$x \in \mathrm{dom}(h_k)$ and if $t' \subseteq t$ and $i \in \{0, 1\}$ then $[t^\frown i] \cap \mathrm{ran}(h_k) = \emptyset$. Since h_k is a homeomorphism we have $s \in Sq$ so that $x \in [s]$ and $h_k([s]) \subseteq [t]$. By the claim, $h_j([s]) \subseteq [t]$ for all $j > k$. Thus $t \subseteq w$ and $t \subseteq z$. But t was arbitrary, so $w = z$.

Now suppose $\forall k (x \notin \mathrm{dom}(h_n))$ and $w \upharpoonright n = z \upharpoonright n$ but $w(n) \neq z(n)$. Let k be odd and so large that if $t \in Sq$ and $\mathrm{dom}(t) \leqslant n+2$ then $\mathrm{ran}(h_k) \cap [t] \neq \emptyset$. Let l be least so that $x \upharpoonright l \notin \mathrm{dom}(h_k)$. Let $s = x \upharpoonright l$; then $s \in M_{k+1}$. Let u_s be as in the definition of h_{k+1}. Then $[u_s] \cap D_k = \emptyset$, so $\mathrm{dom}(u_s) > n+2$ and $u_s \upharpoonright (n+1) \neq w \upharpoonright (n+1)$ or $u_s \upharpoonright (n+1) \neq z \upharpoonright (n+1)$. Say that $u_s \upharpoonright (n+1) \neq w \upharpoonright (n+1)$. Now $h_{k+1}([s]) \subseteq [u_s]$ by definition, and $h_j([s]) \subseteq [u_s \upharpoonright n+1]$ for all $j \geqslant k+1$ by the claim. This contradicts our assumption that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} h'(x_k) = w$.

Thus h is well-defined. Clearly h is continuous. A completely symmetric argument shows that h^{-1} is well-defined (i.e. h is one-one and onto) and continuous. Thus h is a homeomorphism of ${}^{\omega}2$. Clearly $h \upharpoonright \mathrm{dom}(h') = h'$, and thus $h(A) = B$. ∎

Both of the hypotheses that A and B are everywhere properly Γ and that A and B are meager are necessary in Theorem 2. One can, however, replace "meager" by "comeager" by passing to complements. In the case that $\Gamma = \Sigma^1_1$, the hypothesis of $b\Gamma - AD$ can also be shown necessary; this follows from [3] and the fact that any properly Σ^1_1 set is Borel isomorphic to a meager, everywhere properly Σ^1_1 set.

We conjecture that Theorem 2 holds for subsets of the real line. Of course, one must formulate the notion of reasonable closure properly in order to prove this.

References

[1] J. Christensen, *Topology and Borel structure*, New York.
[2] H. Friedman, *Higher set theory and mathematical practice*, Annals of Math. Logic 2 (1971), p. 326.
[3] L. Harrington, *Analytic determinacy and* $0^\#$, to appear.
[4] K. Kuratowski, *Topology*, vol. I, New York–London–Warszawa 1966.
[5] D. Martin, *Borel determinacy*, Annals of Math. 102 (1975), p. 363.
[6] J. Steel, *Ph. D. thesis*, Berkeley 1977.
[7] R. Van Wesep, *Wadge degrees and descriptive set theory*, to appear in: *Proceedings of the* UCIA — *Cal-Tech logic seminar*, Lecture Notes in Mathematics.

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On fine shape theory II

by

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Abstract. Let $\mathfrak{C}_p^B$ be the proper shape category defined by Ball in terms of proper mutation. It is proved that the fine shape category $\mathfrak{C}_f$ is isomorphic to the full subcategory of $\mathfrak{C}_p^B$ whose objects are locally compact metric spaces of the form $X \times R_+$, where X is any compactum and R_+ is the space of non negative reals. The proper movability is defined and a characterization of pointed FANR in terms of proper movability is obtained.

1. Introduction. The notion of proper shape was introduced originally by Ball and Sher [3]. Their presentation paralleled Borsuk's one [1], using a notion of proper fundamental net in place of Borsuk's fundamental sequence. Ball [1] has established proper shape theory modeled on the ANR-systems of Mardešić–Segal [14], on the mutations of Fox [7] or on the shapings of Mardešić [13]. We mean by $\mathscr{C}_p$ the proper shape category in the sense of Ball and Sher [3] and by $\mathscr{C}_p^B$ the proper shape category in the sense of Ball [1]. As presented by Ball [1], whether $\mathscr{C}_p$ and $\mathscr{C}_p^B$ are isomorphic is an open question.

Recently the authors [12] have introduced the fine shape category $\mathscr{C}_f$ consisting of all compacta and proved that $\mathscr{C}_f$ is isomorphic to the full subcategory $\tilde{\mathscr{C}}_p$ of $\mathscr{C}_p$ whose objects consist of space of the form $X \times R_+$, where X is any compactum and R_+ is the space of non-negative reals. In this paper we first prove that $\mathscr{C}_f$ is isomorphic to the full subcategory $\tilde{\mathscr{C}}_p^B$ of $\mathscr{C}_p^B$ consisting of spaces of the form $X \times R_+$, X a compactum. This gives a partial answer to Ball's question mentioned above. In the second part of the paper we shall investigate a characteristic property of a pointed FANR in connection with the categories $\mathscr{C}_f$, $\mathscr{C}_p$ and $\mathscr{C}_p^B$. We use [12] as general reference for notions and notations. Throughout the paper all spaces are metrizable and maps are continuous. If X is a subset of a space M, then we denote by $U(X, M)$ the set of all neighborhoods of X in M.

2. $\mathscr{C}_f$ and $\mathscr{C}_p^B$. Ball has defined the proper shape categories $\mathscr{S}_p^1$, $\mathscr{S}_p^2$ and $\mathscr{S}_p^3$ whose objects consist of locally compact spaces and proved that these three categories are isomorphic to each other. (Cf. [1, §§ 2, 3 and 5, Theorems 4.6 and 5.3].) We shall identify the categories $\mathscr{S}_p^i$, $i = 1, 2, 3$, under Ball's isomorphism and denote it by $\mathscr{C}_p^B$.

THEOREM 1. *Let $\tilde{\mathscr{C}}_p^B$ be the full subcategory of $\mathscr{C}_p^B$ whose objects consist of spaces of the form $X\times R_+$, where X is any compactum and R_+ is the space of non-negative reals. Then there exists a category isomorphism $\Sigma\colon \mathscr{C}_f\to\tilde{\mathscr{C}}_p^B$ such that $\Sigma(X) = X\times R_+$ for every object X of $\mathscr{C}_f$.*

Proof. Let X and Y be compacta and let M and N be compact AR's containing X and Y respectively. We denote by J the set of non-negative integers and by Δ the set of all strictly increasing functions $\delta\colon J\to J$. Let $\{U_i\colon i\in J\}$ be a closed neighborhood basis of X in M such that $U_i\supset U_{i+1}$ and let $\{V_i\colon i\in J\}$ be a neighborhood basis of Y in N such that $V_i\supset V_{i+1}$ and each V_i is an ANR. For each $\delta\in\Delta$, let

$$U_\delta = \bigcup_{i\in J} U_{\delta(i)}\times[i, i+1] \quad\text{and}\quad V_\delta = \bigcup_{i\in J} V_{\delta(i)}\times[i, i+1].$$

Note that $\{U_\delta\colon \delta\in\Delta\}$ and $\{V_\delta\colon \delta\in\Delta\}$ form neighborhood bases of $X\times R_+$ and $Y\times R_+$ in $M\times R_+$ and N, respectively. Define a map $\xi_\delta\colon R_+\to R_+$ by

$$\xi_\delta(t) = i+\frac{t+\delta(0)-\delta(i)}{\delta(i+1)-\delta(i)} \tag{2.1}$$

for $\delta(i)-\delta(0)\leqslant t\leqslant\delta(i+1)-\delta(0)$, $i\in J$. Now we define a functor $\Sigma\colon\mathscr{C}_f\to\tilde{\mathscr{C}}_p^B$ as follows. For an object X of $\mathscr{C}_f$, let $\Sigma(X) = X\times R_+$. Suppose that a fundamental map $F\colon X\to Y$ in M, N is given. By the definition [12, § 1] of a fundamental map there exists a $\delta_F\in\Delta$ such that

$$F(U_{\delta_F(i)}\times[\delta_F(i), \infty))\subset V_i, \quad i\in J. \tag{2.2}$$

For each $\delta\in\Delta$, set

$$W_\delta = \bigcup_{i\in J} U_{\delta_F\delta(i)}\times[\delta_F\delta(i)-\delta_F\delta(0), \delta_F\delta(i+1)-\delta_F\delta(0)]. \tag{2.3}$$

Here $\delta_F\delta$ is a map in Δ defined by $\delta_F\delta(i) = \delta_F(\delta(i))$, $i\in J$. Obviously W_δ is a closed neighborhood of $X\times R_+$ in $M\times R_+$. Define a map $f_\delta\colon W_\delta\to V_\delta$ by

$$f_\delta(x, t) = \{F(x, t+\delta_F\delta(0)), \xi_{\delta_F\delta}(t)\}, \quad (x, t)\in W_\delta. \tag{2.4}$$

Let $(x, t)\in W_\delta$. Then for some $i\in J$ $x\in U_{\delta_F\delta(i)}$ and $\delta_F\delta(i)\leqslant t+\delta_F\delta(0)\leqslant\delta_F\delta(i+1)$. Hence $F(x, t)\in V_{\delta(i)}$ (cf. (2.2)) and $\xi_{\delta_F\delta}(t)\in[i, i+1]$. Thus $f_\delta(x, t)\in V_{\delta(i)}\times[i, i+1]\subset V_\delta$ and f_δ is well defined. Let φ_F be the set of all maps f satisfying the condition:

(2.5) $f\colon U\to V$, where $U\in \boldsymbol{U}(X\times R_+, M\times R_+)$ and $V\in\boldsymbol{U}(Y\times R_+, M\times R_+)$,

(2.6) there exists $\delta\in\Delta$ such that $U\subset W_\delta$, $V_\delta\subset V$ and $f(x) = jf_\delta i(x)$ for $x\in U$, where $i\colon U\to W_\delta$ and $j\colon V_\delta\to V$ are the inclusions.

Obviously φ_F forms a proper mutation of $\boldsymbol{U}(X\times R_+, M\times R_+)$ into $\boldsymbol{U}(Y\times R_+, N\times R_+)$ in the sense of Ball [1, § 5]. Define $\Sigma([F]) = [\varphi_F]$. Here $[F]$ is the f-class determined by a fundamental map F (cf. [12, § 2]) and $[\varphi]$ is the similarity class of a proper mutation φ (cf. [1, § 5]).

To show that Σ is an isomorphism, we have to show that

(2.7) if F and G are fundamental maps of X into Y in M, N, then φ_F and φ_G are similar if and only if $F \underset{f}{\simeq} G$,

(2.8) if $\boldsymbol{f}$ is a proper mutation of $\boldsymbol{U}(X\times R_+, M\times R_+)$ into $\boldsymbol{U}(Y\times R_+, N\times R_+)$, then there exists a fundamental map $F\colon X\to Y$ in M, N such that φ_F is similar to $\boldsymbol{f}$.

The proofs of (2.7) and (2.8) are obtained by a slight modification of the proof of Theorem 2 of [12]. We shall prove only (2.8) and omit the proof of (2.7). For each $k\in J$, let $\hat{V}_k = \bigcup_{i\in J} V_{k+i}\times[i, i+1]$. Since $\boldsymbol{f}$ is a proper mutation, there exists a map $f_k\colon W_k\to\hat{V}_{k+1}$, $f_k\in\boldsymbol{f}$ and $W_k\in\boldsymbol{U}(X\times R_+, M\times R_+)$, $k\in J$, such that for some $\hat{W}_k\in\boldsymbol{U}(X\times R_+, W_{k+1})$

$$f_k\,|\,\hat{W}_k \underset{p}{\simeq} f_{k+1}\,|\,\hat{W}_k \quad\text{in } \hat{V}_{k+1},$$

where $\underset{p}{\simeq}$ means properly homotopic. For $k, i\in J$, take $U_{k,i}\in\boldsymbol{U}(X, M)$ such that

$$\bigcup_{i\in J} U_{k,i}\times[i, i+1]\subset\hat{W}_k \quad\text{and}\quad U_{k+1,i}\subset U_{k,i+1} \quad\text{for each } k, i\in J.$$

Choose $\delta\in\Delta$ such that $U_{\delta(i)}\subset U_{i,0}$, $i\in J$. There exists a proper homotopy

$$H_k\colon (\bigcup_{i\in J} U_{\delta(k+i)}\times[i, i+1])\times I\to\hat{V}_{k+1}$$

such that

$$H_k(x, t, 0) = f_k(x, t) \quad\text{and}\quad H_k(x, t, 1) = f_{k+1}(x, t)$$

for $(x, t)\in\bigcup_{i\in J} U_{\delta(k+i)}\times[i, i+1]$. Let $P_N\colon N\times R_+\to N$ and $P_{R_+}\colon N\times R_+\to R_+$ be the projections. Define $H\colon U_\delta\to\hat{V}_0$ by

$$H(x, t) = \{P_N\cdot H_k(x, 0, t-k), t+P_{R_+}\cdot H_k(x, 0, t-k)\}$$

for $(x, t)\in U_\delta$, $t\in[k, k+1]$, $k\in J$. Since

$$H(U_{\delta(k)}\times[k, k+1])\subset\hat{V}_{k+1} \quad\text{for each } k\in J,$$

by applying homotopy extension theorem H has an extension $\tilde{H}\colon M\times R_+\to N\times R_+$ such that

$$\tilde{H}(U_{\delta(k)}\times[k, \infty))\subset\hat{V}_k \quad\text{for}\quad k\in J.$$

Put $F = P_N\tilde{H}$. It is easy to see that F is a fundamental map of X into Y in M, N and φ_F is similar to $\boldsymbol{f}$. This completes the proof of (2.8).

The following is a consequence of Theorem 1 and [12, Theorem 2] and gives a partial solution to a question of Ball [1].

COROLLARY 1. *Let $\mathscr{C}_p$ be the proper shape category defined by Ball and Sher [3] and let $\tilde{\mathscr{C}}_p$ be the full subcategory of $\mathscr{C}_p$ whose objects consist of spaces of the form*

$X\times R_+$, where X is any compactum. Then there exists a category isomorphism from $\mathscr{C}_p$ onto $\mathscr{C}_p^B$ which is the identity on the set of objects.

For a locally compact space Y, we denote by $\mathrm{Sh}_p^B(Y)$ the proper shape of Y in the sense of Ball [1] which is taken in the category $\mathscr{C}_p^B$.

THEOREM 2. *Let X, X' be compacta and let Y, Y' be locally compact spaces. If $\mathrm{Sh}_p^B(X\times R_+)\geqslant \mathrm{Sh}_p^B(X'\times R_+)$ (or equivalently $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(X')$) and $\mathrm{Sh}_p^B(Y)\geqslant\mathrm{Sh}_p^B(Y')$, then $\mathrm{Sh}_p^B(X\times Y)\geqslant\mathrm{Sh}_p^B(X'\times Y')$. Here $\mathrm{Sh}_f(X)$ is the fine shape of X.*

The theorem is proved by the same way as in the proof of [10, Theorem 5'] and we omit the proof.

For a compactum X and an abelian group G, we mean by ${}^sH_q(X{:}G)$ the q-dimensional Steenrod homology group of X with coefficients in G [18].

THEOREM 3. *Let X and X' be compacta. If $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(X')$, then ${}^sH_q(X'{:}G)$ is a direct factor of ${}^sH_q(X{:}\ G)$ for each q.*

The proof is obvious. For, let $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(X')$. By Theorem 1 of [12], there exist compact AR's M, M' containing X, X' as unstable subsets, respectively, and proper maps $f\colon M-X\to M'-X'$ and $g\colon M'-X'\to M-X$ such that $M-X$ and $M'-X'$ are locally finite simplicial complexes and $gf\underset{p}{\simeq}1_{M-X}$. Note that f and g can be chosen arbitrarily within their proper homotopy classes. By the definition of the homology ${}^sH_*$, the maps f and g induce unique homomorphisms

$$f_*\colon {}^sH_q(X{:}G)\to{}^sH_q(X'{:}G)\quad\text{and}\quad g_*\colon {}^sH_q(X'{:}G)\to{}^sH_q(X{:}G)$$

such that $g_*f_*=$ the identity on ${}^sH_q(X{:}G)$ for each q. This completes the proof.

PROBLEM. In Theorem 3, can the relation "$\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(X)$" be replaced by "$\mathrm{Sh}(X)\geqslant\mathrm{Sh}(X')$"?

Here $\mathrm{Sh}(X)$ is the shape of X in the sense of Borsuk [4].

3. Pointed FANR. A compactum X is said to be a *pointed* FANR if for every $x\in X$ the pointed space (X,x) is an FANR in the pointed shape category in the sense of Borsuk [4, p. 204]. It is known by Siebenmann, Guillou and Hähl [17] and Dydak [5] that X is a pointed FANR if and only if X has the shape of a CW-complex.

Let K be a compactum lying in a compact AR M. A compactum $X\subset K$ is said to be a *fine fundamental retract of K in M* if there exists a fundamental map $r\colon K\to X$ in M, M such that

(3.1) $$r(x,t)=x\quad\text{for each }(x,t)\in X\times R_+\,.$$

A compactum X is said to be a *fine absolute neighborhood retract* (FANR_f) if for every compactum Y containing X and for every compact AR M (or equivalently, for some compact AR M) containing Y there exists a closed neighborhood K of X in Y such that X is a fine fundamental retract of K in M.

Let Y be a compactum contained in a compact AR M as an unstable subset. A compactum $X\subset Y$ is a *fine coretract of Y in M* if there exists a proper map $r\colon M-Y\to M-X$ (called a *fine coretraction from Y to X in M*) satisfying the following condition;

(3.2) for every $U\in \boldsymbol{U}(X,M)$ there exists a $V\in\boldsymbol{U}(X,U)$ such that for every $U_0\in\boldsymbol{U}(X,M)$ there exists a $V_0\in\boldsymbol{U}(X,V)$ and a homotopy $H\colon (V-Y)\times I\to U$ satisfying that $H(y,0)=r(y)$ and $H(y,1)=y$ for $y\in V-Y$ and $H((V_0-Y)\times I)\subset U_0$.

The following is obvious.

(3.3) *Suppose that M and M' are compact* AR's *containing a compactum Y as an unstable subset. If a compactum $X\subset Y$ is a fine coretract of Y in M, then X is also a fine coretract of Y in M'.*

By (3.3) we can define as follows. A compactum X is an *absolute fine coretract* if for every compactum Y containing X and for every (or equivalently, some) compact AR M containing Y unstably there exists a closed neighborhood K of X in Y such that X is a fine coretract of K in M.

A compactum X is *fine movable* if for every (or equivalently, some) AR M containing X and for every $U\in\boldsymbol{U}(X,M)$ there exists a $U_0\in\boldsymbol{U}(X,U)$ satisfying the following condition:

(3.4) For any $V\in\boldsymbol{U}(X,M)$ there exists a homotopy $H\colon U_0\times I\to U$ such that $H(x,0)=x$ and $H(x,1)\in V$ for $x\in U_0$ and $H(x,t)=x$ for $(x,t)\in X\times I$.

This definition is slightly different from the original one [11, (2.2)], but it is easy to see that two definitions are equivalent.

Next, after the model of S. Godlewski [8] we define an absolute (neighborhood) proper shape retract. Let X, Y and P be locally compact spaces such that X is closed in Y and Y is closed in P. A proper mutation $\boldsymbol{r}\colon U(Y,P)\to U(X,P)$ is said to be a *proper mutational retraction* if for every $r\in\boldsymbol{r}$ $r(x)=x$, $x\in X$. A closed subset X of a locally compact space Y is a *mutational proper retract* of Y if there exist a locally compact space P containing Y as a closed set and a proper mutational retraction $\boldsymbol{r}$ from $\boldsymbol{U}(Y,P)$ to $\boldsymbol{U}(X,P)$. A locally compact space X is a *mutational absolute neighborhood proper shape retract* (ANPSR_m) if for every locally compact space Y containing X as a closed subset there exists a closed neighborhood K of X in Y such that X is a proper mutational retract of K.

Finally we extend the concept of movability to the proper shape category. A locally compact space X is said to be *properly movable* if for every (or equivalently, some) locally compact ANR M containing X as a closed subset and for every $U\in\boldsymbol{U}(X,M)$ there exists a closed neighborhood W of X in U such that for every $V\in\boldsymbol{U}(X,U)$ there exists a proper homotopy $H\colon W\times I\to U$ such that

(3.5) $$H(x,0)=x\quad\text{and}\quad H(x,1)\in V\quad\text{for}\quad x\in W\,.$$

The followings are obvious.

(3.6) *Every* ANPSR *in the sense of Sher* [16] *is an* ANPSR_m.

(3.7) *Every* ANPSR_m *is properly movable.*

(3.8) *Every movable compactum in the sense of Borsuk* [4] *is properly movable and every properly movable locally compact space is movable in the sense of Kozlowski and Segal* [9].

(3.9) *Every locally compact* 0*-dimensional space is properly movable.*

(3.10) *The proper movability is a hereditarily proper shape property, that is, if* X, Y *are locally compact,* $\mathrm{Sh}_p^B(X) \leqslant \mathrm{Sh}_p^B(Y)$ *and* Y *is properly movable then* X *is also properly movable.*

Whether every ANPSR_m is an ANPSR is an open question. The following example shows that the converse of (3.7) or (3.8) does not hold.

EXAMPLE. Let us define locally compact spaces X and Y lying the plane R^2 as follows:

$$S_k = \{(x, y) \in R^2 \colon (x - 3/2^{k+1})^2 + y^2 = 1/2^k\},\ k = 1, 2, 3, \ldots;$$

$$S_0 = \{(0, 0) \in R^2\};$$

$$X = \Big(\bigcup_{i=0}^{\infty} S_i\Big) \times R_+;$$

$$Y = \Big(\bigcup_{i=0}^{\infty} S_i \times [0, i]\Big) \cup S_0 \times R_+ .$$

Then X is movable in the sense of Kozlowski and Segal [9] but not properly movable. Also Y is properly movable but not an ANPSR_m.

By the example it is known that the proper movability is not productive. However, the following theorem shows that if X is a compactum and for some APSR K in the sense of Ball [2] $X \times K$ is properly movable then X is a pointed FANR.

THEOREM 4. *Let* X *be a compactum. Then the following are equivalent.*

(i) X *is a pointed* FANR.

(ii) X *is an* FANR_f.

(iii) X *is an absolute neighborhood fine coretract.*

(iv) X *is fine movable.*

(v) $X \times K$ *is an* ANPSR *for every* ANPSR K.

(vi) $X \times K$ *is an* ANPSR_m *for every* ANPSR_m K.

(vii) $X \times K$ *is properly movable for every* ANPSR_m K.

(viii) $X \times K$ *is an* ANPSR *for some* APSR K.

(ix) $X \times K$ *is an* ANPSR_m *for some* APSR K.

(x) $X \times K$ *is properly movable for some* APSR K.

We shall show the following implications:

$$\text{(i)} \to \text{(ii)} \to \text{(iii)} \to \text{(iv)} \to \text{(i)}$$

$$\begin{array}{ccccccccc} \text{(i)} & \longrightarrow & & \longrightarrow & \text{(vi)} & \to & \text{(vii)} & & \\ \downarrow & & & & \downarrow & & \downarrow & & \\ \text{(v)} & \to & \text{(viii)} & \to & \text{(ix)} & \to & \text{(x)} & \to & \text{(iv)} \end{array}$$

The proof of the implications (i) $\leftrightarrow$ (iv) and (i) $\to$ (v), (vi) was given by [11, Theorem 2] and [10, Theorem 6 and Remark 4].

(i) $\to$ (ii) This follows from Dydak, Nowak and Strok [6, Lemma 4].

(ii) $\to$ (iii) Let Y be a compactum containing X. We assume that Y is a subset of the Hilbert cube Q. Since X is an FANR_f, there exist a closed neighborhood K of X in Y and a fundamental map $r \colon K \to X$ in Q, Q satisfying the condition (3.1). Choose a map $\alpha \colon Q \to I$ such that $K = \alpha^{-1}(1)$. Consider the product $Q \times I$ and identify Q with $Q \times \{1\} \subset Q \times I$. Define a map $f \colon Q \times I - K \to Q \times I - X$ by

$$f(x, t) = \{r(x, \mathrm{Min}(\alpha(x), t)/(1 - \mathrm{Min}(\alpha(x), t))), \mathrm{Min}(\alpha(x), t)\}$$

for $(x, t) \in Q \times I - K$. Obviously f is a fine coretraction from K to X in $Q \times I$.

(iii) $\to$ (iv) Suppose that an absolute neighborhood coretract X is contained in the Hilbert cube Q. Identify Q with $Q \times \{1\} \subset Q \times I$. Since Q is an unstable subset of $Q \times I$, there exist a closed neighborhood Y of X in Q and a fine coretraction $r \colon Q \times I - Y \to Q \times I - X$ from Y to X in $Q \times I$. Let $U \in \boldsymbol{U}(X, Q)$. By the definition of a fine coretraction there exist $V_0 \in \boldsymbol{U}(X, Q)$ and $\varepsilon_0 > 0$ such that the neighborhood $V_0 \times [1 - \varepsilon_0, 1] \in \boldsymbol{U}(X, Q \times I)$ satisfies the condition (3.2) for $M = Q$, $U = U \times I$ and $V = V_0 \times [1 - \varepsilon_0, 1]$. Put $U_0 = V_0 \cap Y$. We shall prove that the neighborhood $U_0 \in \boldsymbol{U}(X, Q)$ satisfies (3.4) for $M = Q$. To do it, let V be any open neighborhood of X in Q. From the condition (3.2) there exists $V_1 \in \boldsymbol{U}(X, Q)$, $\varepsilon_1 > 0$ and a homotopy $K \colon (V_0 \times [1 - \varepsilon_0, 1] - Y) \times I \to U \times I$ such that

$$K(x, t, 0) = r(x, t) \quad \text{and} \quad K(x, t, 1) = (x, t)$$

for $(x, t) \in V_0 \times [1 - \varepsilon_0, 1] - Y$, and

$$K\big((V_1 \times [1 - \varepsilon_1, 1] - Y) \times I\big) \subset V \times I .$$

Since r is proper, $r^{-1}(Q \times I - V \times I)$ is a compact set of $Q \times I - Y$. Then there exist $W \in \boldsymbol{U}(Y, Q)$ and $\varepsilon_2 > 0$ such that $r(W \times [1 - \varepsilon_2, 1] - Y) \subset V \times I$. We may assume that $\varepsilon_2 < \varepsilon_1$. Define a homotopy $H' \colon U_0 \times I \to U$ by

$$H'(x, 0) = x \quad \text{for} \quad x \in U_0 ,$$

$$H'(x, s) = P_Q \cdot K(x, (1 - s\varepsilon_2), 1 - s) \quad \text{for} \quad (x, s) \in U_0 \times (0, 1] ,$$

where $P_Q \colon Q \times I \to Q$ is the projection. Then $H'(U_0 \times \{1\} \cup V_1 \times I) \subset V$. Since V is an ANR, by a standard argument we can construct a homotopy $H \colon U_0 \times I \to U$ from H' satisfying the condition:

$$H(x,0) = x \quad \text{and} \quad H(x,1) \in V \quad \text{for} \quad x \in U_0,$$
$$H(x,t) = x \quad \text{for} \quad (x,t) \in X \times I.$$

This completes the proof.

(v) → (viii) → (ix) and (vi) → (ix). Because every APSR is an ANPSR and an ANPSR$_m$ by Sher [16] and (3.6).

(vi) → (vii) and (ix) → (x). It is obvious from (3.7).

(x) → (iv) Let X be a compactum and suppose that there exists an APSR K such that $X \times K$ is properly movable. Since $K \in \mathrm{SUV}^\infty$ by Ball [2, Theorem 4.7], we have $\mathrm{Sh}_p(R_+) \leqslant \mathrm{Sh}_p(K)$ (cf. Sher [15]) and hence $\mathrm{Sh}_p^B(R_+) \leqslant \mathrm{Sh}_p^B(K)$ by Ball [1, Theorem 7.0]. Then $\mathrm{Sh}_p^B(X \times R_+) \leqslant \mathrm{Sh}_p^B(X \times K)$ by Theorem 2. Therefore $X \times R_+$ is properly movable by (3.10).

Let X be a subset of a compact AR M. We use the same notations as in the proof Theorem 1. Let Δ be the set of all strictly increasing functions $\delta\colon J \to J$ and let $\{U_i\colon i \in J\}$ be a neighborhood basis of X in M such that each U_i is closed and $U_i \supset U_{i+1}$, $i \in J$. For each $\delta \in \Delta$, we put

$$U_\delta = \bigcup_{i \in J} U_{\delta(i)} \times [i, i+1].$$

Let U be any neighborhood of X in M. There is a $j_0 \in J$ such that $U_{j_0} \subset U$. Put

$$W = \bigcup_{i \in J} U_{j_0+i} \times [i, i+1].$$

Since $X \times R_+$ is properly movable, we can find a closed neighborhood V_0 of $X \times R_+$ in W such that for every $V \in U(X \times R_+, W)$ there exists a proper homotopy $H\colon V_0 \times I \to W$ such that

$$(3.11) \qquad H(y,0) = y \quad \text{and} \quad H(y,1) \in V \quad \text{for} \quad y \in V_0.$$

We may assume that $V_0 = U_\delta$ for some $\delta \in \Delta$. Put $U_* = U_{\delta(0)}$. Let U' be an arbitrary neighborhood of X in U_*. We shall construct a homotopy $K\colon U_* \times I \to U$ such that

$$(3.12) \qquad \begin{aligned} &K(x,0) = x \quad \text{and} \quad K(x,1) \in U' \quad \text{for} \quad x \in U_*,\\ &K(x,t) = x \quad \text{for} \quad (x,t) \in X \times I.\end{aligned}$$

To do it, take a $k_0 \in J$ such that $U_{k_0} \subset U'$. Set

$$(3.13) \qquad V = \bigcup_{i \in J} U_{k_0+i} \times [i, i+1].$$

By the property of $V_0 = U_\delta$, there exists a proper homotopy $H\colon U_\delta \times I \to W$ satisfying (3.11). Since $H^{-1}(W \cap M \times [0, k_0 - j_0])$ is compact, we can find an $m \in J$ such that

$$H\big((U_\delta \cap M \times [m, \infty)) \times I\big) \subset W \cap M \times [k_0 - j_0, \infty),$$

that is,

$$(3.14) \qquad P_M \cdot H\big((U_\delta \cap M \times [m, \infty)) \times I\big) \subset U_{k_0},$$

where $P_M\colon M \times I \to M$ is the projection. Choose a map $\alpha\colon U_* \to [0, m+1]$ such that

$$(3.15) \qquad \alpha^{-1}([i, m+1]) = U_{\delta(i)}, \qquad i = 0, 1, 2, \ldots, m+1.$$

Define $\beta\colon U_* \to I$ by

$$(3.16) \qquad \beta(x) = \mathrm{Min}\{1, m+1-\alpha(x)\} \quad \text{for} \quad x \in U_*.$$

Finally, let us define a homotopy $K\colon U_* \times I \to U$ by

$$K(x,s) = P_M \cdot H\big(x, \alpha(x), s\beta(x)\big) \quad \text{for} \quad (x,s) \in U_* \times I.$$

By (3.11) $K(x,0) = x$ for $x \in U_*$. Since $\beta(x) = 1$ for $x \in U_* - U_{\delta(m)}$ by (3.15) and (3.16), we have $K(x,1) \in U_{k_0}$ for $x \in U_*$ by (3.13), (3.14) and (3.11). Also, $\beta(x) = 0$ for $U_{\delta(m+1)}$ and hence $K(x,t) = x$ for every $(x,t) \in U_{\delta(m+1)} \times I$. Thus K satisfies (3.12). Therefore X is fine movable. This completes the proof.

Note that we can not replace "every" by "some" in (v), (vi) and (vii) of Theorem 4. Because, consider the discrete space $J = \{0, 1, 2, \ldots\}$. For every movable compactum X, the product $X \times J$ is properly movable. However X is not generally an FANR.

References

[1] B. J. Ball, *Alternative approaches to proper shape theory*, Proc. Charlotte Top. Conference.

[2] — *Proper shape retracts*, Fund. Math. 89 (1975), pp. 177–189.

[3] — and R. B. Sher, *A theory of proper shape for locally compact metric spaces*, Fund. Math. 86 (1974), pp. 163–192.

[4] K. Borsuk, *Theory of Shape*, Warszawa 1975.

[5] J. Dydak, *A simple proof that pointed connected FANR-spaces are regular fundamental retracts of ANR's*, Bull. Acad. Polon. Sci 25 (1977), pp. 55–62.

[6] — S. Nowak and M. Strok, *On the union of two FANR-sets*, Bull. Acad. Polon. Sci. 24 (1976), pp. 485–489.

[7] R. H. Fox, *On shape*, Fund. Math. 72 (1971), pp. 47–71.

[8] S. Godlewski, *Mutational retracts and extensions of mutations*, Fund. Math. 84 (1974), pp. 47–65.

[9] G. Kozlowski and J. Segal, *Locally well-behaved paracompacta in shape theory*, Fund. Math. 95 (1977), pp. 55–71.

[10] Y. Kodama, *On product of shape and a question of Sher*, Pacific J. Math. 92 (1977), pp. 115–134.

[11] — *Fine movability*, to appear.

[12] — and J. Ono, *On fine shape theory*, Fund. Math. 105 (1979), pp. 29–39.

[13] S. Mardešić, *Shapes for topological spaces*, Gen. Top. Appl. 3 (1973), pp. 265–282.

[14] — and J. Segal, *Spaces of compacta and ANR-systems*, Fund. Math. 72 (1971), pp. 41–59,

[15] R. B. Sher, *Property* SUV$^\infty$ *and proper shape theory*, Trans. Amer. Math. Soc. 190 (1974), pp. 345–356.
[16] — *Extensions, retracts and absolute neighborhood retracts in proper shape theory*. Fund. Math. 96 (1977), pp. 149–159.
[17] L. C. Siebenmann, L. Guillou et H. Hähl, *Les voisinages reguliers: criteres homotopiques d'existence*, Ann. Sci E. N. S. 7 (1974), pp. 431–462.
[18] N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. of Math. 41 (1940), pp. 833–851.

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Intersection of sectorial cluster sets and directional essential cluster sets

by

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Abstract. Let $f\colon H\to W$, where H is the upper half plane and W is a second countable topological space, and let R be the real line. It is proved that, except a set of points x on R, which is of the first category and measure zero, every essential directional cluster set of f is a subset of every sectorial cluster set of f at x; and if W is compact and normal, then except a countable set of points x on R, every essential directional cluster set intersects every sectorial cluster set of f at x.

1. Let H denote the open upper half plane and let z denote points of H. Let x denote points on the real line R. For each $x\in R$, $\theta\in(0,\pi)$ and $h>0$, let

$$L_\theta(x) = \{z\colon z\in H;\ \arg|z-x| = \theta\}$$

and

$$L_\theta(x,h) = \{z\colon z\in H;\ |z-x|<h\}\cap L_\theta(x).$$

For each pair of directions θ_1,θ_2, $0<\theta_1<\theta_2<\pi$, $\sigma_{\theta_1\theta_2}$ denotes the sector in H with vertex at the origin, defined by

$$\sigma_{\theta_1\theta_2} = \{z\colon z\in H;\ \theta_1<\arg z<\theta_2\}.$$

If there is no ambiguity, we shall simply write σ instead of $\sigma_{\theta_1\theta_2}$. By $\sigma(x)$ we mean the sector in H with vertex at x and which is obtained by a translation of σ. That is,

$$\sigma(x) = \{z\colon z\in H;\ \theta_1<\arg(z-x)<\theta_2\}.$$

Also for $x\in R$ and $h>0$ we shall write

$$\sigma(x,h) = \sigma(x)\cap\{z\colon z\in H;\ |z-x|<h\}.$$

For $E\subset H$, the upper outer density $\bar{d}^*_\theta(E,x)$ and outer density $d^*_\theta(E,x)$ of E at x in the direction θ are defined by

$$\bar{d}^*_\theta(E,x) = \limsup_{h\to 0}\frac{\mu^*[E\cap L_\theta(x,h)]}{\mu(L_\theta(x,h))} \tag{1}$$

and

$$d^*_\theta(E,x) = \lim_{h\to 0}\frac{\mu^*[E\cap L_\theta(x,h)]}{\mu(L_\theta(x,h))}, \tag{2}$$