \cup L(Y_0, Y_1) \cup Y_1$.*

Proof. X being a separable metric space, it is possible to define a real-valued continuous function μ on 2^X (see [15], p. 245–247] having the following properties:

(4) $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for every point $x \in X$;

(5) if A and B are different elements of 2^X, then $A \subset B$ implies $\mu(A) < \mu(B)$.

Let such a function μ be fixed. Let $A_0, A_1 \in 2^X$. By a segment $[A_0, A_1]$ from A_0 to A_1 in 2^X we mean a continuous mapping from the closed unit interval $[0, 1]$ of reals into 2^X which assigns to each number $t \in [0, 1]$ a set $A_t \in 2^X$ and which satisfies the two conditions (see [8], p. 24):

(6) $$\mu(A_t) = (1-t)\mu(A_0) + t\mu(A_1)\,;$$

(7) $$\text{if} \quad t' < t'', \quad \text{then} \quad A_{t'} \subset A_{t''}\,.$$

It is proved in [8], Lemma 2.3, p. 24 that given $A_0, A_1 \in 2^X$, there exists a segment $[A_0, A_1]$ from A_0 to A_1 if and only if $A_0 \subset A_1$ and every component of A_1 intersects A_0. It is also proved in [8], Lemma 2.6, p. 25 that if $A \in C(X)$, then every segment with A as beginning is contained in $C(X)$. Using these results we can redefine a segment $[A_0, A_1]$ between two subcontinua A_0 and A_1 of X with $A_0 \subset A_1$ as the maximal monotone family of continua which begins at A_0 and ends at A_1, and we see that the segment is an arc or a point in $C(X)$.

Let $C = Y_0 \cup L(Y_0, Y_1) \cup Y_1$. Since Y_0 and Y_1 are subcontinua of C by definition, there exist segments $[Y_0, C]$ and $[Y_1, C]$ in $C(X)$. The union $U = [Y_0, C] \cup [Y_1, C]$ is obviously an arcwise connected subset of $C(X)$. Then an arbitrary arc $\mathscr{B}$ in U having Y_0 and Y_1 as its end points has the required properties. Thus the proof of the lemma is complete.

THEOREM. *Let a metrizable continuum X be selectible with a selection $\sigma\colon C(X) \to X$. Let a non-degenerate continuum $K \subset X$ be the limit of a sequence of subcontinua A_n of X:*

(8) $$K = \operatorname*{Lim}_{n\to\infty} A_n\,,$$

such that

(9) *the intersections $A_n \cap K$ either are empty or have diameters tending to zero when $n \to \infty$.*

Let, for $n = 0, 1, 2, \ldots$, points $a_n \in A_n$ and $b_n \in K$ either be end points of $a_n b_n = L(A_n, K)$ in the case when $A_n \cap K$ is empty, or be arbitrary points of $A_n \cap K$ in the opposite case. Further, assume that

(10) $$\operatorname*{Ls}_{n\to\infty} L(A_n, K) \subset K\,.$$

Then the following statements hold true:

1. *If $a = \lim_{n\to\infty} a_n$, then $\sigma(K) = a$.*
2. *If $b = \lim_{n\to\infty} b_n$, then $\sigma(K) = b$.*
3. *If $\overline{\bigcup_{n=0}^{\infty} L(A_n, K)}$ is connected, then it is a locally connected continuum.*

Proof. It follows from Proposition 1 that X is a dendroid; therefore the irreducible junctions $L(P, Q)$ between any two subcontinua P and Q are uniquely determined in the case when $P \cap Q = \emptyset$. In the opposite case we agree to take $L(P, Q) = P \cap Q$ in all formulas in the sequel.

We shall define now a continuous function α from the unit interval $[0, 1]$ into $C(X)$. Put $\alpha(2^{-n}) = A_n$, where $n = 0, 1, 2, \ldots$ For each such natural n there exists, by the lemma, an arc $\mathscr{B}_n \subset C(X)$ whose end points are $A_n = \alpha(2^{-n})$ and $A_{n+1} = \alpha(2^{-n-1})$ and such that each member of $\mathscr{B}_n$ contains either A_n or A_{n+1} and is contained in $A_n \cup L(A_n, A_{n+1}) \cup A_{n+1}$. We extend α to each closed segment $[2^{-n-1}, 2^{-n}]$ as a homeomorphism onto $\mathscr{B}_n$. Then for every real number $0 < t \leqslant 1$ there exists a natural n such that either

(11) $$A_n \subset \alpha(t) \subset A_n \cup L(A_n, A_{n+1}) \cup A_{n+1}$$

or

(12) $$A_{n+1} \subset \alpha(t) \subset A_n \cup L(A_n, A_{n+1}) \cup A_{n+1}\,.$$

Since the irreducible junction between some two continua in a hereditarily unicoherent continuum is contained in every continuum which contains the two continua (see [3], T22, p. 196), we have

$$L(A_n, A_{n+1}) \subset A_n \cup L(A_n, K) \cup K \cup L(A_{n+1}, K) \cup A_{n+1}\,,$$

and therefore we conclude from (8) and (10) that $\operatorname*{Ls}_{n\to\infty} L(A_n, A_{n+1}) \subset K$. Thus we infer from (11) and (12) that if $t_n \in (0, 1]$ and $t_n \to 0$, then $\alpha(t_n)$ tends to K. Hence if we let $\alpha(0) = K$, then the mapping $\alpha\colon [0, 1] \to C(X)$ is continuous.

It follows from the continuity of mappings α and σ that for every $k = 0, 1, 2, \ldots$ the set $D_k = \sigma(\alpha([0, 2^{-k}]))$ is a locally connected subcontinuum of X that contains points $\sigma(K)$ and $\sigma(A_n)$ for every $n \geqslant k$. Since $\sigma(K) \in K$ and $\sigma(A_n) \in A_n$ by the definition of a continuous selection, we have

(13) $$D_k \cap K \neq \emptyset \neq D_k \cap A_n \quad \text{for every } n \geqslant k \text{ and } k = 0, 1, 2, \ldots$$

Further, observe that for K — which is considered as a point of $C(X)$ — we have

$$K = \alpha(0) = \alpha\Big(\bigcap_{k=0}^{\infty} [0, 2^{-k}]\Big) \in \bigcap_{k=0}^{\infty} \alpha([0, 2^{-k}]) .$$

Thus

$$\sigma(K) \in \sigma\Big(\bigcap_{k=0}^{\infty} \alpha([0, 2^{-k}])\Big) \subset \bigcap_{k=0}^{\infty} \sigma\big(\alpha([0, 2^{-k}])\big) = \bigcap_{k=0}^{\infty} D_k .$$

Since $D_{k+1} \subset D_k$ for every $k = 0, 1, 2, \ldots$ by the definition of D_k, and since $\lim_{k\to\infty} \operatorname{diam} D_k = 0$ by the continuity of σ, the intersection $\bigcap_{k=0}^{\infty} D_k$ is a one-point set. Thus we have

$$\{\sigma(K)\} = \bigcap_{k=0}^{\infty} D_k . \tag{14}$$

Decompose now the set N of all non-negative integers into two disjoint subsets N_1 and N_2 taking $n \in N_1$ if $A_n \cap K = \emptyset$ and $n \in N_2$ in the opposite case. We shall prove statements 1 and 2 separately for $n \in N_1$ and for $n \in N_2$. In other words, we decompose the sequences $\{a_n: n \in N\}$ and $\{b_n: n \in N\}$ into two subsequences each, depending on whether n is in N_1 or in N_2. For every of these subsequences we will show the limit properties of statements 1 and 2 separately, and so these statements will be proved for the whole sequences $\{a_n\}$ and $\{b_n\}$, where $n \in N$.

Assume firstly that $n \in N_1$. Since the dendrite D_k intersects both K and A_n for $n \geqslant k$ (see (12)), hence

(15) $L(A_n, K) \subset D_k$ for $n \in N_1$, $n \geqslant k$ and $k = 0, 1, 2, \ldots$

It follows that a_n and b_n as end points of $L(A_n, K)$ are in D_k for $n \in N_1$ and $n \geqslant k$, where $k = 0, 1, 2, \ldots$ Hence we conclude from (15) that

(16) $\sigma(K)$ is the common limit point of the subsequences $\{a_n: n \in N_1\}$ and $\{b_n: n \in N_1\}$.

Assume secondly that $n \in N_2$. Thus $A_n \cap K \neq \emptyset$, and it follows from (13) that D_k intersects $A_n \cap K$ for $n \in N_2$, $n \geqslant k$ and $k = 0, 1, 2, \ldots$, because otherwise we would have a simple closed curve contained in the union $A_n \cup K \cup D_k$. Thus $A_n \cap K \cap D_k \neq \emptyset$ for $n \in N_2$, $n \geqslant k$ and $k = 0, 1, 2, \ldots$ Since the intersection $A_n \cap K \cap D_k$ is connected by the hereditary unicoherence of X, since a_n and b_n are — by the definition — in $A_n \cap K$ and since $\lim_{u\to\infty} \operatorname{diam}(A_n \cap K) = 0$ by assumption, the intersections $A_n \cap K$ must tend to the same limit point to which D_k's tend. Thus we conlude from (14) that

(17) $\sigma(K)$ is the common limit point of the subsequences $\{a_n: n \in N_2\}$ and $\{b_n: n \in N_2\}$.

The points a and b as the limits of the whole sequences $\{a_n\}$ and $\{b_n\}$ respectively are the limits of any subsequences, we conclude from (16) and (17) by the definitions of the sets N_1 and N_2 that statements 1 and 2 hold.

Observe that the irreducible junctions $L(A_n, K)$ are defined only in the case when $A_n \cap K = \emptyset$, i.e., if $n \in N_1$. Thus it follows from (15) taking $k = 0$ that $L(A_n, K) \subset D_0$ for every $n \in N_1$ and hence $\overline{\bigcup_{n=0}^{\infty} L(A_n, K)} \subset D_0$. Therefore if the set of statement 3 is connected, it is a subcontinuum of the dendrite D_0 which is hereditarily locally connected. Thus the proof of the theorem is complete.

Note that hypothesis (9) on the intersections $A_n \cap K$ is essential in the theorem. Namely, let X be the harmonic fan with the top p, with the limit segment K and with segments K_n emanating from p and tending to K. If we take $A_n = K_n \cup K$, we see that conditions (8) and (10) are satisfied, but (9) is not, and we have $\sigma(K) = p$ according to Lemma 4 of [13], p. 371. Hence if we choose a_n and b_n as sequences of points of $A_n \cap K = K$ which convergent to a point different from p, we see that neither 1 nor 2 is satisfied.

Similarly one can observe that hypothesis (10) cannot be omitted in the theorem, even if the dendroid X is a countable plane fan. To see this let us come back to the example of the fan X described in Proposition 4. Taking $K = \overline{sp_0}$ and $A_n = \overline{r_{2n}p_{2n}} \cup \overline{p_{2n}q_{2n}}$ we have (8). We see that $A_n \cap K = \emptyset$, thus (9) holds. Further, $a_n = r_{2n}$ and $b_n = s$, whence $a = b = s$, but $\sigma(K) = p_0$. Moreover, we have $L(A_n, K) = \overline{r_{2n}p} \cup \overline{ps}$, whence $\overline{\bigcup_{n=1}^{\infty} L(A_n, K)}$ is a harmonic fan. In this way we see that neither of statements 1, 2 and 3 holds.

As a consequence of the theorem observe that the fan X described in [6], p. 95 is not selectible. In fact, take the limit segment K of X as the limit of the vertical segments A_n. Then we see that each b_n is the top of X, while the sequence of a_n's tends to the opposite end point of K, and therefore there is no continuous selection on $C(X)$.

In connection with the theorem let us recall the following concept due to R. B. Bennett [2]. A point p of a dendroid X is called a Q-point if there exists in X a point sequence $\{p_n\}$ such that (i) $\{p_n\}$ converges to p, (ii) the arcs pp_n converge to a non-degenerate limit continuum K, and (iii) if $K \cap pp_n = pq_n$, then the point sequence $\{q_n\}$ converges to p. It is an unproved conjecture that if a dendroid contains a Q-point, then it is not contractible. Let us note that if a dendroid contains a Q-point p for which the condition

$$\lim_{n\to\infty} \operatorname{diam} pq_n = 0 \tag{18}$$

holds, then we can take the arcs pp_n as the continua A_n of the theorem; thus (ii) gives (8), (iii) and (18) give (9), and assumption (10) trivially holds because we have $A_n \cap K \neq \emptyset$. Thus if we substitute q_n and p for a_n and a respectively, then statement 1 gives by (iii) the following.

COROLLARY. *If a dendroid X contains a Q-point p such that* (18) *holds, then for every continuous selection σ: $C(X) \to X$ we have $\sigma(K) = p$ (here K denotes the limit continuum mentioned in the definition of a Q-point).*

The author does not know if condition (18) is essential in the corollary, i.e., if there exists a selectible dendroid containing a Q-point p for which (18) fails and with $\sigma(K) \neq p$. Recently Mr. S. T. Czuba has found a dendroid (even a fan) with a Q-point p for which (18) does not hold, but this example is not selectible.

Consider now the dendroid D_0 described in [5], p. 305. Let X be a continuum obtained from D_0 by shrinking the horizontal straight line segment of D_0 to which the points $p_1, p_2, \ldots$ belong (see the picture of C_0, Fig. 1 on p. 305 of [5]) to a point p. It is evident that X is a countable plane fan with a Q-point p. As it was recently shown by Dr. T. Maćkowiak, the fan X is selectible. Thus the existence of a Q-point in a countable plane fan X does not imply that X is not selectible.

References

[1] D. P. Bellamy and J. J. Charatonik, *The set function T and contractibility of continua*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), pp. 47–49.

[2] R. Bennett, *On some classes of non-contractible dendroids*, Math. Institute of the Polish Academy of Sciences, Mimeographed Paper (1972) (unpublished).

[3] J. J. Charatonik, *Two invariants under continuity and the incomparability of fans*, Fund. Math. 53 (1964), pp. 187–204.

[4] *Problems and remarks on contractibility of curves*, General Topology and its Relations to Modern Analysis and Algebra IV, Proceedings of the Fourth Prague Topological Symposium, 1976, Part B Contributed Papers, Society of Czechoslovak Mathematicians and Physicists, Prague 1977, pp. 72–76.

[5] — and C. Eberhart, *On smooth dendroids*, Fund. Math. 67 (1970), pp. 297–322.

[6] — — *On contractible dendroids*, Colloq. Math. 25 (1972), pp. 89–98.

[7] — and Z. Grabowski, *Homotopically fixed arcs and the contractibility of dendroids*, Fund. Math. 100 (1978), pp. 229-237.

[8] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52 (1942), pp. 22–36.

[9] W. Kuperberg, *Uniformly pathwise connected continua*, Studies in Topology (Proc. Conf., Univ. North Carolina, Charlotte, N. C. 1974; dedicated to Math. Sect. Polish Acad. Sci.), pp. 315–324; New York 1975.

[10] K. Kuratowski, S. B. Nadler, Jr. and G. S. Young, *Continuous selections on locally compact separable metric spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 5–11.

[11] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), pp. 152–182.

[12] S. B. Nadler, Jr., *Problem* 906 *in the New Scottish Book*, dated December 13, 1974 (unpublished).

[13] — and L. E. Ward, Jr., *Concerning continuous selections*, Proc. Amer. Math. Soc. 25 (1970), pp. 369–374.

[14] L. E. Ward, Jr., *Rigid selections on smooth dendroids*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 1041–1043.

[15] H. Whitney, *Regular families of curves*, Ann. of Math. 34 (1933), pp. 244–270.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
Wrocław

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Fixed point theorems for λ-dendroids

by

Roman Mańka (Wrocław)

Abstract. Fixed point theorems are proved for functions whose values and the images as well as the inverse-images of continua are continua.

§ 1. **Introduction.** Throughout this paper X will denote an arbitrary *λ-dendroid*, i.e. a hereditarily decomposable and hereditarily unicoherent metric continuum. We shall consider, under the name *c-functions*, functions having continua $F(p) \subset X$ as values, non-empty for all $p \in X$. If $p \in F(p)$, then the point p will be called a *fixed point* of F.

In [5] I proved that if F is upper semi-continuous, then

(I) *there exists a fixed point of F.*

With the aid of papers [5] and [6] we prove here that the fixed point theorem (I) holds under the same remaining assumptions, even without the upper semi-continuity of the function F.

First, a stronger fixed point theorem (§ 3, Theorem 1) is proved under the following two conditions ([5], p. 113, (I) and (II)):

(II) *for every continuum $K \subset X$ the image $F(K) \subset X$ is a continuum,*

(III) *the property $K \cap F(K) \neq \emptyset$ is inductive for continua $K \subset X$,* where the *image* $F(K)$ means the union $\bigcup \{F(p): p \in K\}$, and a property is called *inductive* provided that for every decreasing sequence of sets having this property their common part also has this property (see [4], p. 54). Then we prove a theorem stating exactly that (I) follows from (II) and (III) (§ 3, Theorem 2), which is next applied to considering the following condition on the sets $F^{-1}(K) = \{p \in X: F(p) \cap K \neq \emptyset\}$:

(IV) *for every continuum $K \subset F(K)$ the set $F^{-1}(K)$ is a continuum.*

Namely, fixed point theorems are proved for c-functions satisfying (IV) or (II) (§ 4, Theorems 3 and 4), which imply a common generalization of the fixed point theorems by Gray [1], by the present author [6] and by Smithson [9], [10] (§ 4, Corollary).

Finally the following property of a non-degenerated subcontinuum E of X will be considered: