The structure $\mathfrak{E}_5^n(\mathfrak{R}R^+) = (R^n, L_{\mathfrak{R}}, D_{\mathfrak{R}}, B_5)$ does not satisfy A4, while it is a m… of $\mathscr{E}^n \cup \{A1, A2, A3, A5'\}$. Indeed, it suffices to verify A5′. If $\{abcd\} \not\subseteq K$ then … is obviously satisfied. Let $a, b, c, d \in K$. Then $a = e_1(x)$, $b = e_1(y)$, $c = e_1…$ $d = e_1(v)$ for some $x, y, z, v \in R$. Assume

$$B^{Kf}(abd) \wedge B^{Kf}(bcd) \wedge bc \underset{\mathfrak{R}}{\equiv} ad;$$

then $B_{\mathfrak{R}R^+}(f(a)f(b)f(d)) \wedge B_{\mathfrak{R}R^+}(f(b)f(c)f(d)) \wedge ||b-c|| = ||a-d||$, thus

$$[f_0(x) \leqslant f_0(y) \leqslant f_0(z) \leqslant f_0(v) \vee f_0(x) \geqslant f_0(y) \geqslant f_0(z) \geqslant f_0(v)] \wedge |y-z| = |x-v|.$$

In turn

$$|y-z| = |x-v| \Rightarrow x+z = y+v \vee x+y = z+v$$
$$\Rightarrow f_0(x)+f_0(z) = f_0(y)+f_0(v) \vee f_0(x)+f_0(y) = f_0(z)+f_0(v),$$

whence $f_0(z) = f_0(v)$ and therefore $c = d$.

Thus $A4 \notin \mathrm{Cn}(A1, A2, A3, A5')$ and so A4 cannot be replaced by A5′.

It is not difficult to check that A4 becomes dependent in the presence of WP. In conclusion

$$\mathcal{O}\mathscr{E}^n = \mathrm{Cn}(\mathscr{E}^n \cup \{A1, A2, A3, WP\}),$$

and so the weak Pasch axiom is the only plane axiom of ordered Euclidean geometry, concerning the betweenness relation.

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Pointed and unpointed shape and pro-homotopy

by

Jerzy Dydak (Warszawa)

Abstract. In the paper we consider whether every unpointed shape morphism can be realized as a pointed shape morphism and whether every pointed shape morphism being an unpointed shape equivalence is also a pointed shape equivalence.

1. Introduction. The main pointed shape invariants, i.e., pointed 1-movability, pointed movability, being pointed FANR are at the same time invariants of the unpointed shape theory (see [10] and [12]). However it is not known whether they are hereditary shape invariants. On the way to attack this problem arise the following questions:

QUESTION 1. *Let (X, x) and (Y, y) be pointed continua and let $f: X \to Y$ be a shape morphism. Does there exist a morphism $g: (X, x) \to (Y, y)$ such that the induced morphism $g': X \to Y$ is equal to f?*

QUESTION 2. *Let (X, x) and (Y, y) be pointed continua and let $f: (X, x) \to (Y, y)$ be a shape morphism such that the induced morphism $f': X \to Y$ is an isomorphism. Is f an isomorphism?*

The analogous questions may be considered in pro-homotopy.

In this paper we consider the above questions. We show that in general the answers to Question 1 and Question 2 (in pro-homotopy) are negative. However they can be positively answered in some special cases.

Specially interesting is Question 2 because the negative answer to it would give a weak proper homotopy equivalence not being a proper homotopy equivalence which existence has been asked by T. A. Chapman and L. C. Siebenmann [7].

2. Notations and terminology. By $\boldsymbol{H}(\boldsymbol{H}_0)$ we denoted the homotopy category of (pointed) connected CW complexes.

For any category $\boldsymbol{C}$ we denote by pro-$\boldsymbol{C}$ its pro-category (see [1] and [19]) and by tow$(\boldsymbol{C})$ we denote a full subcategory of pro-$\boldsymbol{C}$ whose objects are towers i.e. inverse sequences in $\boldsymbol{C}$. (see [11]).

By $\boldsymbol{F}$: pro-$\boldsymbol{H}_0 \to$ pro-$\boldsymbol{H}$ we denote the forgetful functor obtained by suppressing base points.

Each covariant functor $G: C \to D$ induces in a natural way a functor pro-G: pro-$C \to$ pro-D.

Any pro-group consisting of Abelian (free) groups will be called an *Abelian* (*free*) *pro-group*.

A pro-group (G_a, p_a^b, A) is said to be *normal* provided for any $a \in A$ there exists $b \geqslant a$ such that for each $c \geqslant b$ the group $\operatorname{im} p_a^c$ is a normal subgroup of $\operatorname{im} p_a^b$.

T. Watanabe [26] has originally introduced normal inverse sequences as towers (G_n, p_n^{n+1}) of groups such that for any n there exists $k \geqslant n$ such that $\operatorname{im} p_n^m$ is a normal subgroup of G_n for each $m \geqslant k$.

LEMMA 2.1. *Any normal tower (G_n, p_n^{n+1}) of groups is isomorphic to a tower (H_n, q_n^{n+1}) such that $\operatorname{im} q_n^k$ is a normal subgroup of H_n for each $k \geqslant n$.*

Proof. We may assume that $\operatorname{im} p_n^k$ is a normal subgroup of $\operatorname{im} p_n^{n+1}$ for $k \geqslant n+1$. Take $H_n = \operatorname{im} p_n^{n+1}$ and let $q_n^{n+1}: H_{n+1} \to H_n$ be defined by p_n^{n+1}. It is easily seen that (H_n, q_n^{n+1}) satisfies the required conditions.

The idea of considering normal towers of groups comes out from the work of B. I. Gray [13].

The *first derived functor* $\underleftarrow{\lim}^1$: tow(**Gr**) $\to$ **Ens**$_0$ from towers of groups to the category of pointed sets is defined as follows (see [5], p. 251):

Let $G = (G_n, p_n^{n+1})$. If $x = (x_i)$, $g = (g_i) \in \prod_{i=1}^{\infty} G_i$, then let $y = g \cdot x$ be defined by

$$y_i = g_i x_i \cdot p_i^{i+1}(g_{i+1}^{-1}).$$

$\underleftarrow{\lim}^1 G$ is the set of equivalence classes of $\prod_{i=1}^{\infty} G_i$ under the equivalence relation "$\sim$" given by

$$x \sim y \quad \text{iff} \quad y = g \cdot x \quad \text{for some } g.$$

Then $\underleftarrow{\lim}^1 G$ is a pointed one-point set (written $\underleftarrow{\lim}^1 G = *$) iff for each $y = (y_i)$ there exists $g = (g_i)$ such that for each i

$$y_i = g_i \cdot p_i^{i+1}(g_{i+1}^{-1}).$$

By $K(\cdot, 1)$: Gr $\to H_0$ we denote the *Eilenberg–MacLane functor*, i.e., $K(G, 1)$ is a CW complex of type $(G, 1)$ (see [24], pp. 427–428).

If $f, g: X \to Y$ are two maps and α is a path joining $f(x_0)$ and $g(x_0)$, then f is *α-homotopic* to g provided there is a homotopy $H: X \times I \to Y$ joining f and g such that $\alpha(t) = H(x_0, t)$ for each $t \in I$ (see [24], p. 379).

Two maps $f, g: (X, x_0) \to (Y, y_0)$ are said to be *freely homotopic* provided $f, g: X \to Y$ are homotopic.

The tower of groups (G_n, p_n^{n+1}) satisfies the *Mittag-Leffler condition* provided for each n there exists $k \geqslant n$ such that $\operatorname{im} p_n^k = \operatorname{im} p_n^m$ for each $m \geqslant k$ (see [2]). It is well-known that a tower of groups G satisfying M-L is isomorphic to a tower whose bonding maps are onto and $\underleftarrow{\lim}^1 G = *$ (see [5], p. 252–253).

3. Morphisms of pro-H_0 inducing isomorphisms of pro-H. This section is devoted to Question 2 in pro-homotopy. First of all we construct a group A and homomorphisms $h, g: A \to A$ which yield an example of a morphism f of pro-H_0 such that $F(f)$ is an isomorphism of pro-H and f is not an isomorphism of pro-H_0.

Let A be the group generated by an infinite countable set $(x_i)_{i=1}^{\infty}$ of generators with relations

$$x_{2+k}^{-1} x_{i+k} x_{2+k} = x_{1+k}^{-1} x_{i+k} x_{1+k}$$

for all $k \geqslant 0$ and $i \geqslant 3$, i.e.,

$$A = \{x_1, \ldots, x_n, \ldots : x_i^{-1} x_m x_i = x_{m-1}^{-1} x_m x_{m-1} \text{ for } i < m\}.$$

Let

$$A_m = \{x_m, \ldots : x_i^{-1} x_p x_i = x_{p-1}^{-1} x_p x_{p-1} \text{ for } m \leqslant i < p\}.$$

Let B_m be the subgroup of A_m generated by all x_i, $i \geqslant m+1$. Then $\varphi_m: B_m \to A_m$ defined by $\varphi_m(x) = x_m^{-1} x x_m$ is a monomorphism and A_m is obtained from A_{m+1} by adjoining a new generator x_m and relations

$$x_m^{-1} x x_m = \varphi_{m+1}(x) \quad \text{for} \quad x \in B_{m+1}.$$

Thus A_m is an HNN extension of A_{m+1} (see [14] and [23]). In particular we may consider A_{m+1} to be a subgroup B_m of A_m (see [14]).

Let $g, h: A \to A$ be homomorphisms defined by

$$g(x_i) = x_{i+1} \quad \text{and} \quad h(x_i) = x_1^{-1} x_{i+1} x_1 \quad \text{for each } i \geqslant 1.$$

Then

$$gh(x_i) = g(x_1^{-1} x_{i+1} x_1) = x_2^{-1} x_{i+2} x_2$$

and

$$hg(x_i) = h(x_{i+1}) = x_1^{-1} x_{i+2} x_1$$

for each $i \geqslant 1$. Thus $gh = hg$ and h and g are conjugate.

Let us show that $x_k \notin \operatorname{im} h$ for each $k \geqslant 3$.

Suppose on the contrary that

$$x_k = x_1^{-1} x x_1 \quad \text{where} \quad x \in A_2.$$

By using the abelization of $A_1 = A$ we get $x \in A_2 - A_{k+1}$. Take a number n such that $x \in A_n - A_{n+1}$.

Then $2 \leqslant n \leqslant k$ and consequently x can be expressed in terms of x_l, $l \geqslant n$. Therefore

$$x_1^{-1} x x_1 = x_{n-1}^{-1} x x_{n-1} \quad \text{and} \quad x_k^{-1} x_{n-1}^{-1} x x_{n-1} = 1$$

in A_{n-1}. But $x \notin A_{n+1} = B_n$ and by Britton's Lemma (see [6] and [23])

$$x_k^{-1} x_{n-1}^{-1} x x_{n-1} \neq 1.$$

This contradiction shows that $x_k \notin \operatorname{im} h$ for each $k \geqslant 3$.

Let $G_n = A$, $p_n^{n+1} = g$ and $h_n = h$ for each n. Then the diagram

$$\begin{array}{ccc} G_{n+1} & \xrightarrow{h_{n+1}} & G_{n+1} \\ \downarrow{\scriptstyle p_n^{n+1}} & & \downarrow{\scriptstyle p_n^{n+1}} \\ G_n & \xrightarrow[h_n]{} & G_n \end{array}$$

is commutative and consequently h_n generate a morphism $h\colon (G_n, p_n^{n+1}) \to (G_n, p_n^{n+1})$ of pro-**Gr**.

Since $\operatorname{im} g^k$ is not contained in $\operatorname{im} h$ for each k (because $x_{k+1} \in \operatorname{im} g^k$), then h is not an epimorphism of pro-**Gr** (see [20] and [22] for a description of epimorphisms of pro-**Gr**).

By applying the Eilenberg–MacLane functor we get a morphism

$$f\colon (K(G_n, 1), K(p_n^{n+1}, 1)) \to (K(G_n, 1), K(p_n^{n+1}, 1))$$

of pro-$\boldsymbol{H}_0$ not being an isomorphism. However the diagram

$$\begin{array}{ccc} K(A,1) & \xrightarrow{K(h,1)} & K(A,1) \\ \downarrow{\scriptstyle K(g,1)} & \swarrow{\scriptstyle \mathrm{id}_{K(A,1)}} & \downarrow{\scriptstyle K(g,1)} \\ K(A,1) & \xrightarrow[K(h,1)]{} & K(A,1) \end{array}$$

is commutative in $\boldsymbol{H}$ (because h and g are conjugate). Consequently $\boldsymbol{F}(f)$ is an isomorphism of pro-$\boldsymbol{H}$ (see [22]). Moreover $\boldsymbol{F}(f)$ is the identity morphism.

Thus, in general, the answer to Question 2 in pro-homotopy is negative.

Now we pass to consider additional assumptions under which the answer to Question 2 is positive.

THEOREM 3.1. *Let $f\colon X \to Y$ be a morphism of* pro-$\boldsymbol{H}_0$ *such that $\boldsymbol{F}(f)$ is an isomorphism. Then f is an isomorphism of* pro-$\boldsymbol{H}_0$ *iff* pro-$\pi_1 f$ *is an epimorphism.*

Proof. Necessity is obvious. So we prove sufficiency. It is enough to consider the case where f is a special morphism (see [19]), i.e.,

$$X = ((X_a, x_a), [p_a^b], A), \qquad Y = ((Y_a, y_a), [q_a^b], A)$$

and f is generated by a family of maps

$$f_a\colon (X_a, x_a) \to (Y_a, y_a)$$

with

$$f_a p_a^b \simeq q_a^b f_a \text{ rel.} x_b \quad \text{for} \quad b \geqslant a.$$

Since $\boldsymbol{F}(f)$ is an isomorphism of pro-$\boldsymbol{H}$, then for each $a \in A$ there exist $b \geqslant a$ and a map $g\colon (Y_b, y_b) \to (X_a, x_a)$ such that gf_b and p_a^b are freely homotopic. So let α be a loop at x_a such that p_a^b is α-homotopic to gf_b.

Take a map $h\colon (Y_b, y_b) \to (X_a, x_a)$ being α-homotopic to g (see [24], pp. 379–380). Then hf_b is α-homotopic to gf_b and consequently

$$p_a^b \simeq hf_b \text{ rel. } x_b.$$

Thus we get

3.2. *For each $a \in A$ there exist $b \geqslant a$ and a map*

$$h\colon (Y_b, y_b) \to (X_a, x_a) \quad \text{with} \quad p_a^b \simeq hf_b \text{ rel. } x_b.$$

Observe that we have not used that pro-$\pi_1 f$ is an epimorphism of pro-**Gr**. Now, for each $a \in A$, there exist $c \geqslant b \geqslant a$ and a map

$$g\colon (Y_c, y_c) \to (X_b, x_b)$$

such that

$$\operatorname{im} \pi_1 q_a^b \subset \operatorname{im} \pi_1 f_a \quad \text{and} \quad f_b g \simeq q_b^c \quad \text{freely}.$$

So let α be a loop at x_b such that

$f_b g$ is α-homotopic to q_b^c.

Take a loop β at x_a such that

$$f_a \cdot \beta \simeq q_a^b \cdot \alpha.$$

Let $h\colon (Y_c, y_c) \to (X_a, x_a)$ be a map such that

$p_a^b g$ is β-homotopic to h.

Then

$f_a p_a^b g$ is $(f_a \cdot \beta)$-homotopic to $f_a h$,
$q_a^b f_b g$ is $(q_a^b \cdot \alpha)$-homotopic to q_a^c and
$f_a p_a^b g \simeq q_a^b f_b g$ rel. y_c.

Hence

$$f_a h \simeq q_a^c \text{ rel. } y_c.$$

Thus we get

3.3. *For each $b \in A$ there exist $c \geqslant b$ and a map $g\colon (Y_c, y_c) \to (X_b, x_b)$ with $f_b g \simeq q_b^c$ rel. y_c.*

Take data as in 3.2 and 3.3. Let

$$r = hf_b g\colon (Y_c, y_c) \to (X_a, x_a).$$

Then

$$f_a r = f_a(hf_b)g \simeq f_a p_a^b g \simeq q_a^b f_b g \simeq q_a^b q_b^c \simeq q_a^c$$

and

$$rf_c = h(f_b g)f_c \simeq hq_b^c f_c \simeq hf_b p_b^c \simeq p_a^b p_b^c \simeq p_a^c$$

all homotopies preserving base points.

By Morita's characterization of an isomorphisms of pro-categories (see [22], Theorem 1.1) f is an isomorphism of pro-$\boldsymbol{H}_0$ which completes the proof.

In order to apply Theorem 3.1 we need the following

LEMMA 3.4. *Let $r\colon (G_a, p_a^b, A) \to (H_a, q_a^b, A) = H$ be a special morphism of* pro-**Gr** *generated by homomorphisms $r_a\colon G_a \to H_a$. Suppose that for each $a \in A$ there exist $b \geqslant a$ and a homomorphism $s\colon H_b \to G_a$ such that $(r_a s, q_a^b)$ and (sr_b, p_a^b) are pairs of conjugate homomorphisms. If either*

1. *H is a normal pro-group or*
2. *H is a free pro-group,*

then r is an isomorphisms of pro-**Gr**.

Proof. 1. Suppose H is a normal pro-group and $a \in A$. Take $c \geqslant b \geqslant a$ and a homomorphism $s\colon H_c \to G_b$ such that

$(r_b s, q_b^c)$ and (sr_c, p_b^c) are pairs of conjugate homomorphisms and $\operatorname{im} q_a^c$ is a normal subgroup of $\operatorname{im} q_a^b$.

So let $y \in H_b$ and $z \in G_b$ be elements such that

$$r_b s(x) = y^{-1} q_b^c(x) y \quad \text{for each } x \in H_c$$

and

$$sr_c(x) = z^{-1} p_b^c(x) z \quad \text{for each } x \in G_c.$$

Hence $\ker r_c \subset \ker p_b^c$ and r is a monomorphism of pro-**Gr** (see [20] and [22]).

Now

$$r_a p_a^b s(x) = q_a^b r_b s(x) = q_a^b(y)^{-1} q_a^c(x) q_a^b(y) \quad \text{for each } x \in H_c$$

and since $\operatorname{im} q_a^c$ is a normal subgroup of $\operatorname{im} q_a^b$, then

$$\operatorname{im} q_a^c \subset \operatorname{im} r_a.$$

Therefore r is an epimorphism of pro-**Gr** and this implies that r is an isomorphism of pro-**Gr** (see [20] and [22]).

2. Suppose H is a free pro-group. If for any $a \in A$ there exists $b \geqslant a$ such that $\operatorname{im} q_a^b$ is Abelian, then H is a normal pro-group and the result follows from the first part of the proof. So assume there exists $a_0 \in A$ such that $\operatorname{im} q_a^b$ is not Abelian for any $b \geqslant a \geqslant a_0$.

Take $a \geqslant a_0$. Then there exist $b \geqslant a$, homomorphism $s\colon H_b \to G_a$ and elements $y \in G_a$, $z \in H_a$ such that

$$r_a s(x) = z^{-1} q_a^b(x) z \quad \text{for each } x \in H_b$$

and

$$sr_b(x) = y^{-1} p_a^b(x) y \quad \text{for each } x \in G_b.$$

Let $t\colon H_b \to G_a$ be defined by

$$t(x) = y^{-1} s(x) y \quad \text{for } x \in H_b.$$

Then

$$tr_b(x) = p_a^b(x) \quad \text{for each } x \in G_b$$

and

$$r_a t(x) = u^{-1} q_a^b(x) u \quad \text{for each } x \in H_b,$$

where $u = z \cdot r_a(y^{-1})$.

We are going to show that $u = 1$.

Suppose, on the contrary, that $u \neq 1$. Then

$$u^{-1} q_a^b r_b(x) u = r_a t r_b(x) = r_a p_a^b(x) = q_a^b r_b(x) \quad \text{for } x \in G_b.$$

Take two elements $v, w \in \operatorname{im}(q_a^b r_b)$. Then

$$u^{-1} v u = v \quad \text{and} \quad u^{-1} w u = w.$$

Consequently $\{u, v\} = \{v_1\}$ and $\{u, w\} = \{v_2\}$, i.e., the subgroup of H_a generated by u and v is an infinite cyclic group generated by v_1, and the subgroup of H_a generated by u and w is an infinite cyclic group generated by v_2 (see [18], p. 95). Since u is a power of both v_1 and v_2, then the group $\{v_1, v_2\}$ is also an infinite cyclic group. Hence $wv = vw$ and $\operatorname{im}(q_a^b r_b)$ is Abelian.

Take $c \geqslant b$ and a homomorphism $s'\colon H_c \to G_b$ such that $r_b s'$ and q_b^c are conjugate. Then $q_a^b r_b s'$ and q_a^c are conjugate and consequently $\operatorname{im} q_a^c$ is isomorphic to a subgroup of $\operatorname{im}(q_a^c r_b)$. Thus $\operatorname{im} q_a^c$ is Abelian, a contradiction.

Consequently $u = 1$ and r is an isomorphism of pro-**Gr**.

THEOREM 3.5. *Let $f\colon X \to Y$ be a morphism of* pro-$\boldsymbol{H}_0$ *such that $\boldsymbol{F}(f)$ is an isomorphism of* pro-$\boldsymbol{H}$. *If* pro-$\pi_1 Y$ *is isomorphic to a normal (free) progroup, then f is an isomorphism of* pro-$\boldsymbol{H}_0$.

Proof. Let $g\colon$ pro-$\pi_1 Y \to G$ be an isomorphism of pro-**Gr**, where G is a normal (free) pro-group. Let

$$h = g \cdot (\text{pro-}\pi_1 f).$$

By 2.2 of [19] there exist isomorphisms

$$i\colon G_1 \to \text{pro-}\pi_1 X \quad \text{and} \quad j\colon G_1 \to G_2$$

of pro-**Gr** such that $r = jhi$ is a special morphism and G_2 is a normal (free) pro-group. Since $\boldsymbol{F}(f)$ is an isomorphism of pro-$\boldsymbol{H}$, then it is easily seen that r satisfies the condition of Lemma 3.4. Consequently r is an isomorphism of pro-**Gr**. Hence pro-$\pi_1 f$ is an isomorphism of pro-**Gr** and by Theorem 3.1, f is an isomorphism of pro-$\boldsymbol{H}_0$.

As an immediate consequence of Theorem 3.5 we get the following

THEOREM 3.6. *Let $f\colon (X, x) \to (Y, y)$ be a pointed shape morphism inducing an unpointed shape equivalence. If Y is a pointed* 1-*movable continuum or a curve, then f is a pointed shape equivalence.*

4. Realizing morphisms of pro-H as coming from pro-H_0. K. Borsuk [4] has given an example of a curve X such that $\mathrm{Sh}(X, a) \neq \mathrm{Sh}(X, b)$ for some points $a, b \in X$. Consequently, by Theorem 3.6, there is no shape morphism $f\colon (X, a) \to (X, b)$ equal to 1_X when suppressing base points. Moreover the following holds

THEOREM 4.1. *Let X be a curve and $x \in X$. Then (X, x) possesses property*

(*) *for each shape morphism $f\colon Y \to X$ and for any $y \in Y$ there exists $g\colon (Y, y) \to (X, x)$ equal to f when suppressing base points*

iff X is pointed movable or has the shape of some solenoid.

Proof. Sufficiency. Observe that (X, x) possesses Property (*) iff (Z, z) does, where $\mathrm{Sh}(X, x) = \mathrm{Sh}(Z, z)$. If X has the shape of some solenoid S, then by the result from [15] and Theorem 3.6, $\mathrm{Sh}(X, x) = \mathrm{Sh}(S, a)$. It is shown by J. Keesling [15] that (S, a) possesses Property (*).

For X being pointed 1-movable the result follows from Theorem 4.2.

Necessity. Let $(X, x) = \underleftarrow{\lim}((X_n, x_n), p_n^{n+1})$, where X_n are 1-dimensional connected polyhedra. If for each n there exists $m \geqslant n$ such that $\mathrm{im}\,\pi_1(p_n^m)$ is Abelian group, then it is an infinite cyclic group. Consequently p_n^m can be factored through a circle and X has the shape of some solenoid.

So it suffices to consider the case where $\mathrm{im}\,\pi_1(p_n^m)$ is not Abelian for each $m \geqslant n$.

Take an arbitrary point $y = (y_n)_{n=1}^{\infty} \in X$. By Property (*) there exists a shape morphism

$$f\colon (X, y) \to (X, x)$$

equal to 1_X when suppressing base points. Hence there exist an increasing function $a\colon N \to N$ and maps

$$f_n\colon (X_{a(n)}, y_{a(n)}) \to (x_n, x_n)$$

freely homotopic to $p_n^{a(n)}$ such that

$$p_n^{n+1} f_{n+1} \simeq f_n p_{a(n)}^{a(n+1)} \text{ rel. } y_{a(n+1)}.$$

So let f_n be α_n-homotopic to $p_n^{a(n)}$ for some path α_n joining x_n and y_n. Then

$p_n^{n+1} f_{n+1}$ is $(p_n^{n+1}\alpha_{n+1})$-homotopic to $p_n^{a(n+1)}$

and

$f_n p_{a(n)}^{a(n+1)}$ is α_n-homotopic to $p_n^{a(n+1)}$.

Hence

$f_n p_{a(n)}^{a(n+1)}$ is β_n-homotopic to $p_n^{n+1} f_{n+1}$,

where $\beta_n = \alpha_n * p_n^{n+1}(\alpha_{n+1}^{-1})$. Hence

$$\pi_1(p_n^{n+1} f_{n+1})(c) = \pi_1(f_n p_{a(n)}^{a(n+1)})(c) = b_n^{-1} \pi_1(p_n^{n+1} f_{n+1})(c)\, b_n,$$

where $b_n = [\beta_n]$. Since $\mathrm{im}\,\pi_1(p_n^{n+1} f_{n+1})$ is supposed to be non-Abelian, then $b_n = 1$ (see the proof of Lemma 3.4). Hence $\alpha_n \simeq p_n^{n+1} \cdot \alpha_{n+1}$ rel. x_n, y_n. Thus x and y are joinable in the sense of J. Krasinkiewicz and P. Minc [17]. Hence X is pointed 1-movable (see [17]) and X is pointed movable (see [16] and [21]).

Thus the proof of Theorem 4.1 is concluded.

Thus, in general, the answer to Question 1 is negative. However Question 1 can be positively answered in the following special case.

THEOREM 4.2. *Let (X, x) be an object of* $\mathrm{tow}(H_0)$ *such that* $\underleftarrow{\lim}^1 \text{pro-}\pi_1(X, x) = *$. *Then for any object (Y, y) of* $\mathrm{tow}(H_0)$ *and for any morphism $f\colon Y \to X$ of* $\mathrm{tow}(H)$ *there exists a morphism $g\colon (Y, y) \to (X, x)$ of* $\mathrm{tow}(H_0)$ *with $F(g) = f$.*

Proof. We may assume

$$(X, x) = ((X_n, x_n), [p_n^{n+1}]), \quad (Y, y) = ((Y_n, y_n), [q_n^{n+1}])$$

and f is generated by maps

$$f_n\colon (Y_n, y_n) \to (X_n, x_n)$$

such that

$p_n^{n+1} f_{n+1}$ is α_n-homotopic to $f_n q_n^{n+1}$

for some loop α_n at x_n. Since $\underleftarrow{\lim}^1 \text{pro-}\pi_1(X, x) = *$, then there exist loops β_n at x with

$$\alpha_n^{-1} \simeq \beta_n * (p_n^{n+1}\beta_{n+1})^{-1} \text{ rel. } x_n.$$

Take $g_n\colon (Y_n, y_n) \to (X_n, x_n)$ such that

f_n is β_n-homotopic to g_n.

Then

$p_n^{n+1} g_{n+1}$ is $(p_n^{n+1}\beta_{n+1})^{-1}$-homotopic to $p_n^{n+1} f_{n+1}$,

$p_n^{n+1} f_{n+1}$ is α_n-homotopic to $f_n q_n^{n+1}$ and

$g_n q_n^{n+1}$ is β_n^{-1}-homotopic to $f_n q_n^{n+1}$.

Hence

$p_n^{n+1} g_{n+1}$ is $((p_n^{n+1}\beta_{n+1})^{-1} * \alpha_n * \beta_n)$-homotopic to $g_n q_n^{n+1}$,

i.e.,

$$p_n^{n+1} g_{n+1} \simeq g_n q_n^{n+1} \text{ rel. } x_n.$$

It is easy to see that the morphism g generated by g_n satisfies the required conditions.

The author does not know the answer to the following special case of Question 2.

QUESTION 3. *Let $f\colon X \to Y$ be a morphism of* $\mathrm{tow}(H_0)$ *such that $F(f)$ is an isomorphism of* $\mathrm{tow}(H)$ *and* $\underleftarrow{\lim}^1 \text{pro-}\pi_1 Y = *$. *Is f an isomorphism of* $\mathrm{tow}(H_0)$?

However we can prove the following.

THEOREM 4.3. *Let* $f\colon X\to Y$ *be a morphism of* tow$(\boldsymbol{H}_0)$ *such that* $\boldsymbol{F}(f)$ *is an isomorphism of* tow$(\boldsymbol{H})$ *and* $\underleftarrow{\lim}^1\text{pro-}\pi_1 Y = *$. *Then there exists an isomorphism* $g\colon X\to Y$ *of* tow$(\boldsymbol{H}_0)$ *with* $\boldsymbol{F}(g)=\boldsymbol{F}(f)$.

Proof. Assume

$$X=\big((X_n,x_n),[p_n^{n+1}]\big),\qquad Y=\big((Y_n,y_n),[q_n^{n+1}]\big)$$

and f is generated by maps

$$f_n\colon (X_n,x_n)\to(Y_n,y_n)\quad\text{with}\quad f_np_n^{n+1}\simeq q_n^{n+1}f_{n+1}\ \text{rel.}\ x_{n+1}$$

for each n. Since $\boldsymbol{F}(f)$ is an isomorphism of pro-$\boldsymbol{H}$, then we may assume that there exist maps

$$h_n\colon (Y_{n+1},y_{n+1})\to(X_n,x_n)$$

and loops α_n at y_n such that

f_nh_n is α_n-homotopic to q_n^{n+1}

and

$$p_n^{n+1}\simeq h_nf_{n+1}\ \text{rel.}\ x_{n+1}\qquad(\text{see } 3.2)\,.$$

Take loops β_n at y_n such that

$$\alpha_n\simeq\beta_n*(q_n^{n+1}\beta_{n+1})^{-1}\ \text{rel.}\ y_n\,.$$

Let $t_n\colon (Y_n,y_n)\to(Y_n,y_n)$ and $u_n\colon (Y_{n+1},y_{n+1})\to(Y_n,y_n)$ be maps such that

id_{Y_n} is β_n-homotopic to t_n

and

u_n is β_n-homotopic to q_n^{n+1}.

Let $g_n=t_nf_n$. Then

$q_n^{n+1}g_{n+1}=q_n^{n+1}t_{n+1}f_{n+1}$ is $(q_n^{n+1}\cdot\beta_{n+1})^{-1}$-homotopic to $q_n^{n+1}f_{n+1}$
$g_np_n^{n+1}\simeq g_nh_nf_{n+1}=t_nf_nh_nf_{n+1}$ is β_n^{-1}-homotopic to $f_nh_nf_{n+1}$,
$(f_nh_n)f_{n+1}$ is α_n-homotopic to $q_n^{n+1}f_{n+1}$.

Consequently

$$g_np_n^{n+1}\simeq q_n^{n+1}g_{n+1}\ \text{rel.}\ x_{n+1}\,.$$

Thus g_n generate a morphism $g\colon X\to Y$ of pro-$\boldsymbol{H}_0$ with $\boldsymbol{F}(g)=\boldsymbol{F}(f)$.

Let $r_n=h_nu_{n+1}\colon (Y_{n+2},y_{n+2})\to(X_n,x_n)$. Then

$f_nr_n=f_nh_nu_{n+1}$ is α_n-homotopic to $q_n^{n+1}u_{n+1}$,
$q_n^{n+1}u_{n+1}$ is $(q_n^{n+1}\beta_{n+1})$-homotopic to q_n^{n+2}.

Hence

$$g_nr_n=t_nf_nr_n\ \text{is}\ (\beta_n^{-1}*\alpha_n*(q_n^{n+1}\beta_{n+1}))\text{-homotopic to}\ q_n^{n+2}.$$

Consequently

$$g_nr_n\simeq q_n^{n+2}\ \text{rel.}\ y_{n+2}\,.$$

This implies that pro-$\pi_1 g$ is an epimorphism of pro-**Gr** and by Theorem 3.1, g is an isomorphism of pro-$\boldsymbol{H}_0$ which completes the proof.

5. Concluding remarks. The following result strengthens those obtained in [10].

THEOREM 5.1. *Let* $f\colon X\to Y$ *be a shape equivalence* (*shape domination*) *and* $x\in X$, $y\in Y$. *If* Y *is pointed* 1*-movable* (X *and* Y *are pointed* 1*-movable*), *then there exists a pointed shape equivalence* (*domination*)

$$g\colon (X,x)\to(Y,y)$$

equal to f *when suppressing base points.*

Proof. If f is a shape equivalence, then Theorem 5.1 follows from Theorems 3.6 and 4.2.

Suppose f is a shape domination. Take morphisms

$$h_1\colon (X,x)\to(Y,y)\quad\text{and}\quad h_2\colon (Y,y)\to(X,x)$$

such that $\boldsymbol{F}(h_1)=f$ and $\boldsymbol{F}(h_1h_2)=1_Y$ (see Theorem 4.2). By Theorem 3.6, h_1h_2 is an isomorphism. Let

$$g=(h_1h_2)^{-1}h_1\,.$$

Then $gh_2=1_{(Y,y)}$ and $\boldsymbol{F}(g)=\boldsymbol{F}(h_1)=f$ which completes the proof.

Theorems 4.2 and 4.3 imply that the vanishing of $\underleftarrow{\lim}^1(\text{pro-}\pi_1)$ is an unpointed shape invariant. We do not know if it is a hereditary shape invariant. We also do not know the answer to the following

QUESTION 4. *Suppose* $\mathrm{Sh}\,X\geqslant\mathrm{Sh}\,Y$ *and* $\underleftarrow{\lim}^1\text{pro-}\pi_1(X,x)=\underleftarrow{\lim}^1\text{pro-}\pi_1(Y,y)=*$. *Does* $\mathrm{Sh}(X,x)\geqslant\mathrm{Sh}(Y,y)$ *hold*?

Notice that the positive answer to Question 3 would imply the positive answer to Question 4 (see the proof of Theorem 5.1).

As an application of our results we get a group-theoretical one.

THEOREM 5.2. *Let* F *be a tower of finitely generated free groups. Then* $\underleftarrow{\lim}^1F=*$ *iff* F *satisfies the Mittag-Leffler condition.*

Proof. Take a pointed curve (X,x) with pro-$\pi_1(X,x)=F$. Then by Theorems 4.1 and 4.2, (X,x) is pointed movable or has the shape of some solenoid. If (X,x) is pointed movable, then F satisfies M-L.

Suppose (X,x) has the shape of some solenoid. Then F is isomorphic to a tower of countable Abelian groups. Hence by the result of B. I. Gray [13], F satisfies M-L which completes the proof.

Theorem 3.5 indicates that continua whose first pro-homotopy group is isomorphic to a normal pro-group are of some interest. This is confirmed by the following

THEOREM 5.3. *Let X be a 1-movable continuum. If* pro-$\pi_1(X, x)$ *is isomorphic to a normal pro-group for some $x \in X$, then X is pointed 1-movable.*

Proof. Using Lemma 2.5 of [8] we infer that pro-$\pi_1(X, x)$ is isomorphic to a normal tower of groups.

Let $(X, x) = \underleftarrow{\lim}((X_n, x_n), p_n^{n+1})$, where X_n are finite CW complexes. We may assume that there exist:

1. a tower $G = (G_n, q_n^{n+1})$ of groups such that $\operatorname{im} q_n^m$ is a normal subgroup of G_n for $m \geqslant n$,

2. homomorphisms

$$h_n\colon \pi_1(X_n, x_n) \to G_n \quad \text{and} \quad g_n\colon G_{n+1} \to \pi_1(X_n, x_n)$$

with $g_n h_{n+1} = \pi_1(p_n^{n+1})$ and $h_n g_n = q_n^{n+1}$,

3. homomorphisms $s_m^{n+1}\colon \pi_1(X_{n+1}, x_{n+1}) \to \pi_1(X_m, x_m)$ for $m \geqslant n+1$ such that $\pi_1(p_n^m)s_m^{n+1}$ and $\pi_1(p_n^{n+1})$ are conjugate.

Let $t_m^{n+1} = h_m s_m^{n+1} g_{n+1}\colon G_{n+2} \to G_m$ for $m \geqslant n+3$.
Then

$$q_n^m t_m^{n+1} = q_n^m h_m s_m^{n+1} g_{n+1} = h_n \pi_1(p_n^m) s_m^{n+1} g_{n+1}$$

is conjugate to

$$h_n \cdot \pi_1(p_n^{n+1}) g_{n+1} = q_n^{n+1} h_{n+1} g_{n+1} = q_n^{n+1} q_{n+1}^{n+2} = q_n^{n+2}.$$

Since $\operatorname{im} q_n^{n+2}$ is a normal subgroup of G_n, then

$$\operatorname{im} q_n^{n+2} \subset \operatorname{im} q_n^m.$$

Consequently G satisfies M-L. So does pro-$\pi_1(X, x)$ and X is pointed 1-movable.

In [9] the author has proved that if a shape morphism $f\colon (X, x) \to (Y, y)$ of pointed Hausdorff continua induces isomorphisms of all pro-homotopy groups and is an unpointed shape domination, then f is an unpointed shape equivalence provided Y is movable. In view of Theorem 3.1 this can be strengthen as follows

THEOREM 5.4. *Let $f\colon (X, x) \to (Y, y)$ be a shape morphism of pointed Hausdorff continua, where Y is movable. Iff: $X \to Y$ is a shape domination, then $f\colon (X, x) \to (Y, y)$ is a shape equivalence provided of induces isomorphisms of all homotopy pro-groups.*

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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