Every compact T_5 sequential space is Fréchet

by

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Abstract. Among the various results proved in this short paper, are two theorems, both of which imply the assertion of the title separately.

§ 1. General results. The following general lemma and the two theorems succeeding it, are not very interesting on their own right. But they contain the key-ideas out of which all the elegant results of this paper are proved. N denotes the set of natural numbers.

LEMMA 1.1. *Let X be a Hausdorff space, x be an element of X such that $X\setminus\{x\}$ is normal. Let (x_n) be a sequence of elements that are mutually distinct and different from x, converging to x. Then for each n in N, there is an open neighbourhood W_n of x_n such that*

(i) *$W_n \cap W_m$ is empty if $n \neq m$ and*

(ii) *if F_n is a finite subset of $W_n\setminus\{x_n\}$ for each n in N and if $F = \bigcup_{n=1}^{\infty} F_n$, then $\overline{F} \subset F \cup \{x\}$.*

Proof. Step 1. We let C_1 be the set of the sequence (x_n). Then C_1 is a discrete countable subset of X. For each n in N, there are three neighbourhoods V_n^1, V_n^2 and V_n^3 of x_n such that

(i) $x \notin \overline{V_n^1}$,

(ii) $V_n^2 \cap C_1 = \{x_n\}$,

(iii) $\overline{V_n^3} \subset V_n^2 \cap V_n^1$.

(For (i) use Hausdorffness of X to separate x_n and x by disjoint neighbourhoods; for (ii) use the fact that x_n is isolated in the relative topology of C_1. For (iii) use the fact that the space is regular at the point x_n).

Then put $V_n = V_n^3\setminus[\bigcup_{i=1}^{n-1} V_i^3]$ for each $n > 1$ and $V_1 = V_1^3$.

Let $V = \bigcup_{n=1}^{\infty} V_n$ and $C_2 = \overline{V}\setminus V\setminus\{x\}$.

Step 2. C_1 and C_2 are claimed to be disjoint closed subsets of $X\setminus\{x\}$. Now $C_1\cup\{x\}$ is clearly compact and hence closed in X. Also $C_2\cup\{x\} = \overline{V}\setminus V =$ a closed set $\setminus$ an open set; therefore $C_2\cup\{x\}$ is closed in X. Thus C_1 and C_2 are closed in $X\setminus\{x\}$.

For each n, we have $x_n\in V_n$. Therefore $C_1\subset V$. But C_2 is disjoint from V. Therefore $C_1\cap C_2$ is empty.

Step 3. Since $X\setminus\{x\}$ is normal, there is an open set W containing C_1, whose closure in $X\setminus\{x\}$ is disjoint from C_2. We let $W_n = W\cap V_n$ for every n in N. Clearly, these W_n's are pairwise disjoint and $x_n\in W_n$ for every n in N. It remains only to prove that the assertion (ii) of the statement holds.

Step 4. Let F be as in the statement (ii). Now

$$F\subset \bigcup_{n=1}^{\infty} W_n\subset \bigcup_{n=1}^{\infty} V_n = V$$

and hence

$$\overline{F}\subset\overline{V}. \tag{1}$$

Also since V_n is open,

$$\overline{F}\cap V_n = (\overline{F\cap V_n})\cap V_n = \overline{F}_n\cap V_n = F_n$$

(since F_n is finite and hence closed). Therefore

$$\overline{F}\cap V = \bigcup_{n=1}^{\infty} F_n = F. \tag{2}$$

Thirdly, $\overline{F}\subset\overline{W}$ and therefore by the choice of W, $\overline{F}$ is disjoint from $C_2 = \overline{V}\setminus V\setminus\{x\}$. Therefore

$$\overline{F}\cap\overline{V}\subset V\cup\{x\}. \tag{3}$$

Now we have

$$\overline{F} = \overline{F}\cap\overline{V}\subset(\overline{F}\cap V)\cup\{x\} = F\cup\{x\},$$

because of (1), (3) and (2).

THEOREM 1.2. *Let X be a countably compact Hausdorff space. Let x in X be such that $X\setminus\{x\}$ is normal. Let Y be a sequential subspace of X containing x. Then x is a Fréchet point in Y. (That is, whenever $x\in\overline{A}$ and $A\subset Y$, there is a sequence in A converging to x.)*

Proof. Let $A\subset Y$ and $x\in\overline{A}\setminus A$. Then because Y is sequential, there exists a sequence (x_n) of distinct elements different from x converging to x, such that $x_n\in Y\cap\overline{A}$. Assuming that there is no sequence from A converging to x, we get that $x_n\in(Y\cap\overline{A})\setminus A$.

Applying Lemma 1.1 for this sequence, we get a sequence (W_n) of pairwise disjoint open sets in $X\setminus\{x\}$ such that $x_n\in W_n$ for every n and such that $\overline{F}\subset F\cup\{x\}$ whenever $F = \bigcup_{n=1}^{\infty} F_n$, where F_n is a finite subset of $W_n\setminus\{x_n\}$ for every $n\in N$.

Since x_n is an accumulation point of A, and W_n is a neighbourhood of x_n, the set $A\cap W_n$ is nonempty. We choose a point y_n in it and put $F_n = \{y_n\}$. When this is done for every n in N, we let $F = \{y_n|\ n\in N\}$. Then by the property stated above, we have $\overline{F}\subset F\cup\{x\}$.

Therefore either $\overline{F} = F$ or $\overline{F} = F\cup\{x\}$.

Now F is clearly infinite and discrete. Hence it cannot be closed in the countably compact space X. Therefore $\overline{F}$ has to be equal to $F\cup\{x\}$. In this case, x is the only limit point of F in the sequential space Y. Therefore there is a sequence in F (and hence in A) converging to x.

THEOREM 1.3. *Let X be a sequential space such that $X\setminus\{x\}$ is a T_4-space for every x in X. Let r be a positive integer. Let Y be a subspace of X such that $\tilde{Y}$ is homeomorphic to S_r. Then there is a subspace Z of Y (relatively closed in Y) such that Z itself is homeomorphic to S_r. (Here, as in many other places below, $\tilde{Y}$ denotes the sequential coreflection of Y.)* (See [6] for the definition of S_γ.)

Proof. We prove by induction on r. For $r = 1$ this is obvious, because, a Hausdorff space Z is homeomorphic to S_1 if and only if $\tilde{Z}$ is homeomorphic to S_1.

Suppose we have proved it for $r = n$. Now let Y be a subspace of X such that $\tilde{Y}$ is homeomorphic to S_{n+1}. Let y be the point of Y correspoding to the unique point of S_{n+1} having sequential order $n+1$. Let (y_n) be the essentially largest sequence of distinct elements different from y, in Y, converging to y.

Apply Lemma 1.1 to this sequence and obtain a sequence (W_n) of pairwise disjoint open sets such that $x_m\in W_m\forall m$ and such that if F_m is a finite subset of $W_m\setminus\{x_m\}$ for each n and if $F = \bigcup_{m=1}^{\infty} F_m$, then $\overline{F}\subset F\cup\{x\}$.

Now look at $W_m\cap Y$. It can be proved that its sequential coreflection is homeomorphic to S_r. (More generally if V is an open subspace of X, the topology of $\tilde{V}$ is the same as the relative topology from that of $\tilde{X}$. Further, if V is an open subset of S_{n+1} containing only one point of sequential order n and not containing the point of sequential order $n+1$, then V is homeomorphic to S_n.)

Therefore by induction hypothesis, there is a subspace Z_m of $W_m\cap Y$, open in $W_m\cap Y$, such that Z_m itself is homeomorphic to S_n. Now let $Z = \{y\}\cup(\bigcup_{m=1}^{\infty} Z_m)$. This is clearly a closed subspace of Y. We claim that it is homeomorphic to S_{n+1}.

We make use of the following general result: Let P be a topological space written as $P = \bigcup_{n=0}^{\infty} P_n$ where

(i) P_0 is the set of a sequence $\{p_1, p_2, \ldots, p_n \ldots\}$ converging to an element p_0.

(ii) $P_n\cap P_m = \emptyset$ if $0 = n\neq m\neq 0$.

(iii) $P_n\cap P_0 = \{p_n\}$ for every $n>0$ and

(iv) P_n is homeomorphic to S_n for every $n>0$.

Then P is homeomorphic to S_{n+1} if and only if whenever $F = \bigcup_{n=1}^{\infty} F_m$ where F_m is a finite subset of $P_m \setminus \{p_m\}$ for every m, then F is closed.

We leave the proof of the above assertion to the reader and observe that we are in a situation similar to that stated above, by choosing P_m to be Z_m and P_0 to be $\{y_n|\ n = 0, 1, 2, \ldots\}$. We have only to prove that if $F = \bigcup_{m=1}^{\infty} F_m$ where F_m is a finite subset of $Z_m \setminus \{y_m\}$ for every m, then F is closed. Since $Z_m \subset W_m$, it follows from our choice of W_m that $\bar{F} \subset F \cup \{x\}$. On the other hand, F is discrete; if $\bar{F} = F \cup \{x\}$, x will be the only accumulation point of F; F must be sequential, since it is a closed subspace of X; therefore there is a sequence in F converging to x. This contradicts the fact that Y is homeomorphic to S_{n+1}. Therefore x cannot be a limit point of F.

Therefore F is closed. The proof is now complete.

§ 2. Corollaries.

COROLLARIES 2.1. (1) *Let X be a countably compact Hausdorff space and let $X \setminus \{x\}$ be normal for each x in X. If Y is a sequential subspace of X, then Y is Fréchet.*

(2) *In every countably compact T_5 space every sequential subspace is Fréchet. (T_5 spaces are, by definition, the hereditarily normal spaces).*

(3) *(Countably compact$+T_5+$sequential) $\Rightarrow$ Fréchet.*

(4) *Every compact T_5 sequential space is Fréchet.*

Proof. Among these four assertions, each follows easily from the preceding. The first of these assertions follows from Theorem 1.2.

COROLLARIES 2.2. (1) *Let X be a countably compact T_2 space; let $x \in X$ be such that $X \setminus \{x\}$ is normal. Let (x_n) be a sequence of distinct elements different from x converging to x. Then there exists a sequence of pairwise disjoint open sets $W_1, W_2, \ldots, W_n, \ldots$, such that $x_n \in W_n$ for every n and such that every neighbourhood of x contains W_n completely for all but a finite number of values of n.*

(2) *Let X be a subspace of a compact T_5 space. Let $(x_n) \to x$, in X. Then there exists an open neighbourhood W_n of $x_n \forall n$, such that every neighbourhood of x contains W_n for all but a finite number of n.*

(3) *The countable T_5 space S_2 does not admit any T_5 compactification.*

Proof. To prove (1), apply Lemma 1.1 to choose the W_n's. Now let W be any neighbourhood of x. Assume that W does not contain W_n for an infinity of values of n. Then choose some y_n in $W_n \setminus W$ for these values of n and let F be the set thus formed. Then $\bar{F} \subset F \cup \{x\}$. But F is infinite and discrete. Since X is countably compact, F cannot be closed. On the other hand x is not a limit point of F, since W is a neighbourhood of x disjoint from F. Thus we have arrived at a contradiction.

(2) can be deduced from (1), and (3) from (2).

COROLLARY 2.3. *Let X be a sequential space such that $X \setminus \{x\}$ is a T_4-space for every x in X. Let n be a positive integer. Then the following are equivalent:*

1) *The sequential order of X is $\geqslant n$.*

2) *X contains a subspace Y such that $\tilde{Y}$ is homeomorphic to S_n.*

3) *X contains a subspace homeomorphic to S_n.*

4) *X contains a closed subspace homeomorphic to S_n.*

Proof. The equivalence of 1) and 2) has been proved in [1]. It follows from Theorem 1.3 that 2) implies 4). It is obvious that 4) implies 3) and 3) implies 2).

COROLLARIES 2.4. 1) *Let P be any closed-hereditary property not possessed by S_2. Then every sequential T_5-space possessing P is Fréchet.*

2) *Every (countably) compact sequential T_5-space is Fréchet.*

Proof. 1) follows from Corollary 2.3 by taking $n = 2$. Then 2) follows easily from 1).

COROLLARY 2.5. *There is a space Z with the following peculiar properties:*

1) *Z is a T_5 space, but no compactification of Z is T_5.*

2) *Z is a T_5 space; Z can be embedded in a T_4 sequential space; but still, Z cannot be embedded in any T_5 sequential space.*

3) *Z is a countable T_2 space; Z can be embedded in a sequential T_2 space; but Z cannot be embedded in any countable sequential T_2 space.*

Proof. Consider the space ψ^* which is essentially the only known example of a compact Hausdorff sequential non-Fréchet space. (See [2] and [6].) By the result of [1], there is a subspace Z of ψ^* such that $\tilde{Z}$ is homeomorphic to S_2. We claim that this Z has the stated properties.

1) Z is T_3, since ψ^* is so. Z is also countable. Every countable T_3 space is T_5. Therefore Z is a T_5-space. It can be deduced from Corollary 2.2 (2) that Z cannot be embedded in any compact T_5 space.

2) Z is already a subspace of the T_4-space ψ^*. Suppose X is a sequential T_5 space containing Z. Then by Theorem 1.3 there is a subspace Y of Z homeomorphic to S_2. This implies that S_2 can be embedded in ψ^*, which is false as proved in [6] or [4].

3) It can be proved along the lines of [6] that for a countable sequential T_2 space an analogue of Theorem 1.3 holds. Considering this for $r = 2$, we get that Z cannot be embedded in a countable sequential T_2 space; for, as observed already, Z does not contain a subspace homeomorphic to S_2.

§ 3. Remarks.

1) To illustrate Theorem 1.2 with an example, we consider the space ψ^*. Here, there is a unique point x_0 of sequential order 2. When that point is removed, we get the space ψ. It is easy to prove that ψ is not normal (see [3], p. 79). If t is an isolated point of ψ^*, then $\psi^* \setminus \{t\}$ is compact Hausdorff and hence normal. If t is a limit point of ψ^* different from x_0, then there is a compact open neighbourhood V of t in ψ^*, every other point of which, is isolated. Thus $\psi^* \setminus \{t\}$ is the disjoint topological

sum of a compact Hausdorff space and a discrete space. Thus we have $\psi^*\setminus\{t\}$ is normal for every point t different from x_0.

It is also easy to see that every point of ψ^* different from x_0 is a Fréchet-point.

Thus this example helps to appreciate the force of Theorem 1.2. In fact, it is this example that motivated the main result of this paper.

2) It is a good guess that the spaces S_n serve as test-spaces for sequential order in the following sense: A sequential T_2 space has sequential order $\geqslant n$ if and only if it contains a copy of S_n. However this guess is not true. What is true is a very close result: A sequential T_2 space has sequential order $\geqslant n$ if and only if it contains a subspace whose sequential coreflection is homeomorphic to S_2. This has been proved in [1]. Several years later, the falsity of the first guess was proved in [6].

Naturally one likes to know whether the first guess is correct, when we restrict our attention to a fairly nice class of spaces. In this direction [6] gives the first positive result that it is so in the class of countable spaces.

Our Corollary 2.3 supplements these results by showing that the first guess holds good in the class of hereditarily normal spaces.

3) We leave open a simple-looking question: What are all the subspaces of compact T_5 spaces? Every such space must be obviously T_5. Further whenever $(x_n)\to x$, there should exist neighbourhoods V_n of x_n such that every neighbourhood of x must contain V_n for all but a finite number of values of n.

4) See [5] for the definitions of spaces S_n, sequential order at a point, etc.

Added in proof. The following is a noteworthy consequence: For any sequential T_5 space, the sequential order and the k-order coincide.

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Accepté par la Rédaction le 5. 9. 1977

Пространства с единственной точкой экстремальной несвязности

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Abstract. R. Telgársky posed in [4] the following question: Does there exist a completely regular space X with exactly one non-isolated point $*$ of extremal disconnectedness? In this paper there are constructed two dense in itself spaces providing an answer in the positive sense: 1. A countable regular space X, and 2. (CH) A compact Hausdorff space X in which points of X-{$*$} have countable character.

Р. Телгарский [4] поставил следующую проблему:

Существует ли пространство с одной и неизолированной точкой экстремальной несвязности?

Напомним, что в топологическом пространстве (X, τ) точка x называется *точкой экстремальной несвязности* (коротко „э. н."), если $x \notin [V_1] \cap [V_2]$ для любых дизъюнктных $V_1, V_2 \in \tau$.

Основные результаты настоящей работы формулируются следующим образом:

Пример 1. Существует счетное регулярное плотное в себе пространство ровно с одной точкой э. н.

Пример 2 [CH]. Существует плотный в себе бикомпакт ровно с одной точкой э. н., остальные точки имеют счетный характер.

Часть работы, связанная с примером 1, принадлежит В. М. Ульянову, остальное — В. И. Малыхину.

I. Построить хаусдорфово пространство ровно с одной и неизолированной точкой э. н. не составляет никакого труда. Пусть (X, T) — произвольное хаусдорфово не бикомпактное пространство. Следовательно, существует центрированная система $\xi \subset T$, не имеющая в пространстве (X, T) точки прикосновения. Дополним ее до какой-нибудь максимальной центрированной системы открытых множеств η. Положим теперь $X_\eta = X \cup \{\eta\}$, $T_\eta = T \cup \{\{\eta\} \cup \cup A \mid A \in \eta\}$. Легко убедиться, что пространство (X_η, T_η) хаусдорфово и точка $\{\eta\}$ в нем — точка э. н.

Предложение 1. *Пространство (X_η, T_η) регулярно если и только если регулярно*