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Intersection of sectorial cluster sets and directional essential cluster sets

by

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Abstract. Let $f\colon H\to W$, where H is the upper half plane and W is a second countable topological space, and let R be the real line. It is proved that, except a set of points x on R, which is of the first category and measure zero, every essential directional cluster set of f is a subset of every sectorial cluster set of f at x; and if W is compact and normal, then except a countable set of points x on R, every essential directional cluster set intersects every sectorial cluster set of f at x.

1. Let H denote the open upper half plane and let z denote points of H. Let x denote points on the real line R. For each $x\in R$, $\theta\in(0,\pi)$ and $h>0$, let

$$L_\theta(x) = \{z\colon z\in H;\ \arg|z-x| = \theta\}$$

and

$$L_\theta(x,h) = \{z\colon z\in H;\ |z-x|<h\}\cap L_\theta(x)\,.$$

For each pair of directions θ_1,θ_2, $0<\theta_1<\theta_2<\pi$, $\sigma_{\theta_1\theta_2}$ denotes the sector in H with vertex at the origin, defined by

$$\sigma_{\theta_1\theta_2} = \{z\colon z\in H;\ \theta_1<\arg z<\theta_2\}\,.$$

If there is no ambiguity, we shall simply write σ instead of $\sigma_{\theta_1\theta_2}$. By $\sigma(x)$ we mean the sector in H with vertex at x and which is obtained by a translation of σ. That is,

$$\sigma(x) = \{z\colon z\in H;\ \theta_1<\arg(z-x)<\theta_2\}\,.$$

Also for $x\in R$ and $h>0$ we shall write

$$\sigma(x,h) = \sigma(x)\cap\{z\colon z\in H;\ |z-x|<h\}\,.$$

For $E\subset H$, the upper outer density $\bar{d}^*_\theta(E,x)$ and outer density $d^*_\theta(E,x)$ of E at x in the direction θ are defined by

$$\bar{d}^*_\theta(E,x) = \limsup_{h\to 0}\frac{\mu^*[E\cap L_\theta(x,h)]}{\mu(L_\theta(x,h))} \tag{1}$$

and

$$d^*_\theta(E,x) = \lim_{h\to 0}\frac{\mu^*[E\cap L_\theta(x,h)]}{\mu(L_\theta(x,h))}\,, \tag{2}$$

where μ and μ^* denote one-demensional measure and outer measure (Lebesgue). When the sets concerned are measurable, one gets the definitions of upper density $\bar{d}_\theta(E, x)$ and density $d_\theta(E, x)$ of a measurable set E at x in the direction θ by replacing μ^* in (1) and (2) respectively by μ.

Let $f\colon H \to W$, where W is a topological space. The sectorial (directional) cluster set relative to the sector (direction) $\sigma(\theta)$ of f at x, designated by $C(f, x, \sigma)$ $(C(f, x, \theta))$, is defined to be the set of all w in W such that for every open set U in W containing w, x is a limiting point of $f^{-1}(U) \cap \sigma(x)$ $(f^{-1}(U) \cap L_\theta(x))$. Replacing the condition that x is a limiting point of $f^{-1}(U) \cap L_\theta(x)$ by the stronger condition that $\bar{d}^*_\theta(f^{-1}(U), x) > 0$, one gets the definition of directional essential cluster set $C_e(f, x, \theta)$ of f at x in the direction θ. The essential cluster set $C_e(f, x)$ of f at x is the set of all w in W such that for every open set U in W containing w, $\bar{d}^*(f^{-1}(U), x) > 0$, where $\bar{d}^*(E, x)$ is defined as in (1) but $L_\theta(x, h)$ being replaced by

$$S(x, h) = \{z\colon z \in H;\ |z - x| < h\}$$

and the measure here is the two demensional Lebesgue measure.

A set $P \subset R$ is said to be *porous at a point* $x \in R$ if

$$\limsup_{r \to 0} \frac{\gamma(x, r, P)}{r} > 0,$$

where $\gamma(x, r, P)$ is the length of the largest open interval in the complement of P which is contained in $(x-r, x+r)$. A set $P \subset R$ is said to be *porous* if it is porous at each of its points. A set $T \subset R$ is said to be *σ-porous* if it is a countable union of porous sets. Clearly every subset of a porous set is porous and every subset of a σ-porous set is σ-porous. Also it can be verified that a σ-porous set is both of the first category and of measure zero. But there exists a perfect set of measure zero which is not σ-porous (see L. Zajíček, *Časopis pro pěstování matematiky* 101 (1976), pp. 350–359).

2. It is known [6] that if $f\colon H \to R$ is measurable and θ_1 and θ_2 are fixed directions, then except a set of measure zero on R, the set $C_e(f, x, \theta_1)$ is a subset of $C(f, x, \theta_2)$ and if f is continuous then the exceptional set is also of the first category. It is also known [4] that if $f\colon H \to R$ is arbitrary and σ is a fixed sector in H, then the total cluster set $C(f, x)$ of f at x is equal to the set $C(f, x, \sigma)$ except a set of points x of the first category on R (see also [2, 5]). In [1] it is also proved that the set of points x at which there exist two arcs γ_1 and γ_2 in H with the property that the arc cluster sets $C_{\gamma_1}(f, x)$ and $C_{\gamma_2}(f, x)$ are disjoint, is countable (see [3, p. 85]).

In this paper, we prove in Theorem 1 that if f is arbitrary and if $\{\sigma\}$ is the collection of all sectors in H then, except a σ-porous set of points x on R, the set $C(f, x, \theta)$ is a subset of $\bigcap \{C(f, x, \sigma)\colon \sigma \in \{\sigma\}\}$ for every direction θ, $0 < \theta < \pi$. We also prove in Theorem 2 that if f is arbitrary then, except a countable set of points x on R, the sets $C(f, x, \sigma)$ and $C_e(f, x, \theta)$ intersect for every σ and every θ.

Throughout the paper we shall consider $f\colon H \to W$, where W is a topological space with a countable basis. The closure of a set E will be denoted by $\bar{E}$.

3. Let $E \subset H$ be arbitrary and $x \in R$. Let

$$\vartheta(E, x) = \{\theta\colon 0 < \theta < \pi;\ x \in \overline{L_\theta(x) \cap E}\}.$$

This will be used in the sequel.

LEMMA 1. *Let $E \subset H$ be arbitrary and let $\sigma \subset H$ be a fixed sector. Then the set*

$$T(E) = \{x\colon x \in R;\ \vartheta(E, x) \neq \emptyset;\ x \notin \overline{\sigma(x) \cap E}\}$$

is a σ-porous set.

Proof. Let for fixed positive integer m

$$T_m(E) = \{x\colon x \in R;\ \vartheta(E, x) \neq \emptyset;\ \sigma(x, 1/m) \cap E = \emptyset\}.$$

Then clearly

(1) $$T(E) = \bigcup_{m=1}^{\infty} T_m(E).$$

Let, if possible for some m, the set $T_m(E)$ be a non-porous set. Then there is $x_0 \in T_m(E)$ such that

(2) $$\limsup_{r \to 0} \frac{\gamma(x_0, r, T_m(E))}{r} = 0,$$

where $\gamma(x_0, r, T_m(E))$ is the length of the largest open interval in the complement of $T_m(E)$ and is contained in $(x_0 - r, x_0 + r)$. Again since $x_0 \in T_m(E)$,

$$\sigma(x_0, 1/m) \cap E = \emptyset \quad \text{and} \quad \vartheta(E, x_0) \neq \emptyset.$$

Let $\theta_0 \in \vartheta(E, x_0)$. Then since $x_0 \in \overline{L_{\theta_0}(x_0) \cap E}$ and

$$\sigma(x_0, 1/m) \cap E = \emptyset,$$

we have $L_{\theta_0}(x_0) \not\subset \sigma(x_0)$. Let $\sigma_{\alpha\beta}$ be any sector such that $\bar{\sigma}_{\alpha\beta} \subset \sigma$. Then either $0 < \alpha < \beta < \theta_0$ or $\theta_0 < \alpha < \beta < \pi$. Let us first assume that $0 < \alpha < \beta < \theta_0$. Let

$$K = \frac{\sin\theta_0 \sin(\beta - \alpha)}{\sin\beta \sin(\theta_0 - \alpha)}.$$

Then from (2) there is a δ such that

$$0 < \delta < \frac{1}{m \sin\theta_0} \min[\sin(\theta_0 - \alpha), \sin(\theta_0 - \beta)]$$

and

(3) $$\gamma(x_0, r, T_m(E)) < \tfrac{1}{2} K r \quad \text{for} \quad 0 < r < \delta.$$

Then both $L_\alpha(x_0-\delta, 1/m)$ and $L_\beta(x_0-\delta, 1/m)$ intersect $L_{\theta_0}(x_0)$. Let

$$h_0 = \delta\frac{\sin\alpha}{\sin(\theta_0-\alpha)}.$$

Since $x_0 \in \overline{L_{\theta_0}(x_0) \cap E}$, there is $z \in L_{\theta_0}(x_0, h_0) \cap E$. Let $x' \in (x_0-\delta, x_0)$ be such that

$$z = L_\alpha(x', 1/m) \cap L_{\theta_0}(x_0).$$

Let $I(x')$ be the open interval in R whose left end point is x' and right end point is x'' such that

$$z = L_\beta(x'', 1/m) \cap L_{\theta_0}(x_0).$$

By a simple calculation, $\mu(I(x')) = K(x_0-x')$. Hence from (3), since $0<x_0-x'<\delta$, we have

(4) $$\gamma(x_0, x_0-x', T_m(E))<\tfrac{1}{2}\mu(I(x')).$$

Since $I(x')\subset(x', 2x_0-x')$, from (4) and from the definition of $\gamma(x_0, x_0-x', T_m(E))$ it follows that $I(x') \cap T_m(E) \neq \emptyset$. Let $x''' \in I(x') \cap T_m(E)$. Clearly, $z \in \sigma_{\alpha\beta}(x''', 1/m)$. Also since $z \in E$, $z \in \sigma_{\alpha\beta}(x''', 1/m) \cap E$. But since $x''' \in T_m(E)$, $\sigma_{\alpha\beta}(x''', 1/m) \cap E = \emptyset$, which is a contradiction. If $\theta_0<\alpha<\beta<\pi$, then we would also arrive at a similar contradiction by considering the interval $(x_0, x_0+\delta)$. Thus each $T_m(E)$ is porous and hence, from (1), the set $T(E)$ is σ-porous.

THEOREM 1. *If $f: H \to W$ is arbitrary and $\{\sigma\}$ is the collection of all sectors in H, then the set*

$$\Theta(x) = \{\theta:\ 0<\theta<\pi;\ C(f, x, \theta)\not\subset \bigcap\{C(f, x, \sigma):\ \sigma \in \{\sigma\}\}\}$$

is void, except a σ-porous set of points x on R.

Proof. For each pair of rationals α, β, $0<\alpha<\beta<\pi$, denote by $\sigma_{\alpha\beta}$ the sector

$$\sigma_{\alpha\beta} = \{z:\ z \in H;\ \alpha<\arg z<\beta\}$$

and let $\{\sigma_{\alpha\beta}\}$ be the collection of all such sectors. Let $\{V_n\}$ be a countable basis for the topology of W and let

$$E_n = f^{-1}(V_n),$$

$$\vartheta(E_n, x) = \{\theta:\ 0<\theta<\pi;\ x \in \overline{L_\theta(x) \cap E_n}\},$$

$$T_{\alpha\beta}(E_n) = \{x:\ x \in R;\ \vartheta(E_n, x) \neq \emptyset,\ x \notin \overline{\sigma_{\alpha\beta}(x) \cap E_n}\}$$

and

$$T = \{x:\ x \in R,\ \Theta(x) \neq \emptyset\}.$$

Now if $x_0 \in T$, then there is $\theta_0 \in \Theta(x_0)$. Hence

$$C(f, x_0, \theta_0)\not\subset \bigcap\{C(f, x_0, \sigma):\ \sigma \in \{\sigma\}\}.$$

So there is $\sigma_0 \in \{\sigma\}$ such that $C(f, x_0, \theta_0)\not\subset C(f, x_0, \sigma_0)$. Hence there is E_{n_0} such that $x_0 \in \overline{L_{\theta_0}(x_0) \cap E_{n_0}}$ and $x_0 \notin \overline{\sigma_0(x_0) \cap E_{n_0}}$. Let $\sigma_{\alpha_0\beta_0} \in \{\sigma_{\alpha\beta}\}$ be such that $\sigma_{\alpha_0\beta_0}\subset\sigma_0$.

So, $x_0 \notin \overline{\sigma_{\alpha_0\beta_0}(x_0) \cap E_{n_0}}$. Also since $x_0 \in \overline{L_{\theta_0}(x_0) \cap E_{n_0}}$, $\theta_0 \in \vartheta(E_{n_0}, x_0)$. Therefore, $x_0 \in T_{\alpha_0\beta_0}(E_{n_0})$. Thus we have

(5) $$T\subset \bigcup T_{\alpha\beta}(E_n)$$

where the union is taken for all integers n and for all pair of rationals α, β, $0<\alpha<\beta<\pi$. By Lemma 1, each of the sets $T_{\alpha\beta}(E_n)$ is a σ-porous set. So, by (5) the set T is a σ-porous set and this completes the proof the theorem.

COROLLARY 1. *If $f: H \to W$ is arbitrary and if $\{\sigma\}$ is the collection of all sectors in H, then except a σ-porous set of points x on R,*

$$\overline{\bigcup\{C(f, x, \theta):\ 0<\theta<\pi\}}\subset \bigcap\{C(f, x, \sigma):\ \sigma \in \{\sigma\}\}.$$

Proof. Since $C(f, x, \sigma)$ is closed for each $\sigma \in \{\sigma\}$, $\bigcap\{C(f, x, \sigma):\ \sigma \in \{\sigma\}\}$ is closed and so the proof follows from Theorem 1.

COROLLARY 2. *If $f: H \to W$ is measurable and $\{\sigma\}$ is the collection of all sectors in H, then the set*

$$Q = \{x:\ x \in R;\ C_e(f, x)\not\subset \bigcap\{C(f, x, \sigma):\ \sigma \in \{\sigma\}\}\}$$

is of measure zero.

If, further, f is continuous, then this set is also of the first category.

Proof. From [5] it follows that if f is measurable then for a fixed θ, $0<\theta<\pi$, the set

$$S = \{x:\ x \in R;\ C_e(f, x)\not\subset C_e(f, x, \theta)\}$$

s of measure zero and if further, f is continuous then the set S is also of the first category. Since σ-porous set is both of the first category and of measure zero, the proof follows from Theorem 1.

Remark. In Corollary 2, the essential cluster set $C_e(f, x)$ can be replaced by the strong essential cluster set $C_s(f, x)$ to get a stronger result. (For the definition of $C_s(f, x)$ see [7].) The proof is the same except that one is to consider $\theta = \frac{1}{2}\pi$ and replace $C_e(f, x)$ by $C_s(f, x)$ and apply a result of O'Malley [7].

4. Throughout this section W is a compact normal space which have a countable basis. In particular, W may be a compact metric space.

LEMMA 2. *Let $f: H \to W$ be arbitrary and let G be an open subset of W. If $C_e(f, x, \theta)\subset G$ then $d_0^*(f^{-1}(G), x) = 1$.*

Proof. Since $W\setminus G$ is closed and disjoint from $C_e(f, x, \theta)$, there exist points w_i, $i = 1, 2, \ldots, k$, in $W\setminus G$ and neighbourhoods $V(w_i)$ of w_i such that

$$W\setminus G\subset \bigcup_{i=1}^{k} V(w_i)$$

and

$$\lim_{r\to 0}\frac{\mu^*[f^{-1}(V(w_i)) \cap L_\theta(x, r)]}{r} = 0$$

for each $i = 1, 2, \ldots, k$. Hence

$$\lim_{r \to 0} \frac{\mu^*[f^{-1}(W \setminus G) \cap L_\theta(x, r)]}{r} = 0$$

and the proof is complete.

LEMMA 3. *Let $f: H \to W$ be arbitrary, and let G be an open subset of W. If $C(f, x, \sigma) \subset G$, then there is a positive integer n such that*

$$f^{-1}(G) \cap \sigma(x, 1/n) = \sigma(x, 1/n).$$

Proof. Since $W \setminus G$ is closed and disjoint from $C(f, x, \sigma)$, there exists points w_i, $i = 1, 2, \ldots, k$, in $W \setminus G$ and neighbourhoods $V(w_i)$ of w_i such that

$$W \setminus G \subset \bigcup_{i=1}^{k} V(w_i)$$

and

$$x \notin \overline{f^{-1}(V(w_i)) \cap \sigma(x)}$$

for each $i = 1, 2, \ldots, k$. Let $n_1, n_2, \ldots, n_k$ be positive integers such that

$$f^{-1}(V(w_i)) \cap \sigma(x, 1/n_i) = \emptyset$$

for each $i = 1, 2, \ldots, k$. Set

$$n = \max[n_1, n_2, \ldots, n_k].$$

Then

$$f^{-1}(W \setminus G) \cap \sigma(x, 1/n) = \emptyset$$

which proves the lemma.

THEOREM 2. *Let $f: H \to W$ be arbitrary and let E be the set of all points $x \in R$ for which there exist a sector σ_x in H and a direction θ_x, $0 < \theta_x < \pi$, with the property that*

$$C(f, x, \sigma_x) \cap C_e(f, x, \theta_x) = \emptyset.$$

Then E is countable.

Proof. For each pair of rationals α, β, $0 < \alpha < \beta < \pi$, let

$$\sigma_{\alpha\beta} = \{z\colon z \in H;\ \alpha < \arg z < \beta\}$$

and let $\{\sigma_{\alpha\beta}\}$ be the collection of all such sectors in H. Let $\mathscr{B}$ be a countable basis for the topology of W and let $\mathscr{G}$ be the collection of all open sets which can be expressed as a finite union of sets $B \in \mathscr{B}$. Then $\mathscr{G}$ is countable. Let $\mathscr{G}^*$ be the set of all ordered pair (G_1, G_2) of sets of $\mathscr{G}$ with $G_1 \cap G_2 = \emptyset$. So, $\mathscr{G}^*$ is also countable. Let, for $G \in \mathscr{G}$ and $x \in R$,

$$\Theta(f^{-1}(G), x) = \{\theta\colon 0 < \theta < \pi;\ d_\theta^*(f^{-1}(G), x) = 1\}$$

and let for a fixed positive integer n, for a fixed pair $(G_1, G_2) \in \mathscr{G}^*$ and for a fixed sector $\sigma_{\alpha\beta} \in \{\sigma_{\alpha\beta}\}$

$$P(G_1, G_2, \alpha, \beta, n) = \{x\colon x \in R;\ \sigma_{\alpha\beta}(x, 1/n) \cap f^{-1}(G_1) = \sigma_{\alpha\beta}(x, 1/n);\ \Theta(f^{-1}(G_2), x) \neq \emptyset\}.$$

Let $x \in E$. Then there exists a sector σ_x and a direction θ_x, such that $C(f, x, \sigma_x) \cap C_e(f, x, \theta_x) = \emptyset$. Then since the sets $C(f, x, \sigma_x)$ and $C_e(f, x, \theta_x)$ are closed and since W is normal, there exist two open sets G_1^0, G_2^0 in W with $G_1^0 \cap G_2^0 = \emptyset$ such that $C(f, x, \sigma_x) \subset G_1^0$ and $C_e(f, x, \theta_x) \subset G_2^0$. Again, the sets $C(f, x, \sigma_x)$ and $C_e(f, x, \theta_x)$ being closed, they are compact and so there exist two open sets $G_1, G_2 \in \mathscr{G}$ such that $C(f, x, \sigma_x) \subset G_1$ and $C_e(f, x, \theta_x) \subset G_2$ and $G_1 \cap G_2 = \emptyset$. So, $(G_1, G_2) \in \mathscr{G}^*$. Hence, by applying Lemma 3 and Lemma 2, there exists a positive integer n such that

$$\sigma_x(x, 1/n) \cap f^{-1}(G_1) = \sigma_x(x, 1/n)$$

and

$$d_{\theta_x}^*(f^{-1}(G_2), x) = 1,$$

Thus $\Theta(f^{-1}(G_2), x) \neq \emptyset$. Let $\sigma_{\alpha\beta} \in \{\sigma_{\alpha\beta}\}$ be such that $\sigma_{\alpha\beta} \subset \sigma_x$. Then

$$\sigma_{\alpha\beta}(x, 1/n) \cap f^{-1}(G_1) = \sigma_{\alpha\beta}(x, 1/n).$$

Hence, $x \in P(G_1, G_2, \alpha, \beta, n)$. Thus

(6) $$E \subset \bigcup P(G_1, G_2, \alpha, \beta, n)$$

where the union will be taken for all pair $(G_1, G_2) \in \mathscr{G}^*$, for all positive integers n, and for all rationals α, β, $0 < \alpha < \beta < \pi$. Since this is a countable union, the theorem will be proved if we show that each $P(G_1, G_2, \alpha, \beta, n)$ is countable. Let, if possible, for some $(G_1, G_2) \in \mathscr{G}^*$, positive integer n, and rationals α, β, $0 < \alpha < \beta < \pi$, the set $P(G_1, G_2, \alpha, \beta, n)$ be uncountable. Then there is a point $x \in P(G_1, G_2, \alpha, \beta, n)$ which is a limiting point of $P(G_1, G_2, \alpha, \beta, n)$ from both sides. Since

$$x \in P(G_1, G_2, \alpha, \beta, n), \quad \Theta(f^{-1}(G_2), x) \neq \emptyset;$$

so there is θ_x at x such that $d_{\theta_x}^*(f^{-1}(G_2), x) = 1$. We observe that $\theta_x \notin (\alpha, \beta)$. For, if $\theta_x \in (\alpha, \beta)$ then $f^{-1}(G_1) \cap f^{-1}(G_2) \neq \emptyset$, which contradicts the fact that $G_1 \cap G_2 = \emptyset$. Hence we consider the two cases $\beta \leqslant \theta_x < \pi$ and $0 < \theta_x \leqslant \alpha$. Let us first consider the case $\beta \leqslant \theta_x < \pi$. Let $\sigma' \subset \sigma_{\alpha\beta}$, where

$$\sigma'(x) = \{z\colon z \in H;\ \theta_1 < \arg(z - x) < \theta_2;\ \alpha < \theta_1 < \theta_2 < \beta\}.$$

Since x is a limiting point of the set $P(G_1, G_2, \alpha, \beta, n)$ from both sides, there is a sequence $\{x_m\} \subset P(G_1, G_2, \alpha, \beta, n)$ such that $x_{m-1} < x_m < x$ for all m and $\lim_{m \to \infty} x_m = x$. Let m_1 be the first integer such that

$$x_{m_1} \geqslant x - \frac{1}{n \sin\theta_x} \min[\sin(\theta_x - \theta_2), \sin(\theta_x - \theta_1)].$$

Then both sides of $\sigma'(x_m, 1/n)$, namely $L_{\theta_1}(x_m, 1/n)$ and $L_{\theta_2}(x_m, 1/n)$, will intersect $L_{\theta_x}(x)$ for all $m \geqslant m_1$. For $m \geqslant m_1$ let J_m denote the open segment on $L_{\theta_x}(x)$ with endpoints

$$z_m = L_{\theta_1}(x_m, 1/n) \cap L_{\theta_x}(x) \quad \text{and} \quad z'_m = L_{\theta_2}(x_m, 1/n) \cap L_{\theta_x}(x) .$$

Let

$$Q = \bigcup_{m=m_1}^{\infty} J_m .$$

Clearly $Q \subset f^{-1}(G_1)$. Let h_m denote the distance between x and z'_m. Then $J_m \subset Q \cap L_{\theta_x}(x, h_m)$ for all $m \geqslant m_1$. So, by a simple calculation,

$$\mu(Q \cap L_{\theta_x}(x, h_m)) \geqslant \mu(J_m) = \frac{h_m \sin\theta_x \sin(\theta_2 - \theta_1)}{\sin\theta_2 \sin(\theta_x - \theta_1)}$$

which shows that

$$\frac{\mu(Q \cap L_{\theta_x}(x, h_m))}{h_m} \geqslant \frac{\sin\theta_x \sin(\theta_2 - \theta_1)}{\sin\theta_2 \sin(\theta_x - \theta_1)} > 0 .$$

Since $h_m \to 0$ as $m \to \infty$, taking limit as $m \to \infty$ we have

(7) $$\bar{d}_{\theta_x}(Q, x) > 0 .$$

Now since $Q \subset f^{-1}(G_1)$ and $f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$,

$$L_{\theta_x}(x, h_m) \cap f^{-1}(G_2) \subset L_{\theta_x}(x, h_m) \setminus Q .$$

Also since $d^*_{\theta_x}(f^{-1}(G_2), x) = 1$,

(8) $$d_{\theta_x}(L_{\theta_x}(x, h_m) \setminus Q, x) = 1 .$$

But (7) and (8) are contradictory. If $0 < \theta_x \leqslant \alpha$ then we can arrive at a similar contradiction by considering a sequence $\{x_m\} \subset P(G_1, G_2, \alpha, \beta, n)$ such that $x_{m-1} > x_m > x$ for all m and $\lim_{m \to \infty} x_m = x$. This contradiction shows that each of the sets $P(G_1, G_2, \alpha, \beta, n)$ is countable and the proof is complete by (6).

COROLLARY 3. *If $f\colon H \to W$ is arbitrary and σ is a fixed sector in H then the set*

$$\Theta(x) = \{\theta\colon 0 < \theta < \pi;\ C(f, x, \sigma) \cap C_e(f, x, \theta) = \emptyset\}$$

is void, except a countable set of points x on R.

COROLLARY 4. *If $f\colon H \to W$ and if $\{\sigma\}$ is the collection of all sectors in H and ψ be a fixed direction, $0 < \psi < \pi$, then the set*

$$\vartheta(x) = \{\sigma\colon \sigma \in \{\sigma\};\ C(f, x, \sigma) \cap C_e(f, x, \psi) = \emptyset\}$$

is void, except a countable set of points x on R.

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