Проанализируем теперь свойства спектра $\{Y_\alpha, \pi_\alpha^\gamma | \alpha \leqslant \gamma < \omega_1\}$.

1. Каждое пространство Y_α есть компакт.

2. Отображение π_α^γ неприводимо для всех $\alpha \leqslant \gamma < \omega_1$, так как каждое отображение $\pi_\alpha^{\alpha+1}$ неприводимо (см. об этом в [1]). Обозначим через B предел спектра $\{Y_\alpha, \pi_\alpha^\gamma\}$, и для каждого $\beta \in \omega_1$ через $\pi_\beta^{\omega_1}$ — отображение предельного пространства спектра B на Y_β, и пусть $*$ — точка $\{\{y_\alpha | \alpha < \omega_1\}\}$ в B. Докажем, что B — искомый бикомпакт. Очевидно, что если $b \in B \setminus \{*\}$, то точка b имеет счетный характер в B. Докажем, что $*$ — точка э. н. в B. Пусть W_1 и W_2 — два дизъюнктных открытых в B множества и $* \in [W_1]_B \cap \cap [W_2]_B$. Отображение $\psi \equiv \pi_0^{\omega_1}$ бикомпакта B на Y_0 неприводимо, поэтому существуют такие дизъюнктные открытые множества S_1 и S_2 в Y_0, что $\psi^{-1}(S_1) \subseteq W_1$, $[\psi^{-1}(S_1)]_B \supset W_1$, $\psi^{-1}(S_2) \subseteq W_2$, $[\psi^{-1}(S_2)]_B \supset W_2$, тем самым $S_1 \in \sigma$ и $S_2 \in \sigma$ и какое-то из этих двух множеств имеет меньший номер, пусть $S_1 = V_{\alpha_1}$, $S_2 = V_{\alpha_2}$ и $\alpha_1 < \alpha_2$. Так как для любого $\beta < \omega_1$ $y_\beta \in [(\pi_0^\beta)^{-1}(S_1)]_{Y_\beta} \cap \cap [(\pi_0^\beta)^{-1}(S_2)]_{Y_\beta}$, то на некотором шаге γ описанного выше трансфинитного индуктивного процесса при использовании леммы 3 за W будет взято $V_{\alpha_1} = S_1$, и мы получим тогда, что $y_{\gamma+1} \notin [(\pi_0^{\gamma+1})^{-1}(Y_0 \setminus S_1 \setminus \{y_0\})]_{Y_{\gamma+1}}$, тем самым и $* \notin [\psi^{-1}(S_2)]_B$. Противоречие.

Построение примера 2 полностью завершено.

Пример, аналогичный примеру 2, можно построить и использованием лишь аксиомы Мартина, однако это построение громоздко и потому опускается.

Примечание при корректуре. Недавно В. К. Ван Дауэн доказал, что для всякого вполне регулярного не псевдокомпактного пространства X со счетной π-базой в $\beta X \setminus X$ существует далекая точка. Отсюда также следует существование вполне регулярного пространства с единственной и неизолированной точкой э. н.

Литература

[1] В. В. Федорчук, *О бикомпактах с несовпадающими размерностями*, Докл. АН СССР, 1968, т. 182, № 2, стр. 275–277.

[2] — *Вполне замкнутые отображения и совместимость некоторых теорем общей топологии с аксиомами теории множеств*, Математич. сборник, 1976, т. 99, № 1, стр. 3–33.

[3] Е. Г. Скляренко, *Некоторые вопросы теории бикомпактных расширений*, Известия АН СССР, сер. матем., 1962, т. 26, № 3, стр. 427–452.

[4] R. Telgársky, *Scattered compactification and points of extremal disconnectedness*, Bull. Acad. Polon. Sci. 25 (1977), pp. 159–163.

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$L_{\omega_1\omega}$ equivalence between countable and uncountable linear orderings *

by

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Abstract. The complete $L_{\omega_1\omega}$ theory of an arbitrary denumerable (linear) order type $\mathfrak{A}$ is either (i) categorical, (ii) satisfied in all powers $\leqslant 2^{\aleph_0}$, or (iii) satisfied in all infinite powers. This holds even if $\mathfrak{A}$ is permitted to carry unary predicates. Algebraic properties necessary and sufficient for $\mathfrak{A}$ to be of type (i) or (iii) are stated.

§ 1. Preliminaries. Notation and terminology, where not specifically introduced, will follow [1] or [5]. $|A|$ will denote the cardinality of the set A, and $\mathfrak{A} \cong \mathfrak{B}$ that the structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. By an *ordering* we mean a structure $\mathfrak{A} = (A, \leqslant, P_i^{\mathfrak{A}})_{i \in I}$, where $\leqslant$ is a reflexive linear ordering of A, $|I| \leqslant \aleph_0$, and each $P_i^{\mathfrak{A}}$ is a unary predicate on A. When no unary predicates are present $\mathfrak{A}$ will be referred to as an *order type*. Throughout the paper we assume that L is a first-order finitary language for the orderings currently being considered and that $L_{\omega_1\omega}$ and $L_{\infty\omega}$ are the infinitary languages with the same predicate symbols as L. By a *Scott sentence* we mean a complete $L_{\omega_1\omega}$ sentence satisfied by an ordering.

The universes of orderings $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{A}_0, \mathfrak{A}_1, \ldots$ are denoted by $A, B, C, A_0, A_1, \ldots$ An ordering is referred to as *densely ordered* or *dense* if it satisfies

$$\exists x \exists y (x \neq y) \wedge \forall x \forall y (x < y \rightarrow \exists z (x < z < y)).$$

$\mathfrak{B} \subseteq \mathfrak{A}$ is referred to as *dense* in $\mathfrak{A}$ if $\mathfrak{A}$ is dense and satisfies

$$\forall x, y \in A (x < y \rightarrow \exists z \in B (x < z < y)).$$

If $\mathfrak{A}$ contains a densely ordered subset, then $\mathfrak{A}$ is *nonscattered*. Otherwise $\mathfrak{A}$ is *scattered*. $\mathfrak{B}$ is an *interval* of $\mathfrak{A}$ if $\mathfrak{B} \subseteq \mathfrak{A}$ and $\forall x, y \in B \forall z \in A (x < z < y \rightarrow z \in B)$. By a *nontrivial interval* we mean one with at least two elements. An interval without endpoints is *open*. The notations (a, b), $[a, b]$ for intervals are to be interpreted as usual, while $\mathfrak{A}^{\leqslant a}$, $\mathfrak{A}^{<a}$, $\mathfrak{A}^{\geqslant a}$, $\mathfrak{A}^{>a}$ denote the intervals $\{b \in A: b \leqslant a\}$, $\{b \in A: b < a\}$, $\{b \in A: b \geqslant a\}$, $\{b \in A: b > a\}$. Topological notions, such as neighborhood and limit point, are relative to the topology generated by the open intervals of a dense ordering.

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Let $\mathfrak{A}$, $\mathfrak{B}$ be orderings. $G(\mathfrak{A}, \mathfrak{B})$ is the game for two players in which a move consists of a choice by player one of an element from either structure and by player two of a corresponding element from the other structure. If, after ω moves, the correspondence thus established is an isomorphism of the substructures generated by the moves, then player two wins. Otherwise player one wins. If player two has a winning strategy, then $\mathfrak{A}$ and $\mathfrak{B}$ are *game-equivalent*. A proof of game-equivalence will be referred to as a *back and forth argument*.

For a language $\mathfrak{L}$ (e.g., L, $L_{\omega_1\omega}$ or $L_{\infty\omega}$), $\mathfrak{B}$ and $\mathfrak{A}$ are $\mathfrak{L}$-*equivalent* if they satisfy the same sentence of $\mathfrak{L}$. The well-known connections between game-equivalence, $L_{\omega_1\omega}$ and $L_{\infty\omega}$ are as follows.

1.1. THEOREM. *$\mathfrak{A}$ and $\mathfrak{B}$ are game-equivalent if and only if they are $L_{\infty\omega}$-equivalent. If $\mathfrak{A}$ is countable, then there exists a Scott sentence (which we always denote by $\varphi_{\mathfrak{A}}$) such that the following are equivalent*:

(i) $\mathfrak{B} \models \varphi_{\mathfrak{A}}$,

(ii) *for each n, $\mathfrak{B}$ and $\mathfrak{A}$ realize the same $L_{\infty\omega}$ n-types,*

(iii) *$\mathfrak{B}$ is game-equivalent to $\mathfrak{A}$.*

In general, if $\mathfrak{A}$ is countable then

$$\mathfrak{A} <_{\omega_1\omega} \mathfrak{B} \Rightarrow \mathfrak{A} <_{\infty\omega} \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \equiv_{\omega_1\omega} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}.$$

Since we do not need to distinguish between $L_{\infty\omega}$ and $L_{\omega_1\omega}$ in this case, we simply write $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$. We shall use the fact that an embedding $\sigma\colon \mathfrak{A} \to \mathfrak{B}$ is elementary if and only if for each $n \in \omega$, $a_1, \ldots, a_n \in \mathfrak{A}$, $(\mathfrak{A}, a_1, \ldots, a_n)$ is game-equivalent to $(\mathfrak{B}, \sigma(a_1), \ldots, \sigma(a_n))$.

We refer to the class $\{|\mathfrak{B}|\colon \mathfrak{B} \models \varphi_{\mathfrak{A}}\}$ of cardinalities of models of $\varphi_{\mathfrak{A}}$ as the *spectrum* of $\varphi_{\mathfrak{A}}$ (or as the spectrum of $\mathfrak{A}$). Card is the class of infinite cardinals, and $\mathrm{Card}^{<\varkappa}$ the set of infinite cardinals less than $\varkappa$.

$\mathfrak{A} - \mathfrak{B}$ denotes the subordering of $\mathfrak{A}$ with universe $\{a \in A\colon a \notin B\}$, $\mathfrak{A} \cap \mathfrak{B}$ the subordering of $\mathfrak{A}$, $\mathfrak{B}$ with universe $A \cap B$. If ξ is an order type, the sum $\mathfrak{B} = \sum_{\alpha\in\xi} \mathfrak{A}_\alpha$ is formed from copies $\mathfrak{B}_\alpha = (B_{\alpha'} \leqslant_{\alpha'} P_i^{\mathfrak{B}_\alpha})_{i\in I}$ of $\mathfrak{A}_{\alpha'}$ with $\mathfrak{B}_\alpha$ disjoint from each $\mathfrak{B}_{\beta'}$, $\beta \neq \alpha$. $\mathfrak{B}$ has universe $\bigcup_{\alpha\in\xi} B_{\alpha'}$ ordered by $(\bigcup_{\alpha\in\xi} \leqslant_\alpha) \cup (\bigcup_{\alpha<\beta} B_\alpha \times B_\beta)$, with unary predicates $P_i^{\mathfrak{B}} = \bigcup_{\alpha\in\xi} P_i^{\mathfrak{B}_\alpha}$. If $\mathfrak{A}_\alpha \cong \mathfrak{B}_\beta$, all $\alpha, \beta \in \xi$, then $\sum_{\alpha\in\xi} \mathfrak{A}_\alpha$ is denoted by $\mathfrak{A}\cdot\xi$. $\mathfrak{A}_0 + \mathfrak{A}_1 + \ldots + \mathfrak{A}_{n-1}$ denotes $\sum_{i\in n} \mathfrak{A}_i$. An ordering $\mathfrak{A}$ is referred to as *self-additive* if $\mathfrak{A} \equiv \mathfrak{A} + \mathfrak{A}$ (this is a variant of a notion introduced in [8]). Note that if $\mathfrak{A}$ is countable and self-additive, then $\mathfrak{A} \cong \mathfrak{A} + \mathfrak{A}$. A routine back and forth argument shows that if $\mathfrak{A}_\alpha \equiv \mathfrak{B}_\alpha$, $\alpha \in \xi$, then $\sum_{\alpha\in\xi} \mathfrak{A}_\alpha \equiv \sum_{\alpha\in\xi} \mathfrak{B}_\alpha$.

The set of automorphisms of $\mathfrak{A}$ will be denoted by $\mathrm{Aut}\,\mathfrak{A}$. The *orbit* of $a \in A$ in $\mathfrak{A}$ is the set of $b \in A$ such that for some $\sigma \in \mathrm{Aut}\,\mathfrak{A}$, $\sigma(a) = b$, and is denoted by $O(a, \mathfrak{A})$ or simply $O(a)$. We shall make heavy use of the fact, which we now record, that all members of an orbit have the same $L_{\infty\omega}$ properties.

1.2. LEMMA. *If $b \in O(a, \mathfrak{A})$, then $(\mathfrak{A}, a) \equiv (\mathfrak{A}, b)$.*

§ 2. Statement of the Main Theorem. While a countable structure $\mathfrak{A}$ is isomorphic to any countable model of its Scott sentence $\varphi_{\mathfrak{A}}$, $\varphi_{\mathfrak{A}}$ may have uncountable models. Following a suggestion of Makkai, we refer to $\mathfrak{A}$ as *absolutely characterizable* if every model of $\varphi_{\mathfrak{A}}$ is isomorphic to $\mathfrak{A}$, that is, if the spectrum of $\varphi_{\mathfrak{A}}$ is $\{\aleph_0\}$. Countable ordinals are absolutely characterizable, while the order type η of the rational numbers is $L_{\infty\omega}$-equivalent to any dense order type without endpoints. In [6] it is shown that countable scattered order types are absolutely characterizable. Malitz and Baumgartner [7], [3] have classified the spectra of Scott sentences for arbitrary countable structures. In the theorem below, the principal result of this paper, we extend Makkai's theorem and provide a description of the relation between structural properties of countable orderings and the spectra of their Scott sentences.

2.1. THEOREM. *Let $\mathfrak{A}$ be a denumerably infinite ordering with Scott sentence $\varphi_{\mathfrak{A}}$. The spectrum of $\varphi_{\mathfrak{A}}$ is as described below.*

(i) *$\{\aleph_0\}$ if and only if each orbit of $\mathfrak{A}$ is scattered.*

(ii) *All infinite cardinals if and only if $\mathfrak{A}$ has a self-additive interval.*

(iii) *All infinite cardinals $\varkappa \leqslant 2^{\aleph_0}$ if neither* (i) *nor* (ii) *applies. Each of the cases* (i), (ii), (iii) *does occur.*

From Theorem 2.1 we may immediately draw the following conclusions.

2.2. COROLLARY. (i) *If a complete $L_{\omega_1\omega}$ sentence ψ is satisfied by an uncountable linear ordering, then ψ is satisfied by an ordering of power $2^{\aleph_0}$.*

(ii) *The Hanf number for Scott sentences of linear ordering is $(2^{\aleph_0})^+$.*

A version of Theorem 2.1 can be stated and proved which replaces all mention of infinitary languages with game-theoretic terminology. In this way the theorem can be viewed as a description of the properties of the relation of game-equivalence between countable and uncountable orderings. Although the core of our proof of Theorem 2.1 is almost purely combinatorial, we chose to emphasize connections with $L_{\omega_1\omega}$ and $L_{\infty\omega}$. This is in keeping with our intention of exploring the model theory of $L_{\omega_1\omega}$ in a concrete setting. Barwise and Eklof [2] have examined the infinitary properties of Abelian torsion groups, but relatively little work of this kind has appeared.

The core of the proof of the main theorem is in §§ 4, 5, while § 6 completes it by showing that each of the cases (i), (ii), (iii) of Theorem 2.1 does occur.

§ 3. Countable orderings in which every orbit is scattered. In this section we make a routine application of the theorem of Makkai [6] cited in § 2 to establish part of Theorem 2.1. The guiding idea is that if $\mathfrak{A}$ is a union of absolutely characterizable substructures (in this case, orbits) which are definable, then $\mathfrak{A}$ is absolutely characterizable.

3.1. THEOREM. *If $\mathfrak{A}$ is countable and each orbit of $\mathfrak{A}$ is scattered, then $\mathfrak{A}$ is absolutely characterizable.*

Proof. For each $a \in A$, let ψ_a be a Scott sentence for the subordering $O(a, \mathfrak{A})$ and θ_a a formula of $L_{\omega_1\omega}$ with one free variable whose interpretation $\theta_a^{\mathfrak{A}}$ in $\mathfrak{A}$ is

$O(a, \mathfrak{A})$. Let $\psi_a^{\theta_a}$ denote the relativization of ψ_a to θ_a so that $\mathfrak{B} \models \psi_a^{\theta_a}$ if and only if $\theta_a^{\mathfrak{B}} \models \psi_a$. $\mathfrak{A}$ satisfies the sentence $\forall x(\bigvee_{a\in A} \theta_a(x) \wedge \bigwedge_{a\in A} \psi_a^{\theta_a})$, which we denote by ζ. If $\mathfrak{B} \models \zeta$, the first conjunct implies that $B \subseteq \bigcup_{a\in A} \mathfrak{B}^{\theta_a}$ and the second, by Makkai's theorem, that each $\mathfrak{B}^{\theta_a}$ is countable.

To see that Theorem 3.1 extends Makkai's theorem, we now show that there exist nonscattered orderings in which every orbit is scattered. Let $\{a_i: i \in \omega\}$ be an enumeration without repetitions of η, the order type of the rationals. Let $\mathfrak{A}_{a_i}$ be a copy of the finite ordinal $i+1$. Then $\mathfrak{A} = \sum_{a_i \in \eta} \mathfrak{A}_{a_i}$ is a nonscattered ordering whose only automorphism is the trivial one.

§ 4. Denumerable orderings with a nonscattered orbit.

Our goal in this section is to prove the following theorem.

4.1. THEOREM. *If $\mathfrak{A}$ is a denumerable ordering with a nonscattered orbit, then $\varphi_{\mathfrak{A}}$ has a model of power $2^{\aleph_0}$.*

For the present we fix countable $\mathfrak{A} = (A, \leqslant, P_i^{\mathfrak{A}})_{i\in I}$ with nonscattered orbit $O(a_0, \mathfrak{A})$. We begin by defining a "factor" ordering $\mathfrak{A}^*$ via an equivalence relation on $\mathfrak{A}$, and then showing that it suffices to prove Theorem 4.1 for $\mathfrak{A}^*$. The proof then proceeds by analyzing the relation "dense in" between orbits of $\mathfrak{A}^*$, showing that it is almost a linear ordering (Order Lemma), and then using this fact in conjunction with back and forth arguments. We begin with a useful fact about orbits, omitting the routine proof.

4.2. LEMMA. *If $\mathfrak{C}$ is a subordering of $\mathfrak{B}$, σ an automorphism of $\mathfrak{B}$, $b \in \mathfrak{B}$, then $O(b, \mathfrak{B}) \cap \mathfrak{C} \cong O(b, \mathfrak{B}) \cap \sigma(\mathfrak{C})$. In particular, $O(b, \mathfrak{B}) \cap \mathfrak{C}$ is empty if and only if $O(b, \mathfrak{B}) \cap \sigma(\mathfrak{C})$ is empty.*

We now begin the construction of the ordering $\mathfrak{A}^*$ mentioned above.

4.3. DEFINITION. For each element a of $\mathfrak{A}$, let a^* denote the set $\bigcup\{\mathfrak{B}: \mathfrak{B}$ is an interval of $\mathfrak{A}$, $a \in \mathfrak{B}$, and $\mathfrak{B} \cap O(a, \mathfrak{A})$ is scattered$\}$.

Some straightforward but useful consequences of the definition are as follows.

4.4. LEMMA. *Let a, b be elements of $\mathfrak{A}$.*

1. *If $a \in O(b, \mathfrak{A})$, then either $a^* \cap b^*$ is empty or $a^* = b^*$.*
2. *a^* is the largest interval $\mathfrak{B}$ containing a such that $\mathfrak{B} \cap O(a, \mathfrak{A})$ is scattered.*
3. *If $\sigma \in \operatorname{Aut}\mathfrak{A}$, then $\sigma(a^*) = (\sigma(a))^*$. Thus if $b \in O(a, \mathfrak{A})$, b^* is isomorphic to a^*.*

We now put an equivalence relation $\sim$ on $\mathfrak{A}$.

4.5. DEFINITION. For b, c in $\mathfrak{A}$, $b \sim c$ if and only if either 1 or 2 holds.

1. For some $a \in O(a_0)$, $b, c \in a^*$.
2. Neither b nor c is in $\bigcup\{a^*: a \in O(a_0)\}$ and for every d in $\bigcup\{a^*: a \in O(a_0)\}$, $b<d$ if and only if $c<d$.

An important property of $\sim$ which follows from Lemma 4.4 is as follows.

4.6. LEMMA. *Let $\sigma \in \operatorname{Aut}\mathfrak{A}$, $a, b \in \mathfrak{A}$. Then $a \sim b$ if and only if $\sigma(a) \sim \sigma(b)$.*

Denote the equivalence class of $a \in \mathfrak{A}$ by $[a]$. Let f be the map carrying a to $[a]$, and A^* the set $\{[a]: a \in A\}$. To avoid ambiguity we write $[a]$ for the substructure of $\mathfrak{A}$ and $f(a)$ to denote the corresponding element of A^*. It follows from Lemma 4.6 that for each $\sigma \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$, $f(\sigma(a)) = \sigma(f(a))$.

4.7. DEFINITION. Let $\mathfrak{A}^*$ be any structure with universe A^*, linear ordering $\leqslant^*$, and unary predicates $Q_0^{\mathfrak{A}^*}, Q_1^{\mathfrak{A}^*}, \ldots$, satisfying the following conditions.

(i) $f(a) \leqslant^* f(b)$ if and only if either $[a] = [b]$ or $\forall x \in [a]\, \forall y \in [b] (x \leqslant y)$.

(ii) The unary predicates $Q_0^{\mathfrak{A}^*}, Q_1^{\mathfrak{A}^*}, \ldots$, as many as needed, have the following properties.

1. For each $a \in \mathfrak{A}$, $\mathfrak{A}^* \models Q_0 f(a)$ if and only if $\exists b \in O(a_0)(b^* = [a])$.
2. For each $a \in \mathfrak{A}$ there is a unique $i \in \omega$ such that $\mathfrak{A}^* \models Q_i f(a)$.
3. For each $a, b \in \mathfrak{A}$, if $\mathfrak{A}^* \models Q_i f(a)$, then $\mathfrak{A}^* \models Q_i f(b)$ if and only if for some $\sigma \in \operatorname{Aut}\mathfrak{A}$, $[b] = \sigma([a])$.

We leave it to the reader to verify that $\mathfrak{A}^*$ exists and note that the orbits of $\mathfrak{A}^*$ are the $Q_i^{\mathfrak{A}^*}$. Some other properties of $\mathfrak{A}^*$ are as follows.

4.8. LEMMA. (1) *For every $\sigma \in \operatorname{Aut}\mathfrak{A}$ ($\sigma^* \in \operatorname{Aut}\mathfrak{A}^*$) there is $\sigma^* \in \operatorname{Aut}\mathfrak{A}^*$ ($\sigma \in \operatorname{Aut}\mathfrak{A}$) such that $f\sigma \doteq \sigma^* f$.*

(2) *For any $a \in \mathfrak{A}^*$, $O(f(a), \mathfrak{A}^*) = f(O(a, \mathfrak{A})) = \{[b]: b \in O(a, \mathfrak{A})\}$.*

(3) *$O(f(a_0), \mathfrak{A}^*)$ is dense in $\mathfrak{A}^*$.*

(4) *$(A^*, \leqslant^*)$ is isomorphic to an interval of the rationals.*

Proof. (1) Let $\sigma^*(f(x))$ be $f(\sigma(x))$. Given $\sigma^* \in \operatorname{Aut}\mathfrak{A}^*$, we let σ be any map so that for each $a \in \mathfrak{A}$, $\sigma \restriction [a]$ is an isomorphism of $[a]$ with $f^{-1}(\sigma^*(f(a)))$. (2) follows from (1) and (3) from (2). From (3) it follows that $(A^*, \leqslant^*)$ is densely ordered and a routine back and forth argument then gives (4).

The proof of Theorem 2.1 could be carried out in $\mathfrak{A}$, but the presence of an orbit dense in $\mathfrak{A}^*$ makes it more convenient to work there, as we now show that we may do.

4.9. LEMMA. *If $\mathfrak{A}^* \equiv \mathfrak{B}$, then there is an ordering $\mathfrak{C}$, $|\mathfrak{C}| = |\mathfrak{B}|$; such that $\mathfrak{A} \equiv \mathfrak{C}$.*

Proof. For each $b \in \mathfrak{B}$ such that $b \in Q_i^{\mathfrak{B}}$, let $\mathfrak{C}_b$ be a copy of the isomorphism type of any equivalence class $[a]$ such that $\mathfrak{A}^* \models Q_i f(a)$. Let $\mathfrak{C} = \sum_{b\in\mathfrak{B}} \mathfrak{C}_b$. A back and forth argument lifts player two's winning strategy in $G(\mathfrak{A}^*, \mathfrak{B})$ to $G(\mathfrak{A}, \mathfrak{C})$. If player one chooses $a \in \mathfrak{A}$ ($y \in \mathfrak{C}_b$), player two responds in $\mathfrak{C}_{b'}$ ($[a']$), where b' ($[a']$) is the response to $[a]$ (b) dictated by his winning strategy in $G(\mathfrak{A}^*, \mathfrak{B})$. The fact that $[a] \cong \mathfrak{C}_{b'}$ ($[a'] \cong \mathfrak{C}_b$) assures that player two has a winning strategy.

Since we shall work in $\mathfrak{A}^*$ and no longer need to refer frequently to $\mathfrak{A}$, a_0, or f, we denote $f(a_0)$ by d, and free $\mathfrak{A}$ and a_0 from the fixed meanings we have given them in this section.

4.10. Definition. Let a, b be any elements of $\mathfrak{A}^*$. $N(a, b)$ will denote the interval $\{a\} \cup (\bigcup \{\mathfrak{C}$: $\mathfrak{C}$ is an open neighborhood of a in $\mathfrak{A}^*$ and $\mathfrak{C} \cap O(b, \mathfrak{A}^*)$ is empty$\})$. If $S \subseteq A^*$, let $N(a, S)$ denote the interval $\bigcap_{x \in S} N(a, x)$.

Note that if $c \in O(b, \mathfrak{A}^*)$, then $N(a, b) = N(a, c)$. We term an interval with at most one point *trivial*. For each $b \in \mathfrak{A}^*$ and $a \in O(d, \mathfrak{A}^*)$, $N(a, b)$ is trivial if and only if a is a limit point of $O(b, \mathfrak{A}^*)$. If $N(a, b)$ is nontrivial, then it is the largest open neighborhood of a which does not meet $O(b, \mathfrak{A}^*)$.

4.11. Lemma. *Let $a \in O(b, \mathfrak{A}^*)$, $c, c_1 \in \mathfrak{A}^*$, and let $\sigma \in \operatorname{Aut} \mathfrak{A}^*$ carry a to b.*

(i) *a is a limit point of $O(c)$ if and only if b is.*

(ii) *σ is an isomorphism between $N(a, c)$ and $N(b, c)$, and between $N(a, c) - N(a, c_1)$ and $N(b, c) - N(b, c_1)$.*

(iii) *If $N(a, c) \cap N(b, c)$ is nonempty, then $N(a, c) = N(b, c)$.*

Proof. (i) Let U be a neighborhood of a. Then $\sigma(U)$ is a neighborhood of b and, by Lemma 4.2, $U \cap O(c)$ is empty if and only if $\sigma(U) \cap O(c)$ is empty. For (ii) we use the fact that $\sigma^{-1}\sigma$ is the identity map. Since $N(a, c) \cap O(c) = \emptyset$, $\sigma(N(a, c)) \cap O(c) = \emptyset$. Since $N(a, c)$ is the largest open neighborhood of a not meeting $O(c)$, $\sigma(N(a, c)) \subseteq N(b, c)$. Similarly $\sigma^{-1}(N(b, c)) \subseteq N(a, c)$, and thus $\sigma^{-1}\sigma(N(a, c)) \subseteq N(a, c)$. Equality holds only if $\sigma(N(a, c)) = N(b, c)$. The second part of (ii) follows from the first. For (iii) we may assume that $N(a, c)$, $N(b, c)$ are nontrivial, and that $N(a, c) \cap N(b, c)$ is nonempty. Then $N(a, c) \cup N(b, c)$ is an open neighborhood of b not meeting $O(c)$, and so contained in $N(b, c)$. Similarly, $N(b, c) \subseteq N(a, c)$.

4.12. Orbit Lemma. *For $a, b \in \mathfrak{A}^*$, $c \in N(a, b)$, $O(c, N(a, b)) = O(c, \mathfrak{A}^*) \cap N(a, b)$.*

Proof. To obtain $O(c, N(a, b)) \subseteq O(c, \mathfrak{A}^*) \cap N(a, b)$, let τ be an automorphism of $N(a, b)$ carrying c to c_1. $\sigma \in \operatorname{Aut} \mathfrak{A}^*$ such that $\sigma(c) = c_1$ is obtained by letting $\sigma \restriction N(a, b)$ be τ and $\sigma \restriction (\mathfrak{A}^* - N(a, b))$ be the identity map. To obtain $O(c, N(a, b)) \supseteq O(c, \mathfrak{A}^*) \cap N(a, b)$, let $\sigma \in \operatorname{Aut} \mathfrak{A}^*$ with $\sigma(c) \in N(a, b)$. By Lemma 4.11 $\sigma \restriction N(a, b) \in \operatorname{Aut} N(a, b)$ and $\sigma(c) \in O(c, N(a, b))$.

4.13. Order Lemma. *Let $d_1 \in O(d, \mathfrak{A}^*)$, $a, b \in \mathfrak{A}^*$. Then either $N(d_1, a) \subseteq N(d_1, b)$ or $N(d_1, b) \subseteq N(d_1, a)$. Thus $\{N(d_1, x)\colon x \in \mathfrak{A}^*\}$ is linearly ordered by inclusion.*

Proof. To obtain a contradiction we assume that neither of $N(d_1, a)$, $N(d_1, b)$ is contained in the other. Clearly neither is trivial. We may assume without loss of generality that $N(d_1, a)^{\leqslant d_1} = \{x \in N(d_1, a)\colon x \leqslant d_1\}$ is properly contained in $N(d_1, b)^{\leqslant d_1}$ and that $N(d_1, a)^{\geqslant d_1}$ properly contains $N(d_1, b)^{\geqslant d_1}$. Let I be $N(d_1, b)^{\leqslant d_1} - N(d_1, a)^{\leqslant d_1}$ and J be $N(d_1, a)^{\geqslant d_1} - N(d_1, b)^{\geqslant d_1}$. Since $N(d, a)$ and $N(d, b)$ are open, I and J are nontrivial. There are two possibilities for I: (i) $(\exists a_1 \in O(a, \mathfrak{A}^*) \cap I)\ \forall x \in I (x \leqslant a_1)$. (ii) There is a strictly increasing sequence $\{a_i\colon i \in \omega\}$ contained in $O(a, \mathfrak{A}^*) \cap I$ which is cofinal in I. It follows that for any $y \in I$ such that $y \notin O(a, \mathfrak{A}^*)$, there is an $x \in I \cap O(a, \mathfrak{A}^*)$ such that $y < x$.

Since $O(d, \mathfrak{A}^*)$ is dense in $\mathfrak{A}^*$ there is $d_2 \in I \cap O(d, \mathfrak{A}^*)$. Since $d_2 \notin O(a, \mathfrak{A}^*)$, there is $x \in I \cap O(a, \mathfrak{A}^*)$ such that $d_2 < x$. Now $d_2 \in N(d_1, b) \cap N(d_2, b)$, so that by Lemma 4.11 (iii), $N(d_1, b) = N(d_2, b)$.

Let $\sigma \in \operatorname{Aut}(\mathfrak{A}^*)$ carry d_2 to d_1. Then by 4.11 (ii), σ carries $N(d_1, b)$ onto itself. In particular, $\sigma(x)$ lies in $N(d_1, b)^{\geqslant d_1}$ and in $O(a, \mathfrak{A}^*)$, violating the assumption that $N(d_1, a)^{\geqslant d_1}$ contains $N(d_1, b)^{\geqslant d_1}$. ■

Let S_d be the set $\{a \in A^*\colon d$ is not a limit point of $O(a, \mathfrak{A}^*)\}$ and recall that $N(d, S_d)$ is defined to be $\bigcap \{N(d, s)\colon s \in S_d\}$. $N(d, S_d)$ is an interval disjoint from S_d. If $a \in N(d, S_d)$, then d is a limit point of $O(a)$, while if $a \in S_d$, $N(d, a)$ is an open neighborhood of d. For each $d' \in O(d)$, 4.11 (i) implies that $S_{d'} = S_d$ and from 4.11 (ii) that $N(d', S_{d'}) \cong N(d, S_d)$. Since $O(d)$ is dense in $(A^*, \leqslant)$, we see that $S_d = \{a \in A^*\colon O(a)$ is not dense (equivalently, nowhere dense) in $(A^*, \leqslant)\}$.

4.14. Lemma. *Either $N(d, S_d) = \{d\}$ or $N(d, S_d)$ is a neighborhood of d.*

Proof. If $N(d, S_d)$ is not a neighborhood of d, then d is a limit point of S_d, and so of $S_d^{<d}$ or of $S_d^{>d}$. Suppose the former. Then there is a strictly increasing sequence $\{s_i\colon i \in \omega\}$ contained in $S_d^{<d}$ with limit d. Let $b \in (\mathfrak{A}^*)_d^{>d}$, $d_1 \in (d, b) \cap O(d)$, and let $\sigma \in \operatorname{Aut} \mathfrak{A}^*$ carry d to d_1. Then all but finitely many elements of $\{\sigma(s_i)\colon i \in \omega\}$ lie in (d, b), d is a limit point of $S_d^{>d}$, and $N(d, S_d) = \{d\}$. ■

To complete the proof of Theorem 4.1, we now consider separately the cases $N(d, S_d) = \{d\}$ and $N(d, S_d) \neq \{d\}$, beginning with the latter.

4.15. Definition. An ordering $\mathfrak{B} = (B, \leqslant, P_i^{\mathfrak{B}})_{i \in I}$ is a *shuffle* if $(B, \leqslant)$ is densely ordered without endpoints and satisfies the following conditions.

(i) $\forall x \bigvee_{i \in I} P_i x$.

(ii) $\forall x \bigwedge_{i \neq j} \sim(P_i x \wedge P_j x)$.

(iii) If $P_i^{\mathfrak{B}}$ is nonempty, then it is dense in $\mathfrak{B}$.

Routine back and forth arguments establish all parts of the next lemma.

4.16. Lemma. *Let $\mathfrak{B} = (B, \leqslant, P_i^{\mathfrak{B}})_{i \in I}$ be a shuffle.*

(i) *The orbits of $\mathfrak{B}$ are the nonempty $P_i^{\mathfrak{B}}$.*

(ii) *If $\mathfrak{C}$ is an open interval of $\mathfrak{B}$, then $\mathfrak{C} \equiv \mathfrak{B}$.*

(iii) *$\mathfrak{B} \equiv \mathfrak{B} \cdot \xi$ for any nonempty order type ξ.*

4.17. Lemma. *If $N(d, S_d) \neq \{d\}$, then each of its open intervals is a shuffle.*

Proof. (i) and (ii) of 4.15 are immediate. For (iii), let $a \in N(d, S_d)$, so that d is a limit point of $O(a, \mathfrak{A}^*)$. It follows that any point of $O(d, \mathfrak{A}^*)$ is such a limit point, and that $O(a, \mathfrak{A}^*)$ is dense in $\mathfrak{A}^*$ and $N(d, S_d)$. It now suffices to notice that if Q_i is the predicate carried by a, $O(a, \mathfrak{A}^*) \subseteq Q_i^{\mathfrak{A}^*}$. ■

Lemma 4.17 implies that if $N(d, S_d) \neq \{d\}$, then $\mathfrak{A}^*$ has the form $\mathfrak{A}_0 + \mathfrak{A}_1 + \mathfrak{A}_2$, where $\mathfrak{A}_1$ is a shuffle. Since $\mathfrak{A}_1 \equiv \mathfrak{A}_1 \cdot \xi$ for any nonempty order type ξ, $\mathfrak{A}^*$ is $L_{\infty\omega}$-equivalent to arbitrarily large orderings of the form $\mathfrak{A}_0 + \mathfrak{A}_1 \cdot \xi + \mathfrak{A}_2$.

It remains to produce a model $\mathfrak{B}$ of $\varphi_{\mathfrak{A}^*}$, $|\mathfrak{B}| = 2^{\aleph_0}$, in case $N(d, S_d) = \{d\}$. Since the avenue of enlarging $N(d, S_d)$ is not open in this case, we turn to the orbits of $\mathfrak{A}^*$. If $\mathfrak{B} \models \varphi_{\mathfrak{A}^*}$, then $Q_i^{\mathfrak{B}} \equiv Q_i^{\mathfrak{A}^*}$, $i \in \omega$. Since the $Q_i^{\mathfrak{A}^*}$, $i>0$, may be scattered, they may be characterized absolutely by $\varphi_{\mathfrak{A}^*}$ and then necessarily be isomorphic to the $Q_i^{\mathfrak{B}}$, $i>0$. But $Q_0^{\mathfrak{A}^*}$ has order type η, game-equivalent to order types of all infinite powers, leaving open the possibility that $Q_0^{\mathfrak{A}^*}$ may be enlarged to produce an uncountable model of $\varphi_{\mathfrak{A}^*}$. We show now that such an enlargement may be carried out.

Let $\mathfrak{B}^*$ be a copy of $\mathfrak{A}^*$ and σ_0 and isomorphism carrying $\mathfrak{A}^*$ onto $\mathfrak{B}^*$. Denote $\sigma_0(d)$ by e and $\sigma_0(S_d)$ by $\mathfrak{T}_e$. Note that $\mathfrak{T}_e$ is $\{b \in B^*\colon e$ is not a limit point of $O(b, \mathfrak{B}^*)\}$, and that for any $a \in \mathfrak{A}^*$, $A \subseteq A^*$, $\sigma_0(N(a, A)) = N(\sigma_0(a), \sigma_0(A))$. As a special case, $N(e, \mathfrak{T}_e) = \sigma_0(N(d, S_d))$. Lemmas 4.11–4.14 and 4.17 hold for $\mathfrak{B}^*$ (with the obvious modifications) and we shall use them freely. Since each $x \in O(d, \mathfrak{A}^*)$ is a limit point of S_d and $O(d, \mathfrak{A}^*)$ is dense in $\mathfrak{A}^*$, S_d is also dense in $\mathfrak{A}^*$ and $\mathfrak{T}_e$ in $\mathfrak{B}^*$.

Let $(B^+, \leqslant)$ be the (unique, up to order-isomorphism) interval of the reals in which $(B^*, \leqslant)$ can be embedded as a dense subset. For any interval U of B^*, let U^+ be the smallest interval of B^+ containing U. Note that if U is open in B^*, then U^+ is open in B^+. Any set dense in U, e.g. $O(e, \mathfrak{B}^*)$ or $\mathfrak{T}_{e}$, will also be dense in U^+. We construct $\mathfrak{B}$ by adding points of B^+ to $Q_0^{\mathfrak{B}^*}$. The simplest such enlargement of B^* would be to let $Q_0^{\mathfrak{B}}$ be $Q_0^{\mathfrak{B}^*} \cup (B^+ - B^*)$, but this approach is demonstrably not always successful (see § 7). Instead, we add a selected subset of $B^+ - B^*$ to $Q_0^{\mathfrak{B}^*}$. Each element e_1 of $O(e, \mathfrak{B}^*)$ has the property that for any $t \in \mathfrak{T}_e$ there is $e_2 \in O(e, \mathfrak{B}^*)$, $e_2 \neq e_1$, with $e_1 \in N(e_2, t)$. Since this can be stated in $L_{\omega_1\omega}$, and $Q_0^{\mathfrak{B}}$ should in $\mathfrak{B}$ play the part of $O(e, \mathfrak{B}^*)$, we single out the points with the best chance of having this property. Let $\mathfrak{E}$ be the set of elements $x \in B^+$ such that for any $t \in \mathfrak{T}_e$ there is $e' \in O(e, \mathfrak{B}^*)$ with $x \in N(e', t)^+$, i.e.

$$\mathfrak{E} = \bigcap_{t \in \mathfrak{T}_e} \bigcup_{e_1 \in O(e, \mathfrak{B}^*)} N(e_1, t)^+ .$$

Let $Q_0^{\mathfrak{B}} = Q_0^{\mathfrak{B}^*} \cup (\mathfrak{E} - B^*)$, and $B = Q_0^{\mathfrak{B}} \cup \bigcup_{i>0} Q_i^{\mathfrak{B}^*}$ with the ordering $\leqslant_{\mathfrak{B}}$ inherited from $(B^+, \leqslant)$. Let $\mathfrak{B}$ be $(B, \leqslant_{\mathfrak{B}}, Q_0^{\mathfrak{B}}, Q_i^{\mathfrak{B}^*})_{i>0}$.

For $|Q_0^{\mathfrak{B}}|$ to be $2^{\aleph_0}$ it suffices that $|\mathfrak{E}| = 2^{\aleph_0}$. For each $t \in \mathfrak{T}_e$,

$$\bigcup \{N(e_1, t)^+ \colon e_1 \in O(e, \mathfrak{B}^*)\}$$

is open and dense in $(B^+, \leqslant)$, so that $\mathfrak{G}$ is not only a G_δ set but also of second category (comeager) in $(B^+, \leqslant)$. It follows from the Baire Category Theorem that $\mathfrak{G}$ is uncountable. Since any uncountable G_δ set of reals has power $2^{\aleph_0}$, $|\mathfrak{E}| = 2^{\aleph_0}$.

Our next theorem shows that $\mathfrak{B} \models \varphi_{\mathfrak{A}^*}$.

4.18. Theorem. *If $\mathfrak{B}^* \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$, then $\mathfrak{B}' \equiv \mathfrak{A}^*$.*

We prove the theorem for $\mathfrak{B}' = \mathfrak{B}$. It will be apparent how to adapt the back and forth argument to $\mathfrak{B}' \neq \mathfrak{B}$. Let a_i and b_i be the elements chosen from $\mathfrak{A}^*$ and $\mathfrak{B}$ on move i and $A_n = \{a_i\colon i \leqslant n\}$, $B_n = \{b_i\colon i \leqslant n\}$. We show that after n moves player two can maintain the following condition, stronger than the required partial isomorphism.

4.19. There is an isomorphism σ_n carrying $\mathfrak{A}^*$ onto the substructure $\mathfrak{B}^* \cup B_n$ of $\mathfrak{B}$ such that for each $a_i \in A_n$, $\sigma_n(a_i) = b_i$.

In the back and forth argument, we can let $\sigma_{n+1} = \sigma_n$ except when player one chooses b_{n+1} from $B - (B^* \cup B_n)$. In this case it will suffice to alter σ_n on $\sigma_n^{-1}(U)$, where U^+ is an appropriate neighborhood of b_{n+1} in B^+.

Let b_r, b_s with $b_r < b_{n+1} < b_s$ be the nearest elements of B_n to b_{n+1}. Since $b_{n+1} \in \mathfrak{E}$, there exists, for each $t \in \mathfrak{T}_e$, an element $e(t)$ in $O(e, \mathfrak{B}^*)$ such that $b_{n+1} \in N(e(t), t)^+$.

4.20. Lemma. (i) $\bigcap_{t \in \mathfrak{T}_e} N(e(t), t)^+ = \{b_{n+1}\}$.

(ii) $\bigcap_{t \in \mathfrak{T}_e} N(e(t), t)$ *is void.*

Proof. (ii) follows from (i). Suppose without loss of generality that $x < b_{n+1}$, $x \in B^+$. Since $\mathfrak{T}_e$ is dense in B^+, there is $t \in \mathfrak{T}_e$ such that $x < t < b_{n+1}$ and $x \notin N(e(t), t)$. ■

4.21. Lemma. *There are $t_i \in \mathfrak{T}_e$, $i \in \omega$, such that*

(i) $\bigcup_{i \in \omega} O(t_i, \mathfrak{B}^*)$ *is dense in* $\mathfrak{B}^*$.

(ii) *For* $i \in \omega$, $N(e, t_{i+1}) \subsetneqq N(e, t_i)$.

(iii) *For* $i \in \omega$, $N(e(t_{i+1}), t_{i+1}) \subsetneqq N(e(t_i), t_i)$.

(iv) $\bigcap_{i \in \omega} N(e, t_i) = \{e\}$.

(v) $\bigcap_{i \in \omega} N(e(t_i), t_i)^+ = \{b_{n+1}\}$ *and* $\bigcap_{i \in \omega} N(e(t_i), t_i)$ *is void.*

Proof. Since $\mathfrak{T}_e$ is dense in $\mathfrak{B}^*$ there are sequences $x_0 < x_1 < \ldots$ and $y_0 > y_1 > \ldots$ with limit e such that $\{x_i\colon i \in \omega\} \cup \{y_i\colon i \in \omega\}$ is a subset of $\mathfrak{T}_e$. We first show how the t_i may be selected. Let $t_0 = x_0$. Suppose that $t_0, \ldots, t_k$ have been selected. If $k>0$ we assume that (ii) and (iii) hold for each $i<k$ and that $N(e, t_k)$, $\subseteq (x_k, y_k)$. Let j be the least index greater than k so that $(x_j, y_j) \subseteq N(e, t_k)$. By the Order Lemma, either $N(e, x_j) \subseteq N(e, y_j)$ or $N(e, y_j) \subseteq N(e, x_j)$. In the former case let $t_{k+1} = x_j$. Otherwise let $t_{k+1} = y_j$. Then

$$N(e, t_{k+1}) \subsetneqq (x_j, y_j) \subsetneqq N(e, t_k) \quad \text{and} \quad N(e, t_{k+1}) \subsetneqq (x_j, y_j) \subseteq (x_{k+1}, y_{k+1}) .$$

Since $b_{n+1} \in N(e(t_k), t_k) \cap N(e(t_{k+1}), t_k)$, 4.11 (iii) implies that $N(e(t_k), t_k) = N(e(t_{k+1}), t_k)$. From this and 4.11 (ii) it follows that $N(e(t_{k+1}), t_{k+1}) \subsetneqq N(e(t_k), t_k)$. It follows from the construction that e is a limit point of $\bigcup_{i \in \omega} O(t_i, \mathfrak{B}^*)$, so that this set is dense in B^* and B^+, (i) and (iv) follow. For (v) suppose that $x < b_{n+1}$ $[x > b_{n+1}]$. Then for some $i \in \omega$ there is $y \in O(t_i, \mathfrak{B}^*)$ such that $x < y < b_{n+1}$ $[x > y > b_{n+1}]$. Thus $x \notin N(e(t_i), t_i)^+$. ■

We not again make free use of lemmas of this section for structures isomorphic to those for which they are stated. We also use the variant of Lemma 4.11 (ii) (whose statement is left to the reader) in which σ is an isomorphism.

Let s_i, $i \in \omega$, by defined by $s_i = \sigma_n^{-1}(t_i)$. 4.21 (iii) and (v) imply that for some $k \in \omega$, $N(e(t_k), t_k) \subseteq (b_r, b_s)$. Let $d_0 = \sigma_n^{-1}(e(t_k))$. By the variant of Lemma 4.11 (ii), $\sigma_n^{-1}(N(e(t_k), t_k)) = N(\sigma_n^{-1}(e(t_k)), \sigma_n^{-1}(t_k)) = N(d_0, s_k)$, so that $N(e(t_k), t_k) \cong N(d_0, s_k)$. Also by 4.11 (ii), for $i \geqslant k$,

$$N(d_0, s_{k+i}) - N(d_0, s_{k+i+1}) \cong N(e(t_k), t_{k+i}) - N(e(t_k), t_{k+i+1})$$

via restrictions of σ_n^{-1}. Again by 4.11 (ii),

$$\begin{aligned} N(e(t_k), t_{k+1}) - N(e(t_k), t_{k+i+1}) &\cong N(e(t_{k+i+1}), t_{k+i}) - N(e(t_{k+i+1}), t_{k+i+1}) \\ &= N(e(t_{k+i}), t_{k+i}) - N(e(t_{k+i+1}), t_{k+i+1}) . \end{aligned}$$

By the transitivity of the isomorphism relation there is for each $i \in \omega$ an isomorphism τ_i from $N(d_0, s_{k+i}) - N(d_0, s_{k+i+1})$ onto $N(e(t_{k+i}), t_{k+i}) - N(e(t_{k+i+1}), t_{k+i+1})$. Now $\bigcup_{i \in \omega} (N(d_0, s_{k+i}) - N(d_0, s_{k+i+1})) = N(d_0, s_k) - \{d_0\}$ and, by 4.21 (v),

$$\bigcap_{i \in \omega} N(d_0, s_{k+i}) = \{d_0\} .$$

Similarly, $\bigcup_{i \in \omega} (N(e(t_{k+i}), t_{k+i}) - N(e(t_{k+i+1}), t_{k+i+1})) = N(e(t_k), t_k)$ and by 4.21 (v), $\bigcap_{i \in \omega} (N(e(t_{k+i}), t_{k+i}) - N(e(t_{k+i+1}), t_{k+i+1}))$ is void. Thus $\bigcup_{i \in \omega} \tau_i$ is an isomorphism of $N(d_0, s_0) - \{d_0\}$ with $N(e(t_0), t_0)$. We can now describe player two's response, σ_{n+1}, to b_{n+1}, completing the proof of 4.18. It is that σ_{n+1} should be

$$\sigma_n \restriction (A^* - N(d_0, s_0)) \cup \bigcup_{i \in \omega} \tau_i \cup \{(d_0, b_{n+1})\} ,$$

agreeing with σ_n outside of $N(d_0, s_0)$ and matching b_{n+1} with d_0. ∎

§ 5. Countable orderings $L_{\omega_1\omega}$-equivalent to arbitrarily large orderings. In this section we determine, in Theorem 5.1, that the Hanf number for Scott sentences (of countable orderings) is $(2^{\aleph_0})^+$ and find two algebraic definitions of $\{\mathfrak{A}: |\mathfrak{A}| = \aleph_0$ & $\varphi_{\mathfrak{A}}$ has arbitrarily large models$\}$. The effect is to establish (ii) and (together with § 4) (iii) of Theorem 2.1. The main combinatorial tool for the Hanf number proof is the Erdös-Rado theorem (see [4]) whose use was prompted by a suggestion of S. Shelah (personal communication).

5.1. THEOREM. *Let $|\mathfrak{A}| = \aleph_0$. The following are equivalent.*

(i) *$\varphi_{\mathfrak{A}}$ has a model of power at least $(2^{\aleph_0})^+$.*

(ii) *$\varphi_{\mathfrak{A}}$ has arbitrarily large models.*

(iii) *There are elements $a_1 < a_2 < a_3$ of $\mathfrak{A}$ such that $(\mathfrak{A}, a_1, a_2) \cong (\mathfrak{A}, a_1, a_3) \cong (\mathfrak{A}, a_2, a_3)$.*

(iv) *$\mathfrak{A}$ has a self-additive interval.*

First we show that (iii) and (iv) are equivalent. If (iii) holds, then to see that $(a_1, a_2]$ is self-additive, note that $(a_1, a_3] \cong (a_1, a_2] + (a_2, a_3]$ and that $(a_1, a_2] \cong (a_2, a_3] \cong (a_1, a_3]$.

For the converse, let $\mathfrak{B}$ be a self-additive interval of $\mathfrak{A}$, $b \in \mathfrak{B}$. For $i \in \{0, 1, 2, 3, 4\}$ let $(\mathfrak{B}_i, a_i)$ be a copy of $(\mathfrak{B}, b)$. Let $\mathfrak{B}' = \sum_{0 \leqslant i \leqslant 4} \mathfrak{B}_i$. Since $\mathfrak{B}' \cong \mathfrak{B}$, it suffices to show that

$$(\mathfrak{B}', a_1, a_2) \cong (\mathfrak{B}', a_1, a_3) \cong (\mathfrak{B}', a_2, a_3) .$$

$(\mathfrak{B}')^{\leqslant a_1} \cong \mathfrak{B}_0 + \mathfrak{B}_1^{\leqslant a_1}$, $\mathfrak{B}_0 \cong \mathfrak{B}_0 + \mathfrak{B}_1$, and $\mathfrak{B}_1^{\leqslant a_1} \cong \mathfrak{B}_2^{\leqslant a_2}$, so that $(\mathfrak{B}')^{\leqslant a_1} \cong (\mathfrak{B}')^{\leqslant a_2}$. Similarly, $(\mathfrak{B}')^{\geqslant a_2} \cong (\mathfrak{B}')^{\geqslant a_3}$. We also need $(a_1, a_2) \cong (a_2, a_3)$, which is immediate, and $(a_1, a_2) \cong (a_1, a_3)$. $\mathfrak{B}_i$ splits into intervals $\mathfrak{B}_i^l$ (left) and $\mathfrak{B}_i^r$ (right) so that $\forall x \in \mathfrak{B}_i^l \forall y \in \mathfrak{B}_i^r (x < y)$ and each is isomorphic to $\mathfrak{B}_i$. We may assume that $\forall i \leqslant 4 (a_i \in \mathfrak{B}_i^l)$ or $\forall i \leqslant 4 (a_i \in \mathfrak{B}_i^r)$. Assume the former. Then

$$(a_1, a_2) \cong \mathfrak{B}_1^{>a_1} + \mathfrak{B}_2^{<a_2} \cong (\mathfrak{B}_1^l)^{>a_1} + \mathfrak{B}_1^r + \mathfrak{B}_2^{<a_2} \cong (\mathfrak{B}_1^l)^{>a_1} + \mathfrak{B}_1^r + \mathfrak{B}_1^r + \mathfrak{B}_2^{<a_2} .$$

Now, since $(\mathfrak{B}_1^l)^{>a_1} + \mathfrak{B}_1^r \cong \mathfrak{B}_1^{>a_1}$ and $\mathfrak{B}_2^{<a_2} \cong \mathfrak{B}_3^{<a_3}$, the last expression is isomorphic to $\mathfrak{B}_1^{>a_1} + \mathfrak{B}_1^r + \mathfrak{B}_3^{<a_3} \cong (a_1, a_3)$. It follows that $(a_1, a_2) \cong (a_1, a_3)$. With these results we have

$$\begin{aligned} (\mathfrak{B}', a_1, a_2) &= (\mathfrak{B}')^{\leqslant a_1} + (a_1, a_2) + (\mathfrak{B}')^{\geqslant a_2} \cong (\mathfrak{B}')^{\leqslant a_1} + (a_1, a_3) + (\mathfrak{B}')^{\geqslant a_3} \\ &\cong (\mathfrak{B}', a_1, a_3) \end{aligned}$$

and

$$\begin{aligned} (\mathfrak{B}', a_1, a_2) &= (\mathfrak{B}')^{\leqslant a_1} + (a_1, a_2) + (\mathfrak{B}')^{\geqslant a_2} \cong (\mathfrak{B}')^{\leqslant a_2} + (a_2, a_3) + (\mathfrak{B}')^{\geqslant a_3} \\ &\cong (\mathfrak{B}', a_2, a_3) . \end{aligned}$$

We now show (i) ⇒ (iii) ⇒ (ii). Since (ii) ⇒ (i) is immediate, this will complete the proof of Theorem 5.1.

Assuming (i), let $\mathfrak{B} \models \varphi_{\mathfrak{A}}$, $|\mathfrak{B}| \geqslant (2^{\aleph_0})^+$. By the downward Löwenheim–Skolem–Tarski theorem for $L_{\omega_1\omega}$, we may assume that $\mathfrak{B}$ has power exactly $(2^{\aleph_0})^+$. For each $n \in \omega$, $\mathfrak{B}$ realizes the same $L_{\omega_1\omega}$ n-types as $\mathfrak{A}$. There are $(2^{\aleph_0})^+$ increasing sequences of length two from B, and these realize at most $\aleph_0$ distinct $L_{\omega_1\omega}$ 2-types. The Erdös-Rado theorem (see [4]) in the form $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$ implies that there is a set $B' \subseteq B$ of power $\aleph_1$ such that all increasing sequences of length two from B' realize the same $L_{\omega_1\omega}$ type in $\mathfrak{B}$. Let $\mathfrak{B}_0$ be a countable $L_{\omega_1\omega}$ elementary submodel of $\mathfrak{B}$ containing three points $b_0 < b_1 < b_2$ of B'. Since $\mathfrak{A} \cong \mathfrak{B}_0$, $\mathfrak{A}$ has points $a_0 < a_1 < a_2$ satisfying (iii).

In the remainder of § 5 we complete the proof of Theorem 5.1 by showing that (iii) ⇒ (ii). Assume (iii) with a_0, a_1, a_2. Let $\psi(v_0, v_1)$ be an $L_{\omega_1\omega}$ formula such that for any countable $\mathfrak{C}$ and $c_0, c_1 \in C$,

$$\mathfrak{C} \models \psi(c_0, c_1) \Rightarrow (\mathfrak{C}, c_0, c_2) \cong (\mathfrak{A}, a_0, a_1) .$$

5.2. Lemma. *Let* $\mathfrak{A} \models \psi(a, b)$.

(i) *For some* $c \in A$, $\mathfrak{A} \models \psi(a, c) \wedge \psi(c, b)$.

(ii) *For some* $c \in A$, $\mathfrak{A} \models \psi(c, a) \wedge \psi(c, b)$.

(iii) *For some* $c \in A$, $\mathfrak{A} \models \psi(a, c) \wedge \psi(b, c)$.

Proof. For (i), we have $\mathfrak{A} \models \exists v(\psi(a_0, v) \wedge \psi(v, a_2))$ and $(\mathfrak{A}, a_0, a_2) \cong (\mathfrak{A}, a, b)$. (ii) and (iii) of 5.2 are similar. ■

5.3. Lemma. *The interpretation* $\psi^{\mathfrak{A}}$ *of* ψ *in* $\mathfrak{A}$ *is a transitive relation.*

Proof. Let $\mathfrak{A} \models \psi(a, b) \wedge \psi(b, c)$. It suffices to show that $(\mathfrak{A}, a, b) \cong (\mathfrak{A}, a, c)$.

$$\begin{aligned}(\mathfrak{A}, a, b) &\cong \mathfrak{A}^{\leqslant a} + (a, b) + \mathfrak{A}^{\geqslant b} \\ &\cong \mathfrak{A}^{\leqslant a} + (a_0, a_2) + \mathfrak{A}^{\geqslant c} \cong \mathfrak{A}^{\leqslant a} + (a_0, a_1] + (a_1, a_2) + \mathfrak{A}^{\geqslant c} \\ &\cong \mathfrak{A}^{\leqslant a} + (a, b] + (b, c) + \mathfrak{A}^{\geqslant c} \cong (\mathfrak{A}, a, c). \ ■\end{aligned}$$

We now find a subordering X of A with the order type η of the rationals such that if $x_1 < x_2$ are from X, then $\mathfrak{A} \models \psi(x_1, x_2)$. Let X_0 be $\{a_0, a_1, a_2\}$, and if X_n is $c_0 < \ldots < c_k$, with $\mathfrak{A} \models \psi(c_i, c_{i+1})$, then Lemmas 5.2 and 5.3 guarantee that there are elements $d_0, d_1, \ldots, d_{k+1}$ of A such that $d_0 < c_0 < d_1 < c_1 < \ldots < c_k < d_{k+1}$ and $\mathfrak{A} \models \psi(x, y)$ for $x < y$, $x, y \in \{d_0, c_0, \ldots, c_k, d_{k+1}\}$. Let $X = \bigcup_{n \in \omega} X_n$. X has order type η and if $x < y$, $x, y \in X$, then $\mathfrak{A} \models \psi(x, y)$.

Let $\bar{X}$ be the interval $\{a \in A\colon \exists b, c \in X(b \leqslant a \leqslant c)\}$ of $\mathfrak{A}$, so that $\mathfrak{A}$ has the form $\mathfrak{A}_0 + \bar{X} + \mathfrak{A}_1$ for some (possibly void) intervals $\mathfrak{A}_0, \mathfrak{A}_1$. Let ξ be an arbitrary nonempty order type, and for each $\alpha \in \xi$, let $\bar{X}_\alpha$ and $X_\alpha \subseteq \bar{X}_\alpha$ be isomorphic copies of $\bar{X}$, X.

5.4. Lemma. $\mathfrak{A} \equiv \mathfrak{A}_0 + \sum_{\alpha \in \xi} \bar{X}_\alpha + \mathfrak{A}_1$.

Proof. If suffices to show that player two has a winning strategy in $G(\bar{X}, \sum_{\alpha \in \xi} \bar{X}_\alpha)$, since he may play the identity map elsewhere. Assume that after n moves the points chosen from $\bar{X}$ are $a_0 < \ldots < a_{n-1}$ and those from $\sum_{\alpha \in \xi} \bar{X}_\alpha$ are $b_0 < b_1 < \ldots < b_{n-1}$. Player two can win by maintaining the following situation throughout the game.

5.5. For each α such that the set of elements $b_k, b_{k+1}, \ldots, b_m$ chosen so far from $\bar{X}_\alpha$ is nonempty, there are

(i) $s_\alpha < b_k$, $t_\alpha > b_m$, $s_\alpha, t_\alpha \in X_\alpha$,

(ii) $u_\alpha, v_\alpha \in X$, $u_\alpha < a_k$, $v_\alpha > a_k$, so that if $\beta > \alpha$ and u_β has been chosen, $v_\alpha < u_\beta$,

(iii) there is an isomorphism σ_α of $[u_\alpha, v_\alpha]$ with $[s_\alpha, t_\alpha]$ such that $\sigma_\alpha(a_i) = b_i$ for $k \leqslant i \leqslant m$.

5.5 holds before any moves are made. Whenever player one moves in $[u_\alpha, v_\alpha]$ or $[s_\alpha, t_\alpha]$, player two's response is dictated by σ_α. We leave it to the reader to verify that in the remaining cases player two may provide a new interval or extend an existing one, as required. ■

§ 6. A denumerable order type whose Scott sentence has spectrum $\mathrm{Card}^{\leqslant 2^{\aleph_0}}$. In the preceding sections we have established that the spectrum of the Scott sentence $\varphi_{\mathfrak{A}}$ is described by one of the three cases listed in the main theorem, but not that each of these does occur. Any scattered ordering provides an instance of case (i), and the order type η one of case (ii). But instances of (iii) are not as readily available. In this section we construct one.

6.1. Theorem. *There is a denumerable ordering* $\mathfrak{A}$ *such that the spectrum of* $\varphi_{\mathfrak{A}}$ *is* $\mathrm{Card}^{\leqslant 2^{\aleph_0}}$.

We begin the proof with the construction of $\mathfrak{A} = (A, \leqslant, P_i^{\mathfrak{A}})_{i \in \omega}$, which will be the union of the disjoint unary predicates $P_i^{\mathfrak{A}}$, $i \in \omega$. The construction is in ω stages. After the nth stage, the subordering $\bigcup_{i \leqslant n} P_i^{\mathfrak{A}}$ will have been constructed. Let $P_0^{\mathfrak{A}}$ have order type η, $P_1^{\mathfrak{A}}$ the order type $\omega^* + \omega$ of the integers, occurring coinitially and cofinally in $P_0^{\mathfrak{A}} \cup P_1^{\mathfrak{A}}$ with $P_0^{\mathfrak{A}}$ dense in $P_0^{\mathfrak{A}} \cup P_1^{\mathfrak{A}}$. Assume now that the nth stage, giving $\bigcup_{i \leqslant n} P_i^{\mathfrak{A}}$, has been completed. Between each pair of consecutive elements a, b, with $a < b$, of $P_n^{\mathfrak{A}}$, let the elements of $P_{n+1}^{\mathfrak{A}}$ lying between a and b have order type $\omega^* + \omega$, be conitial and cofinal in $\{x \in \bigcup_{i \leqslant n+1} P_i^{\mathfrak{A}}\colon a < x < b\}$ and such that $P_0^{\mathfrak{A}}$ is dense in $\bigcup_{i \leqslant n+1} P_i^{\mathfrak{A}}$.

6.2. Lemma. (i) *For* $n > 0$ *the order type of* $P_n^{\mathfrak{A}}$ *is scattered. (In fact, it is* $(\omega^* + \omega)^n$*).*

(ii) A, $P_0^{\mathfrak{A}}$, *and* $\bigcup_{i > 0} P_i^{\mathfrak{A}}$ *have order type* η *and are dense in* $\mathfrak{A}$.

(iii) *The* $P_n^{\mathfrak{A}}$ *are the orbits of* $\mathfrak{A}$.

(i) and (ii) follow immediately from the construction and (iii) from a routine back and forth argument. The three parts together imply that the scattered orbits of $\mathfrak{A}$ are dense in $\mathfrak{A}$.

6.3. Lemma. *If* $\mathfrak{C}$ *satisfies* $\varphi_{\mathfrak{A}}$, *then* $|\mathfrak{C}| \leqslant 2^{\aleph_0}$.

We show that $\mathfrak{C}$ has a countable dense subset. For each i, $P_i^{\mathfrak{C}} \equiv P_i^{\mathfrak{A}}$. For $i > 0$, $P_i^{\mathfrak{A}}$ is scattered. Thus $P_i^{\mathfrak{C}} \cong P_i^{\mathfrak{A}}$, and $\bigcup_{i > 0} P_i^{\mathfrak{C}}$ is countable. Clearly $\bigcup_{i > 0} P_i^{\mathfrak{C}}$ is dense in $\mathfrak{C}$.

To complete the proof of Theorem 6.1 we notice that $\mathfrak{A}$ is densely ordered with a nonscattered orbit $P_0^{\mathfrak{A}}$, dense in $\mathfrak{A}$. Thus the construction of $\mathfrak{A}^*$ just reproduces $\mathfrak{A}$, and we may identify them. By Theorem 4.1, $\varphi_{\mathfrak{A}}$ has a model $\mathfrak{B}$ of power $2^{\aleph_0}$.

§ 7. Concluding remarks. We have determined the Hanf number for Scott sentences of countable orderings, but not for the subset consisting of Scott sentences of order types (orderings without unary predicates) nor for the (larger) set of all $L_{\omega_1\omega}$ sentences whose models are orderings. In the former case Theorem 2.1 implies that the Hanf number is at most $(2^{\aleph_0})^+$. In fact, it is exactly $(2^{\aleph_0})^+$: from the ordering $\mathfrak{A}$ constructed in § 6 an order type $\mathfrak{A}'$ can be defined so that $(\mathfrak{A}')^*$ (in the notation of § 4) is isomorphic to $\mathfrak{A}$ and $\varphi_{\mathfrak{A}'}$, has spectrum $\mathrm{Card}^{\leqslant 2^{\aleph_0}}$. In the latter case, the construction of § 6 implies that the Hanf number is at least $(2^{\aleph_0})^+$, but the proof of Theorem 5.1 does not carry over as written. The application of the Erdös–Rado theorem uses the fact that an arbitrary model of a Scott sentence has at most $\aleph_0$ $L_{\infty\omega}$ 2-types. For models of an arbitrary $L_{\omega_1\omega}$ sentence this cannot be assumed.

In § 6 we invoked Theorem 4.1 to produce $\mathfrak{B}$ of power $2^{\aleph_0}$ satisfying $\varphi_{\mathfrak{A}}$. It turns out that (in the notation of § 4) $N(d, S_d) = \{d\}$ in $\mathfrak{A} = \mathfrak{A}^*$ and, in the construction of $\mathfrak{B}$, $\mathfrak{C}$ is all of $B^+ - B$. To find a case where, by contrast, the simple expedient $Q_0^{\mathfrak{B}} = B^+ - B^*$ does not suffice one may examine $\mathfrak{A} - P_1^{\mathfrak{A}}$. The gaps created by removing $P_1^{\mathfrak{A}}$ reappear in B^+, but may not be added to $Q_0^{\mathfrak{B}}$, as a back and forth argument confirms. In general, $\mathfrak{C}$ is the largest subset of B^+ which may be added to $Q_0^{\mathfrak{A}^*}$ without modifying the $Q_i^{\mathfrak{A}^*}$, $i>0$.

It is not difficult to show that the model $\mathfrak{B}$ of $\varphi_{\mathfrak{A}}$ in the case $N(d, S_d) = \{d\}$ of § 4 has at least $2^{\aleph_0}$ nonisomorphic elementary submodels. It follows that for any denumerable $\mathfrak{A}$, if $\varphi_{\mathfrak{A}}$ has more than one (nonisomorphic) model, then it has at least $2^{\aleph_0}$. In the case of the $\mathfrak{A}$ of § 6 it can be shown that $\mathfrak{B}$ has a "universal" property: if $\mathfrak{C} \models \varphi_{\mathfrak{A}}$, then $\mathfrak{C} < \mathfrak{B}$. We do not know whether all $\varphi_{\mathfrak{A}}$ with spectrum $\mathrm{Card}^{\leq 2^{\aleph_0}}$ possess this property. Another question about $\varphi_{\mathfrak{A}}$ with spectrum $\mathrm{Card}^{\leq 2^{\aleph_0}}$ stems from the fact that in Theorem 2.1 we are able to give structural characterizations of orderings whose spectra are Card and $\{\aleph_0\}$, but none for these. It is clear from our analysis that orderings in which the union of the scattered orbits is dense, yet which have a nonscattered orbit, possess this spectrum. However, there are also dense orderings with the same spectrum but no scattered orbit.

References

[1] J. Barwise, *Back and forth through infinitary logic*, Studies in Model Theory, Math. Assoc. of America (1973), pp. 5–39.

[2] — and P. Eklof, *Infinitary properties of Abelian torsion groups*, Ann. Math. Logic 2 (1970), pp. 25–68.

[3] J. Baumgartner, *The Hanf number for complete $L_{\omega_1\omega}$ sentences (without* GCH), J. Symb. Logic 39 (1974), pp. 575–578.

[4] H. J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, 1971.

[5] D. W. Kueker, *Back-and-forth arguments and infinitary logics*, in Infinitary Logic: in Memoriam Carol Karp, Springer-Verlag, 1975, pp. 17–72.

[6] M. Makkai, *Structures elementarily equivalent relative to infinitary languages to models of higher power*, Acta Math. Acad. Scientarum Hungaricae 21 (1970), pp. 283–295.

[7] J. Malitz, *The Hanf number for complete $L_{\omega_1\omega}$ sentences*, The Syntax and Semantics of Infinitary Languages, Springer-Verlag, 1968, pp. 166–181.

[8] M. Rubin, *Theories of linear order*, Israel J. Math. 17 (4) (1974), pp. 392–443.

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Continuous monotone decompositions of planar curves

by

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Abstract. In this paper we prove that if X is a planar curve and $f\colon X \to Y$ is a monotone open surjection onto a nondegenerate continuum Y such that either (1) $f^{-1}(y)$ is a λ-dendroid for each $y \in Y$, or (2) $f^{-1}(y)$ is locally connected for each $y \in Y$, then f is a homeomorphism. We give also some examples showing that the theorem is in a sense the best possible.

1. Introduction. In this paper we are going to discuss continuous monotone decomposition of certain metric continua. It is well-known that the investigation of continuous monotone decompositions of continua is equivalent to investigation of continuous monotone and open transformations of continua. All the results of this paper are expressed in the language of mappings. In [12] B. Knaster showed that there is a continuous monotone and open map f from an irreducible continuum onto the unit interval $I = \{0, 1\}$ such that each fiber $f^{-1}(t)$ is nondegenerate. Since then there has been a remarkable interest in investigations of structure of the fibers for such mappings defined on irreducible continua (see e.g. [10] and [17]). These investigations were in some sense closed by E. Dyer [8] who proved that for every continuous monotone and open surjection with nondegenerate fibres from an irreducible continnum onto a nondegenerate continuum there is an indecomposable fibre.

From this result it follows in particular that in the above Knaster's example there must be some $t \in I$ such that $f^{-1}(t)$ is indecomposable. There are examples of irreducible and nonirreducible continua admitting monotone open surjections onto nondegenerate continua such that all fibers of the surjections are indecomposable (even pseudoarcs) (see [1], [4] and [11]). It should be noted that the set of indecomposable fibres in such situations is of a particular Borel type. In fact, we have the following theorem easily resulting from a theorem of Mazurkiewicz.

1.1. Theorem. *Let $f\colon X \to Y$ be a continuous monotone open surjection from a continuum X. Then the set of all $y \in Y$ such that $f^{-1}(y)$ is indecomposable (hereditarily indecomposable) is a G_δ-subset of Y.*