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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

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Every compact $\tau_1$ sequential space is Fréchet

by

V. Kannan (Madurai)

Abstract. Among the various results proved in this short paper, are two theorems, both of which imply the assertion of the title separately.

§ 1. General results. The following general lemma and the two theorems succeeding it, are not very interesting on their own right. But they contain the key-ideas out of which all the elegant results of this paper are proved. $N$ denotes the set of natural numbers.

**Lemma 1.1.** Let $X$ be a Hausdorff space, $x$ be an element of $X$ such that $X \setminus \{x\}$ is normal. Let $(x_n)$ be a sequence of elements that are mutually distinct and different from $x$, converging to $x$. Then for each $n$ in $N$, there is an open neighbourhood $W_n$ of $x_n$ such that

1. $W_n \cap W_m$ is empty if $n \neq m$ and
2. if $F = \bigcup_{n=1}^{\infty} F_n$, then $F \subseteq F \cup \{x\}$.

**Proof.** Step 1. We let $C_1$ be the set of the sequence $(x_n)$. Then $C_1$ is a discrete countable subset of $X$. For each $n$ in $N$, there are three neighbourhoods $V_1^n$, $V_2^n$, and $V_3^n$ of $x_n$ such that

1. $x \notin V_1^n$,
2. $V_2^n \cap C_1 = \{x_n\}$,
3. $V_3^n \subseteq V_2^n \cap V_3^n$.

(For (i) use Hausdorffness of $X$ to separate $x_n$ and $x$ by disjoint neighbourhoods; for (ii) use the fact that $x_n$ is isolated in the relative topology of $C_1$. For (iii) use the fact that the space is regular at the point $x_n$.)

Then put $V_n = V_1^n \setminus \bigcup_{m=1}^{n-1} V_1^m$ for each $n > 1$ and $V_1 = V_1^1$.

Let $V = \bigcup_{n=1}^{\infty} V_n$ and $C_2 = \cap_n V_n \setminus \{x\}$.
Step 2. $C_1$ and $C_2$ are claimed to be disjoint closed subsets of $X \setminus \{x\}$. Now
$C_1 \cup \{x\}$ is clearly compact and hence closed in $X$. Also $C_2 \cup \{x\} = \overline{V_1 \cup V_2}$ is a closed set in an open set; therefore $C_2 \cup \{x\}$ is closed in $X$. Thus $C_1$ and $C_2$ are closed in $X \setminus \{x\}$.

For each $n$, we have $x_n \in V_n$. Therefore $C_1 \subseteq V_n$. But $C_2$ is disjoint from $V_n$. Therefore $C_1 \cap C_2$ is empty.

Step 3. Since $X \setminus \{x\}$ is normal, there is an open set $W$ containing $C_1$, whose closure in $X \setminus \{x\}$ is disjoint from $C_2$. We let $W_n = W \cap V_n$ for every $n$ in $N$. Clearly, these $W_n$'s are pairwise disjoint and $x_n \in W_n$ for every $n$ in $N$. It remains only to prove that the assertion (ii) of the statement holds.

Step 4. Let $P$ be as in the statement (ii). Now
$$= \bigcup_{n=1}^\infty W_n \subseteq \bigcup_{n=1}^\infty V_n = V$$
and hence

(1) $$F \subseteq V.$$  

Also since $V_n$ is open,

$$F \cap V_n = (F \cap V_n) \cap V_n = F \cap V_n = F_n$$

(since $F_n$ is finite and hence closed). Therefore

(2) $$F \cap V = \bigcup_{n=1}^m F_n = F.$$  

Thirdly, $F \subseteq W$ and therefore by the choice of $W$, $F$ is disjoint from $C_2 = \overline{V_1 \cup V_2 \cup \{x\}}$. Therefore

(3) $$F \cap \overline{V_1 \cup V_2 \cup \{x\}} = \emptyset.$$  

Now we have

$$F = F \cap \overline{V_1 \cup V_2 \cup \{x\}} = F \cap \overline{V_1 \cup V_2 \cup \{x\}} = \emptyset,$$

because of (1), (3) and (2).

**Theorem 1.2.** Let $X$ be a countably compact Hausdorff space. Let $x$ be a point at which $X \setminus \{x\}$ is normal. Let $Y$ be a sequential subspace of $X$ containing $x$. Then $x$ is a Fréchet point in $Y$. (That is, whenever $x \in X$ and $A \subseteq Y$, there is a sequence in $A$ converging to $x$)

Proof. Let $A \subseteq Y$ and $x \in A \setminus A$ be a point in $A \setminus A$. Then, because $Y$ is sequential, there exists a sequence $(x_n)$ of distinct elements different from $x$ converging to $x$, such that $x_n \in Y \cap A$. Assuming that there is no sequence from $A$ converging to $x$, we get that $x_n \in (Y \cap A) \setminus A$.

Applying Lemma 1.1 for this sequence, we get a sequence $(W_n)$ of pairwise disjoint open sets in $X \setminus \{x\}$ such that $x_n \in W_n$ for every $n$ and such that $F \subseteq \overline{V_1 \cup V_2}$ whenever $F = \bigcup_{n=1}^\infty F_n$, where $F_n$ is a finite subset of $W_n \setminus \{x_n\}$ for every $n \in N$. Since $x_n$ is an accumulation point of $A$, and $W_n$ is a neighbourhood of $x_n$, the set $A \cap W_n$ is nonempty. Choose a point $y_n$ in it and put $F_n = \{y_n\}$. When this is done for every $n$ in $N$, we let $F = \bigcup_{n} y_n$. Then by the property stated above, we have $F \subseteq \overline{V_1 \cup V_2}$.

Therefore either $F = F \cap F = F \cup \{x\}$.

Now $F$ is clearly infinite and discrete. Hence it cannot be closed in the countably compact space $X$. Therefore $F$ has to be equal to $F \cup \{x\}$. In this case, $x$ is the only limit point of $F$ in the sequential space $Y$. Therefore there is a sequence in $F$ (and hence in $A$) converging to $x$.

**Theorem 1.3.** Let $X$ be a sequential space such that $X \setminus \{x\}$ is a $T_2$-space for every $x \in X$. Let $r$ be a positive integer. Let $Y$ be a subspace of $X$ such that $Y$ is homeomorphic to $S_r$. Then there is a subspace $Z$ of $Y$ (relatively closed in $Y$) such that $Z$ itself is homeomorphic to $S_r$. (Here, as in many other places below, $Y$ denotes the sequential completion of $Y$.) (See [6] for the definition of $S_r$.)

Proof. We prove by induction on $r$. For $r = 1$ this is obvious, because

a Hausdorff space $Z$ is homeomorphic to $S_r$ if and only if $Z$ is homeomorphic to $S_r$.

Suppose we have proved it for $r = n$. Now let $Y$ be a subspace of $X$ such that $Y$ is homeomorphic to $S_{n+1}$. Let $y$ be any point of $Y$ corresponding to the unique point of $S_{n+1}$ having sequential order $n+1$. Let $(y_n)$ be the essentially largest sequence of distinct elements different from $y$, in $Y$, converging to $y$.

Applying Lemma 1.1 to this sequence and obtaining a sequence $(W_n)$ of pairwise disjoint open sets such that $x_n \in W_n \setminus \{x_n\}$ and such that if $F \subseteq \bigcup_{n=1}^\infty F_n$, then $F \subseteq F \cup \{x\}$.

Now look at $W_n \cap Y$. It can be proved that its sequential closure is homeomorphic to $S_n$. (More generally if $V$ is an open subspace of $X$, the topology of $V$ is the same as the relative topology from that of $X$.) Further, if $V$ is an open subspace of $S_{n+1}$ containing only one point of sequential order $n$ and not containing the point of sequential order $n+1$, then $V$ is homeomorphic to $S_n$.

Therefore by induction hypothesis, there is a subspace $Z_n \subseteq W_n \cap Y$ open in $W_n \cap Y$, such that $Z_n$ itself is homeomorphic to $S_n$. Now let $Z = \bigcup_{n=1}^\infty Z_n$. This is clearly a closed subspace of $Y$. We claim that it is homeomorphic to $S_{n+1}$.

We make use of the following general result: Let $P$ be a topological space written as $P = \bigcup_{n=0}^\infty P_n$, where

(i) $P_0$ is the set of a sequence $(p_1, p_2, \ldots, p_n)$ converging to an element $p_0$.
(ii) $P_n \cap P_m = \emptyset$ if $n \neq m$.
(iii) $P_n \cap P_m = \{p_n\}$ for every $n > 0$.
(iv) $P_n$ is homeomorphic to $S_n$.
Then $P$ is homeomorphic to $S_{n+1}$ if and only if whenever $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n$ is a finite subset of $P_n \setminus \{p_n\}$ for every $m$, then $F$ is closed.

We leave the proof of the above assertion to the reader and observe that we are in a situation similar to that stated above, by choosing $P_n$ to be $Z_n$ and $P_m$ to be $\{y_k\}_{n=0, 1, 2, \ldots}$. We have only to prove that if $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n$ is a finite subset of $Z_n \setminus \{y_k\}$ for every $m$, then $F$ is closed. Since $Z_n = W_n$, it follows from our choice of $W_n$ that $F = F \cup \{x\}$. On the other hand, $F$ is discrete; if $F = F \cup \{x\}$, $x$ will be the only accumulation point of $F$; $F$ must be sequential, since it is a closed subspace of $X$; therefore there is a sequence in $F$ converging to $x$. This contradicts the fact that $Y$ is homeomorphic to $S_{n+1}$. Therefore $x$ cannot be a limit point of $F$.

Therefore $F$ is closed. The proof is now complete.

§ 2. Corollaries.

Corollaries 2.1. (1) Let $X$ be a countably compact Hausdorff space and let $X \setminus \{x\}$ be normal for each $x$ in $X$. If $Y$ is a sequential subspace of $X$, then $Y$ is Fréchet.

(2) In every countably compact $T_3$ space every sequential subspace is Fréchet. ($T_3$ spaces are, by definition, the hereditarily normal spaces).

(3) (Countably compact $+ T_3$ + sequential) $\Rightarrow$ Fréchet.

(4) Every compact $T_3$ sequential space is Fréchet.

Proof. Among these four assertions, each follows easily from the preceding. The first of these assertions follows from Theorem I.2.

Corollaries 2.2. (1) Let $X$ be a countably compact $T_3$ space; let $x \in X$ be such that $X \setminus \{x\}$ is normal. Let $(x_n)$ be a sequence of pairwise disjoint open sets $W_1, W_2, \ldots, W_n, \ldots$, such that $x_n \in W_n$ for every $n$ and such that every neighbourhood of $x$ contains $W_n$ completely for all but a finite number of values of $n$.

(2) Let $X$ be a subspace of a compact $T_3$ space. Let $(x_n) \rightarrow x$, in $X$. Then there exists an open neighbourhood $W_x$ of $x$ such that every neighbourhood of $x$ contains $W_x$ for all but a finite number of $n$.

(3) The countable $T_3$ space $S_3$ does not admit any $T_3$ compactification.

Proof. To prove (1), apply Lemma 1.1 to choose the $W_n$’s. Now let $W$ be any neighbourhood of $x$. Assume that $W$ does not contain $W_n$ for an infinite number of values of $n$. Then choose some $y_n$ in $W_\infty \setminus W$ for these values of $n$ and let $F$ be the set thus formed. Then $F = F \cup \{x\}$. But $F$ is infinite and discrete. Since $X$ is countably compact, $F$ cannot be closed. On the other hand, $x$ is not a limit point of $F$, since $W$ is a neighbourhood of $x$ disjoint from $F$. Thus we have arrived at a contradiction.

(2) can be deduced from (1), and (3) from (2).

Corollary 2.3. Let $X$ be a sequential space such that $X \setminus \{x\}$ is a $T_\infty$-space for every $x$ in $X$. Let $n$ be a positive integer. Then the following are equivalent:

1) The sequential order of $X$ is $\geq n$.

2) $X$ contains a subspace $Y$ such that $Y$ is homeomorphic to $S_n$.

3) $X$ contains a subspace homeomorphic to $S_n$.

4) $X$ contains a closed subspace homeomorphic to $S_n$.

Proof. The equivalence of 1) and 2) has been proved in [1]. It follows from Theorem 1.3 that 2) implies 4). It is obvious that 4) implies 3) and 3) implies 2).

Corollaries 2.4. 1) Let $P$ be any closed-hereditary property not possessed by $S_2$. Then every sequential $T_\infty$-space possessing $P$ is Fréchet.

2) Every (countably) compact sequential $T_\infty$-space is Fréchet.

Proof. 1) follows from Corollary 2.3 by taking $n = 2$. Then 2) follows easily from 1).

Corollary 2.5. There is a space $Z$ with the following peculiar properties:

1) $Z$ is a $T_3$ space, but no compactification of $Z$ is $T_3$.

2) $Z$ is a $T_3$ space; $Z$ can be embedded in a $T_3$ sequential space; but still, $Z$ cannot be embedded in any $T_3$ sequential space.

3) $Z$ is a countable $T_3$ space; $Z$ can be embedded in a sequential $T_3$ space; but $Z$ cannot be embedded in any countable sequential $T_3$ space.

Proof. Consider the space $\phi^*(z)$ which is essentially the only known example of a compact Hausdorff sequential non-Fréchet space. (See [2] and [6].) By the result of [1], there is a subspace $Z$ of $\phi^*(z)$ such that $Z$ is homeomorphic to $S_2$. We claim that this $Z$ has the stated properties.

1) $Z$ is $T_3$, since $\phi^*$ is so. $Z$ is also countable. Every countable $T_3$ space is $T_3$. Therefore $Z$ is a $T_3$ space. It can be deduced from Corollary 2.2 (2) that $Z$ cannot be embedded in any compact $T_3$ space.

2) $Z$ is already a subspace of the $T_\infty$-space $\phi^*$. Suppose $X$ is a sequential $T_3$ space containing $Z$. Then by Theorem 1.3 there is a subspace $Y$ of $Z$ homeomorphic to $S_2$. This implies that $S_2$ can be embedded in $\phi^*$, which is false as proved in [6] or [4].

3) It can be proved along the lines of [6] that for a countable sequential $T_3$ space an analogue of Theorem 1.3 holds. Considering this for $r = 3$, we get that $Z$ cannot be embedded in a countable sequential $T_3$ space; for, as observed already, $Z$ does not contain a subspace homeomorphic to $S_3$.

§ 3. Remarks.

1) To illustrate Theorem 1.2 with an example, we consider the space $\phi^*$. Here, there is a unique point $x_0$ of sequential order 2. When that point is removed, we get the space $\psi$. It is easy to prove that $\psi$ is not normal (see [3], p. 79). If $x$ is an isolated point of $\psi^*$, then $\psi^* \setminus \{x\}$ is compact Hausdorff and hence normal. If $x$ is a limit point of $\psi^*$ different from $x_0$, then there is a compact open neighbourhood $V$ of $x$ in $\psi^*$, every other point of which is isolated. Thus $\psi^* \setminus \{x\}$ is the disjoint topological
sun of a compact Hausdorff space and a discrete space. Thus we have \( \psi^* (t) \) is

normal for every point \( t \) different from \( x_0 \).

It is also easy to see that every point of \( \psi^* \) different from \( x_0 \) is a Fréchet-point.

Thus this example helps to appreciate the force of Theorem 1.2. In fact, it is this example that motivated the main result of this paper.

2) It is a good guess that the spaces \( S_n \) serve as test-spaces for sequential order in the following sense: A sequential \( T_2 \) space has sequential order \( \geq n \) if and only if it contains a copy of \( S_n \). However this guess is not true. What is true is a very close result: A sequential \( T_2 \) space has sequential order \( \geq n \) if and only if it contains a subspace whose sequential coreflection is homeomorphic to \( S_n \). This has been proved in [1]. Several years later, the falsity of the first guess was proved in [6].

Naturally one likes to know whether the first guess is correct, when we restrict our attention to a fairly nice class of spaces. In this direction [6] gives the first positive result that it is so in the class of countable spaces.

Our Corollary 2.3 supplements these results by showing that the first guess holds good in the class of hereditarily normal spaces.

3) We leave open a simple-looking question: What are all the subspaces of compact \( T_2 \) spaces? Every such space must be obviously \( T_2 \). Further whenever \( (x_n) \to x \), there should exist neighbourhoods \( V_n \) of \( x_n \) such that every neighbourhood of \( x \) must contain \( V_n \) for all but a finite number of values of \( n \).

4) See [3] for the definitions of spaces \( S_n \), sequential order at a point, etc.

Added in proof. The following is a noteworthy consequence: For any sequential \( T_2 \) space, the sequential order and the \( k \)-order coincide.

References


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\[ \text{Пространства с единственной точкой экстремальной несвязности} \]

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Abstract. R. Telgársky posed in [4] the following question: Does there exist a completely regular space \( X \) with exactly one non-isolated point of extremal disconnectedness? In this paper there are constructed two dense in itself spaces providing an answer in the positive sense: 1. A countable regular space \( X \) and 2. \( (CH) \) A compact Hausdorff space \( X \) in which points of \( X \cdot \{x\} \) have countable character.

Р. Телгарский [4] поставил следующую проблему:

Существует ли пространство с одной и неизолированной точкой экстремальной несвязности?

Нам известно, что в топологическом пространстве \( (x, t) \) точка \( x \) называется точкой экстремальной несвязности (корректно "т. н."), если \( x \notin \{V_1 \cap \{V_2\} \) для любых дизъюнктивных \( V_1 \), \( V_2 \in \tau \).

Основные результаты настоящей работы формулируются следующим образом:

Пример 1. Существует регулярно упакованное в себе пространство редко с одной точкой т. н.

Пример 2 [CH]. Существует плотный в себе бикомпакт редко с одной точкой т. н., остальные точки имеют счетный характер.

Часть работы, связанная с примером 1, произведена В. М. Ульяновым, остальное — В. И. Малыхину.

1. Построить хаусдорфское пространство редко с одной и неизолированной точкой т. н. не составляющий низкого труда. Пусть \( (X, T) \) — произвольный хаусдорфский не бикомпактный пространство. Следовательно, существует центральная система \( \xi \subset T \), не имеющая в пространстве \( (X, T) \) точки приближения. Дополним ее до какой-нибудь максимальной центральной системы открытых множеств т. н. Положим теперь \( X_\eta = X \cup \{\eta\} \), \( T_\eta = T \cup \{\eta\} \cup \cup A \in \eta \). Легко убедиться, что пространство \( (X_\eta, T_\eta) \) хаусдорфское и точка \( \eta \) не является точкой т. н.

Предложение 1. Пространство \( (X_\eta, T_\eta) \) регулярно если и только если регулярно