Subhomeotopy groups of the 2-sphere with n holes

by

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Abstract. Let S_n denote the 2-sphere with n holes and let $H(S_n)$ denote the group of isotopy classes (rel ∂S_n) of homeomorphisms of S_n onto itself which are the identity on ∂S_n. A classical result of J. W. Alexander is that $H(S_1)$ is trivial. H. Gluck has shown that $H(S_2) \cong Z$ and J. P. Lee has shown $H(S_3) \cong Z \times Z \times Z$. In this paper spin homeomorphisms and twist homeomorphisms are used to obtain a presentation for $H(S_n)$ for $n > 3$. In particular, it is shown that $H(S_n)$ made abelian is the free group on $[(n-1)(n-2)/2]-1+n$ generators.

1. Introduction. Let X be a compact 2-dimensional manifold with boundary. The group of isotopy classes (rel ∂X) of homeomorphisms of X onto itself which are the identity on ∂X is referred to by L. V. Quintas as a subhomeotopy group of X. In the case X is a 2-sphere with n holes, J. W. Alexander [1] has shown this group is trivial for $n = 1$. H. Gluck [3] has shown this group is isomorphic to Z for $n = 2$ and J. P. Lee [4] has shown it is isomorphic to Z^3 for $n = 3$. In this paper we will investigate the structure of this group for $n > 3$.

2. Notation and preliminaries. Let $S_n = S^2 - \bigcup_{k=1}^{n} \operatorname{Int}(D_k)$ where $D_1, \ldots, D_n$ are disjoint closed disks in S^2. Let $G(S_n)$ denote the group of homeomorphisms of S_n which leave ∂S_n pointwise fixed. Let $G_0(S_n)$ denote the normal subgroup of $G(S_n)$ consisting of those homeomorphisms which are isotopic to the identity. Let $G_1(S_n)$ denote the normal subgroup of $G(S_n)$ consisting of those homeomorphisms, h, which are isotopic to the identity by an isotopy which leaves ∂S_n pointwise fixed (denoted $h \cong 1$ (rel S_n)). Let $H(S_n) = G(S_n)/G_1(S_n)$ and $H'(S_n) = G(S_n)/G_0(S_n)$. Finally let $K(S_n)$ denote the kernal of the natural homomorphism $d\colon H(S_n) \to H'(S_n)$. Note that a typical non-trivial element in $K(S_n)$ is represented by a homeomorphism of S_n which leaves ∂S_n fixed and is isotopic to the identity, but is not isotopic to the identity by an isotopy which leaves ∂S_n fixed.

The desired presentation for $H(S_n)$ will be obtained by first obtaining presentations for $K(S_n)$ and $H'(S_n)$ and then showing that $H(S_n) \cong K(S_n) \oplus H'(S_n)$.

3. Presentation for $K(S_n)$. In this section we will show that for $n \geq 3$, $K(S_n) \cong Z^n$.

Let A be an annulus in the plane parametrized by (r, θ) where $1 \leq r \leq 2$ and θ is a real number mod 2π. Let $s\colon A \to A$ be defined by $s(r, \theta) = (r, \theta - 2\pi r)$.

LEMMA 3.1. *Let $f\colon A \to A$ be a homeomorphism of A such that $f/\partial A$ is the identity, then there is a unique integer r such that f is isotopic to s^r by an isotopy which is fixed on ∂A.*

Proof. This is a consequence of Theorem 7.2 of [3].

Remark. s^r can be considered as resulting from "spinning" one component of ∂A r-times while holding the other boundary component fixed.

THEOREM 3.2. *If $n \geqslant 3$, $K(S_n) \cong Z^n$.*

Proof. For each boundary component C_i, $1 \leqslant i \leqslant n$, of S_n, let A_i be a collar neighborhood of C_i with $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $s_i \in G(S_n)$ be the homeomorphism obtained by letting $s_i/A_i = s$ and extending by the identity. s_i is called a spin homeomorphism of S_n.

If f represents an element in $K(S_n)$, then there exists an isotopy H_t of S_n with $H_0 = 1$ and $H_1 = f$. Let $h_t = H_t^{-1}/\partial S_n$ and extend h_t to an isotopy G_t of S_n which is the identity on $S_n - \bigcup_{i=1}^{n} \operatorname{Int} A_i$ for all t and is such that $G_0 = 1$. To construct G_t merely "unwind" the isotopy H_t/C_i across the annulus A_i for each i. More precisely, if we let $h_t(x) = x$ for $t<0$, then G_t can be defined by $G_t(x, v) = (h_{t-v}(x), v)$ for (x, v) in $C_i \times I \cong A_i$ and $G_t(x) = x$ for x not in $\bigcup_{i=1}^{n} A_i$. Now $G_1 f^{-1} \simeq 1$ $(\operatorname{rel}\partial S_n)$ by the isotopy $G_t H_t$, hence we can take G_1 as a representative of the equivalence class of f in $K(S_n)$. G_1 is the identity on $S_n - \bigcup_{i=1}^{n} \operatorname{Int} A_i$ and by Lemma 3.1 $G_1/A_i \simeq s_i^{k_i}/A_i$ $(\operatorname{rel}\partial A_i)$ for each i. Since $A_i \cap A_j = \emptyset$ for $i \neq j$ we have $G_1/\bigcup A_i \simeq s_n^{k_n} \ldots s_1^{k_1}$ $(\operatorname{rel}\partial(\bigcup A_i))$. Thus the equivalence class of G_1, and hence also that of f, in $K(S_n)$ can be represented by a product of spin homeomorphisms. Next we show this representation is unique for $n \geqslant 3$.

Let $n \geqslant 3$ and assume $s_n^{k_n} \ldots s_1^{k_1} \simeq 1$ $(\operatorname{rel}\partial S_n)$ where $k_p \neq 0$ for some p. Let $x_0 \in C_p$ and $q \neq p$. Let α_p be a simple closed curve based at x_0 which transverses the boundary component C_p once in a clockwise direction. Let α_q be a simple closed curve based at x_0 which loops once around the boundary component C_q in a clockwise direction and is such that α_q and C_q form the boundary of an annulus whose interior is disjoint from $\bigcup_{k=1}^{n} C_k$. In addition, we assume α_q is disjoint from A_i for $i \neq p$. Note that α_q and α_p can be considered as representatives of 2 of the $n-1$ generating elements of the free group $\Pi_1(S_n, x_0)$. Since α_q is disjoint from A_i for $i \neq p$, $(s_n^{k_n} \ldots s_1^{k_1})(\alpha_q) = s_p^{k_p}(\alpha_q)$. Now $s_p^{k_p}(\alpha_q)$ represents the element $[\alpha_p]^{k_p}[\alpha_q][\alpha_p]^{-k_p}$ in $\Pi_1(S_n, x_0)$. On the other hand, since $s_n^{k_n} \ldots s_1^{k_1} \simeq 1$ $(\operatorname{rel}\partial S_n)$, $s_p^{k_p}(\alpha_q)$ must also represent the element $[\alpha_q]$ in $\Pi_1(S_n, x_0)$. This is impossible for $n \geqslant 3$. Thus the representation of elements of $K(S_n)$ as products of spin homeomorphisms is unique and the theorem is proved.

Remark. Lemma 3.1 shows $K(S_2) \cong Z$ and clearly $K(S_1) \cong 1$, since $H(S_1) \cong 1$. The reason we do not get Z^n in these cases is that although the spin homeomorphisms

still generate the given groups, in the case of the annulus we have $s_1 \simeq s_2$ $(\operatorname{rel}\partial S_2)$ while in the case of the disk we have $s_1 \simeq 1$ $(\operatorname{rel}\partial S_1)$.

4. Presentation of $H'(S_n)$. Let $p_1, \ldots, p_n$ be points in S^2 with p_i in $\operatorname{Int} D_i$ for $1 \leqslant i \leqslant n$. Let $F_n = \{p_1, \ldots, p_n\}$. Define $H'(S^2, F_n)$ to be the group of isotopy classes $(\operatorname{rel} F_n)$ of homeomorphisms of S^2 which are the identity on F_n. As shown in Parts 3.c and 4 of the proof of Theorem 6 of [7], the function $\Psi\colon H'(S_n) \to H'(S^2, F_n)$ which sends the equivalence class of a homeomorphism h in $H'(S_n)$ to the equivalence class of the homeomorphism of S^2 obtained by taking the "cone" of h is an isomorphism.

For $1 \leqslant i \leqslant j$, let α_{ij} denote a simple closed curve in S^2 which encloses the points p_i and $p_j, \ldots, p_n$ and is disjoint from $\bigcup_{i=1}^{n} D_i$. That is, one of the disks bounded by α_{ij} contains $\{p_i, p_j, \ldots, p_n\}$ and the other disk contains $F_n - \{p_i, p_j, \ldots, p_n\}$. Let A_{ij} be an annulus in S^2 which has α_{ij} as one boundary component and has its other boundary component inside the disk bounded by α_{ij} which contains p_i. Assume $A_{ij} \cap D_k = \emptyset$ for $1 \leqslant k \leqslant n$. Let $e_{ij}\colon A \to A_{ij}$ be an embedding of the annulus A into S^2 which is such that $e_{ij}(2, \theta) = \alpha_{ij}(\theta/2\pi)$. Define $a_{ij}\colon S^2 \to S^2$ by letting $a_{ij} = e_{ij} s e_{ij}^{-1}$ on A_{ij} and extending by the identity. Note that the effect of a_{ij} is to give one twist to the annulus A_{ij}. A discussion of "twist" homeomorphisms is given in [2]. In particular, the homeomorphism a_{ij} corresponds to the "η-twist" homeomorphism of [2].

Let $\bar{a}_{ij}$ denote the equivalence class of a_{ij} in $H'(S^2 F)$. The following theorem is an immediate consequence of Lemma 3.14 and Theorem 3.15 of [6].

THEOREM 4.1. *Let $G_n = \bigcup_{k=3}^{n} \{\bar{a}_{ik}\colon 1 \leqslant i < k\}$. If $n \geqslant 3$, then $H'(S^2, F_n)$ is generated by G_n and in terms of these generators, $H'(S^2, F_n)$ has a presentation in which a complete set of relations is given as follows:*

1. *If $p \leqslant q$ and $i = r$* or *if $p<q$ and $i<r \geqslant p$*, then
$$\bar{a}_{ip}\bar{a}_{rq}\bar{a}_{ip}^{-1} = \bar{a}_{rq}.$$
2. *If $p<q$ and $i>r$, then*
$$\bar{a}_{ip}\bar{a}_{rq}\bar{a}_{ip}^{-1} = (\bar{a}_{iq}(\bar{a}_{pq} \ldots \bar{a}_{q-1,q}))\bar{a}_{rq}(\bar{a}_{iq}(\bar{a}_{pq} \ldots \bar{a}_{q-1,q}))^{-1}.$$
3. *If $p<q$ and $i<r<p$, then*
$$\bar{a}_{ip}\bar{a}_{rq}\bar{a}_{ip}^{-1} = ((\bar{a}_{pq} \ldots \bar{a}_{q-1,q})\bar{a}_{iq})\bar{a}_{rq}((\bar{a}_{pq} \ldots \bar{a}_{q-1,q})\bar{a}_{iq})^{-1}.$$

Moreover, all the isotopes used to establish the above relations can be taken to be fixed on $\bigcup_{k=1}^{n} D_k$.

Since each a_{ij} is fixed on $\bigcup_{k=1}^{n} D_k$, if we let b_{ij} equal the restriction of a_{ij} to S_n, then b_{ij} is a homeomorphism of S_n fixed on ∂S_n. In fact a_{ij} is the cone of b_{ij}. This means that the isomorphism $\Psi\colon H'(S_n) \to H'(S^2, F_n)$ sends the equivalence class of b_{ij} to the equivalence class of a_{ij}. Since the isotopies used to establish Theorem 4.1

can be taken to be fixed on $\bigcup_{k=1}^{n} D_k$, Ψ^{-1} sends these isotopies onto isotopies of S_n which are fixed on ∂S_n. Thus if we let $\bar{b}_{ij}$ denote the equivalence class of b_{ij} in $H'(S_n)$, then the isomorphism Ψ^{-1} establishes the following result directly from Theorem 4.1.

COROLLARY 4.2. *Let* $G_n = \bigcup_{k=3}^{n} \{\bar{b}_{ik} | \ 1 \leqslant i < k\}$. *If* $n \geqslant 3$, *then* $H'(S_n)$ *is generated by* G_n *and in terms of these generators* $H'(S_n)$ *has a presentation in which a complete set of relations is obtained by replacing each* $\bar{a}_{ij}$ *by* $\bar{b}_{ij}$ *in the presentation for* $H'(S^2, F_n)$. *Moreover, the isotopies used to establish the relations in* $H'(S_n)$ *can be chosen so as to be fixed on* ∂S_n.

5. Presentation for $H(S_n)$.

THEOREM 5.1. *If* $n \geqslant 3$, $H(S_n) \cong K(S_n) \times H'(S_n)$.

Proof. Consider the short exact sequence

$$1 \to K(S_n) \xrightarrow{i} H(S_n) \xrightarrow{d} H'(S_n) \to 1\,.$$

Let h be a homeomorphism of S_n fixed on ∂S_n. Let $[h]$ represent the equivalence class of h in $H'(S_n)$. Let A be the subgroup of $H(S_n)$ generated by $\{[s_r] | \ 1 \leqslant r \leqslant n\}$ and let B be the subgroup of $H(S_n)$ generated by $\bigcup_{k=3}^{n} \{[b_{ik}] | \ 1 \leqslant i < k\}$. Note that s_r and b_{ik} commute for all choices of i, k, r because their supports are disjoint. Hence the elements of A commute with the elements of B. Moreover, $A \cap B = \{1\}$ as can be seen by the following argument. Suppose $w[s_r] = w'[b_{ij}]$ where $w[s_r]$ is a word involving only elements of A and $w'[b_{ij}]$ is a word involving only elements of B. Since $A = I_m(i)$, we have

$$1 = d(w[s_r]) = d[w'(b_{ij})] = w(\bar{b}_{ij})\,.$$

But by Corollary 4.2. the isotopies used to establish the relations in $H'(S_n)$ can be taken to be fixed on ∂S_n, thus $w(\bar{b}_{ij}) = 1$ if and only if $w[b_{ij}] = 1$. This shows that $A \cap B = \{1\}$ and hence $H(S_n) \cong A \times B$. The proof is completed by noting that $A \cong K(S_n)$ and $B \cong H'(S_n)$.

COROLLARY 5.2. *If* $n \geqslant 3$, $H(S_n)$ *abelianized is the free abelian group on* $[(n-1)(n-2)/2]-1+n$ *generators.*

Proof. This result follows immediately from the fact that $K(S_n) \simeq Z^n$ and $H'(S_n)$ abelianized is the free abelian group on $2+3+\ldots+(n-1)$ generators.

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