Concerning the order and the semi-order of n-dimensional Euclidean space *

by

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In the Proceedings of International Colloquium held in Potsdam in 1973, a paper of Wanda Szmielew appeared (see [1]). The author announced the results and promised the proofs to be published in another paper. Wanda Szmielew died a year ago. The present paper is a complement of [1], based on the author's notes.

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Abstract. The n-dimensional Euclidean geometry is understood as the elementary theory of the equidistance and betweenness relations in the n-dimensional Cartesian vector space over a formally real and Pythagorean field. The following two questions are answered: which properties of the semi-betweenness have to be postulated to obtain the semi-ordered Euclidean geometry, and then, what sentences have to be added to obtain the ordered Euclidean geometry.

1. We start by recalling the terminology and notation (see [1]).

Let $\mathbf{F}$ be the class of all formally real and Pythagorean commutative fields

$$\mathfrak{F} = (F, 0, 1, +, \cdot).$$

Given a field $\mathfrak{F} \in \mathbf{F}$, the set $P \subseteq F$ is a *semi-positive cone of* $\mathfrak{F}$ iff

(i) $$P \cup -P = F,$$

(ii) $$P \cap -P = \{0\},$$

(iii) $$P + P \subseteq P.$$

If moreover

(iv) $$P \cdot P \subseteq P,$$

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then P is a *positive cone of* $\mathfrak{F}$. Then

(v) $$1 \in P\,.$$

A *normal semi-positive cone* P besides (i)–(iii) satisfies the condition (v) as well.

Let

$$\mathbf{SOF} = \{(\mathfrak{F}P)\colon\ \mathfrak{F} \in \mathbf{F} \text{ and } P \text{ is a normal semi-positive cone of } \mathfrak{F}\}\,,$$
$$\mathbf{OF} = \{\mathfrak{F}P)\colon\ \mathfrak{F} \in \mathbf{F} \text{ and } P \text{ is a positive cone of } \mathfrak{F}\}\,.$$

We refer to the couple $(\mathfrak{F}P) \in \mathbf{SOF}$ as a *semi-ordered (formally real and Pythagorean) field* and to the $(\mathfrak{F}P) \in \mathbf{OF}$ as an *ordered (formally real and Pythagorean) field.*

Given a semi-ordered field $(\mathfrak{F}P) \in \mathbf{SOF}$, we define a *semi-norm* $||\ \ ||$ ($||\ \ ||_P$) in the n-dimensional Cartesian vector space $\mathfrak{V}^n_C(\mathfrak{F})$, $(n \geqslant 2)$:

$$||a|| \in P \quad \text{and} \quad ||a||^2 = a\cdot a\,.$$

In this way we get the *n-dimensional Cartesian semi-metric vector space* $\mathfrak{V}^n_C(\mathfrak{F}P)$.

In terms of $\mathfrak{V}^n_C(\mathfrak{F})$ we define in the usual way the collinearity and the equidistance relations, $L_{\mathfrak{F}}$ and $D_{\mathfrak{F}}$. In terms of $\mathfrak{V}^n_C(\mathfrak{F}P)$ we define the *semi-betweenness relation*:

$$B_{\mathfrak{F}P}(abc) \Leftrightarrow ||a-b||+||b-c|| = ||a-c||\,.$$

Thus any field $\mathfrak{F} \in \mathbf{F}$ generates the *n-dimensional Euclidean space over* $\mathfrak{F}$,

$$\mathfrak{E}^n(\mathfrak{F}) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}})\,,$$

and any semi-ordered field $(\mathfrak{F}P) \in \mathbf{SOF}$ generates the *n-dimensional semi-ordered Euclidean space over* $(\mathfrak{F}P)$,

$$\mathfrak{E}^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_{\mathfrak{F}P})\,.$$

If, in particular, $(\mathfrak{F}P) \in \mathbf{OF}$, then $\mathfrak{E}^n(\mathfrak{F}P)$ is the *n-dimensional ordered Euclidean space over* $(\mathfrak{F}P)$. Let

$$\mathbf{E}^n = \{\mathfrak{E}^n(\mathfrak{F})\colon \mathfrak{F} \in \mathbf{F}\}\,,$$
$$\mathbf{SOE}^n = \{\mathfrak{E}^n(\mathfrak{F}P)\colon\ (\mathfrak{F}P) \in \mathbf{SOF}\}\,,$$
$$\mathbf{OE}^n = \{\mathfrak{E}^n(\mathfrak{F}P)\colon\ (\mathfrak{F}P) \in \mathbf{OF}\}$$

and let $\mathcal{E}^n$, $\mathcal{SOE}^n$ and $\mathcal{OE}^n$ be elementary theories of the classes $\mathbf{E}^n$, $\mathbf{SOE}^n$ and $\mathbf{OE}^n$ respectively. We refer to the theory $\mathcal{E}^n$ as the (*n-dimensional*) *Euclidean geometry*, and to the theories $\mathcal{SOE}^n$ and $\mathcal{OE}^n$ as to (*n-dimensional*) respectively *semi-ordered* and *ordered Euclidean geometries.*

2. Consider the following four sentences ([1]):

A1. $B(abc) \Rightarrow B(cba)$,

A2. $B(abd) \wedge B(bcd) \Rightarrow B(abc)$,

([1]) Throughout the whole paper we omit the universal quantifiers which should be placed in front of a formula to bind all the free variables occurring in it.

A3. $L(abc) \Leftrightarrow B(abc) \vee B(bca) \vee B(cab)$,

A4. $abc \equiv ab'c' \wedge B(abc) \Rightarrow B(ab'c')$ ([2]),

and the weak Pasch axiom:

WP. $B(pbc) \wedge B(aqb) \Rightarrow \exists r(L(pqr) \wedge L(arc))$.

Let

$$\mathcal{X} = \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{A4}\} \quad \text{and} \quad \mathcal{Y} = \{\mathrm{WP}\}\,.$$

We are going to prove the following two theorems:

THEOREM 1. $\mathbf{SOE}^n$ *coincides (up to isomorphism) with the class of models* $\mathfrak{M}(\mathcal{E}^n \cup \mathcal{X})$.

THEOREM 2. $\mathbf{OE}^n$ *coincides (up to isomorphism) with the class* $\mathfrak{M}(\mathcal{SOE}^n \cup \mathcal{Y})$.

Let us notice first that the following seven sentences are derivable from $\mathcal{E}^n \cup \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}\}$ ([3]).

T1. $B(aaa)$ (from A3 and $\mathcal{E}^n$),

T2. $B(aab)$ (from A1, A3, and T1),

T3. $B(aba) \Rightarrow a = b$

suppose $B(aba) \wedge a \neq b$; by $\mathcal{E}^n$, $a \neq b \models\!\Rightarrow \exists c \sim L(abc)$; by T2, A2, A1, A3, $B(aba) \models\!\Rightarrow B(caa) \wedge B(aba) \models\!\Rightarrow B(cab) \models\!\Rightarrow L(abc))$,

T4. $B(abc) \wedge B(acb) \Rightarrow b = c$ (by A1, A2, and T3),

T5. $B(apb) \wedge B(arb) \Rightarrow B(apr) \vee B(arp)$ (by $\mathcal{E}^n$ and A3),

T6. $B(abp) \wedge B(abr) \wedge a \neq b \Rightarrow B(apr) \vee B(arp)$ (by $\mathcal{E}^n$, A3, and T4),

T7. $B(abc) \wedge B(bcd) \wedge b \neq c \Rightarrow B(abd)$ (by $\mathcal{E}^n$ and A1–A3).

In turn, using A1–A3 and T4–T7, we can easily prove wo lemmas concerning halflines of an arbitrary line K. The *halfline from a through b* ($a \neq b$), is defined as usually:

$$\mathbf{HL}(ab) = \{p\colon\ B(apb) \vee B(abp)\}\,.$$

L1. *Let* $a, b \in K$, $a \neq b$, $A = \mathbf{HL}(ab)$. *Then*

(1) $$p, r \in A \wedge B(pqr) \Rightarrow q \in A$$

and

(2) $$q \in A - \{a\} \wedge B(aqr) \Rightarrow r \in A\,.$$

L2. *Let* $a, b, c \in K$, $a \neq b \neq c$, $B(abc)$, $A = \mathbf{HL}(ba)$, $C = \mathbf{HL}(bc)$. *Then*

(1) $$A \cup C = K, \quad A \cap C = \{b\}\,,$$

(2) $$p, r \in K - \{b\} \wedge B(pbr) \Rightarrow [p \in A \Leftrightarrow r \in C],$$

([2]) We use the formula $pq \equiv p'q'$ instead of $D(pqp'q')$ and the abbreviation $pqr \equiv p'q'r'$ for the conjunction $pq \equiv p'q' \wedge pr \equiv p'r' \wedge qr \equiv q'r'$.

([3]) We use two different symbols: the implication symbol $\Rightarrow$ and the inference symbol $\models\!\Rightarrow$.

and

(3) $$p \in A \wedge r \in C \Rightarrow B(pbr).$$

Among the consequences of $\mathscr{E}^n \cup \{\text{A1–A4}\}$ (till now we have not used A4) let us distinguish the following two:

A5. $B(abc) \wedge ab \equiv ac \Rightarrow b = c$

and (stronger)

A5′. $B(abd) \wedge B(bcd) \wedge bc \equiv ad \Rightarrow c = d$

(comp. [1] and [2]). By $\mathscr{E}^n$, A1, A4, A1, A2, and T3

$$B(abc) \wedge ab \equiv ac \mathrel{|\!\!\Rightarrow} B(abc) \wedge abc \equiv acb \mathrel{|\!\!\Rightarrow} B(cba) \wedge B(acb)$$
$$\mathrel{|\!\!\Rightarrow} B(cbc) \Rightarrow b = c.$$

Thus $\text{A5} \in \text{Cn}(\mathscr{E}^n \cup \{\text{A1–A4}\})$.

It is easy to see that on the base of $\mathscr{E}^n \cup \{\text{A1–A4}\}$ A5 is equivalent to the statement

T8. $B(b\, b{\oplus}c\; c)$,

which says that the midpoint $b \oplus c$ of the segment bc lies between its ends.

Let us show that on the base of $\mathscr{E}^n \cup \{\text{A1, A2, A3, A5}\}$ A5′ is equivalent to the following linear case of A4:

T9. $B(abc) \Rightarrow B(a\,\sigma_a(b)\,\sigma_a(c))$,

where σ_a is the symmetry with respect to a. Assume first A5′. Let $B(abc)$ and suppose that $\sim B(a\,\sigma_a(b)\,\sigma_a(c))$. Then, by $\mathscr{E}^n \cup \{\text{A1–A3}\}$ and A5, $B(\sigma_a(b)\,\sigma_a(c)\,a)$, and thus, taking in A5′ $\sigma_a(b), \sigma_a(c), b, c$ for a, b, c, d, we get $b = c$, which implies $B(a\,\sigma_a(b)\,\sigma_a(c))$. Assume now T9 and let $B(abd) \wedge B(acd) \wedge bc \equiv ad$; since $bc \,\|\, ad$, we get $a{\oplus}b = c{\oplus}d \vee a{\oplus}c = b{\oplus}d$. If $a{\oplus}b = c{\oplus}d$, then $a = b = c = d$; if $a{\oplus}c = b{\oplus}d = p$, then $c = \sigma_p(a) \wedge d = \sigma_p(b) \wedge B(pba)$, whence, by T9, $B(pdc)$, which together with $B(pcd)$ implies $c = d$. Thus $(\text{T9} \Leftrightarrow \text{A5}') \in \text{Cn}(\mathscr{E}^n \cup \{\text{A1, A2, A3, A5}\})$.

As a result, A5 and A5′, as well as T8 and T9, are derivable from $\mathscr{E}^n \cup \{\text{A1–A4}\}$.

In turn, the outer invariance law T9 is equivalent to the following invariance law:

T10. $a, b, c, p \in K \wedge B(abc) \Rightarrow B(\sigma_p(a)\,\sigma_p(b)\,\sigma_p(c))$

(see [3], Th. 7.6.2), and thus implies the inner invariance law:

T11. $B(abc) \Rightarrow B(\sigma_b(a)\, b\sigma_b(c))$.

Let us prove

T12. $a, b, c, a', b', c' \in K \wedge abc \equiv a'b'c' \wedge B(abc) \Rightarrow B(a'b'c')$.

Indeed, let $p = b{\oplus}b'$; then $a'b'c' \equiv abc \equiv \sigma_p(a)b'\sigma_p(c)$, and thus (since either $\sigma_p(a) = \sigma_{b'}(a')$ and $\sigma_p(c) = \sigma_{b'}(c')$ or $\sigma_p(a) = a'$ and $\sigma_p(c) = c'$), applying T10 and T11, we get

$$B(abc) \mathrel{|\!\!\Rightarrow} B(\sigma_p(a)b'\sigma_p(c)) \mathrel{|\!\!\Rightarrow} B(\sigma_{b'}(a')b'\sigma_{b'}(c')) \mathrel{|\!\!\Rightarrow} B(a'b'c'). \blacksquare$$

The statement T12 may be generalized to

T13. $abc \equiv a'b'c' \wedge B(abc) \Rightarrow B(a'b'c')$.

In fact, let $a \neq a'$ and let K be the line through a and a'; there are $b_1, c_1, b_2, c_2 \in K$ such that $abc \equiv ab_1c_1 \wedge a'b'c' \equiv a'b_2c_2$. Applying in turn A4, T12, A4, we get

$$B(abc) \mathrel{|\!\!\Rightarrow} B(ab_1c_1) \mathrel{|\!\!\Rightarrow} B(a'b_2c_2) \mathrel{|\!\!\Rightarrow} B(a'b'c'). \blacksquare$$

By T13, T6, A2 and A5, we obtain

T14. $ab \equiv a'b' \wedge bc \equiv b'c' \wedge B(abc) \wedge B(a'b'c') \Rightarrow ac \equiv a'c'$.

Indeed, take abc and let $ab \equiv a'b'$. If $a = b$ then $a' = b'$. Let $a \neq b$; then $a' \neq b'$ and there is a point c'' such that $abc \equiv a'b'c''$. Assume $B(abc) \wedge bc \equiv b'c' \wedge B(a'b'c')$. By T13

$$B(abc) \mathrel{|\!\!\Rightarrow} B(a'b'c''),$$

by T6

$$B(a'b'c') \wedge B(a'b'c'') \wedge a' \neq b' \mathrel{|\!\!\Rightarrow} B(b'c'c'') \vee B(b'c''c').$$

By A5

$$(B(b'c'c'') \vee B(b'c''c')) \wedge b'c'' \equiv bc \equiv b'c' \mathrel{|\!\!\Rightarrow} c' = c''.$$

Thus $ac \equiv a'c'$. ■

The last two statements may be expressed together as

T15. $ab \equiv a'b' \wedge bc \equiv b'c' \wedge B(abc) \Rightarrow [B(a'b'c') \Leftrightarrow ac \equiv a'c']$.

3. Let us prove now

THEOREM 1. $\mathbf{SOE}^n \underset{\text{iso}}{=} \mathfrak{M}(\mathscr{E}^n \cup \mathscr{X})$.

Proof. $\subseteq$: Let

$$\mathfrak{E}^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_{\mathfrak{F}P}) \in \mathbf{SOE}^n.$$

Then $(\mathfrak{F}P) \in \mathbf{SOF}$ and thus the reduct $(F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}})$ is a model of $\mathscr{E}^n$. Hence it suffices to verify A1–A4. By the remarks in [1] pp. 72_8–73^8, it follows that the semi-betweenness relation $B_{\mathfrak{F}P}$ restricted to an arbitrary line K of the space $\mathfrak{E}^n(\mathfrak{F}P)$ coincides with the betweenness relation restricted to K. Thus $B_{\mathfrak{F}P}$ satisfies the axioms A1–A3. Since every congruence preserves the relation $B_{\mathfrak{F}P}$ (see [1], p. 73^{9-21}), the axiom A4 is satisfied as well.

$\supseteq$: Let $\mathfrak{S} = (S, L, D, B)$ be a model of $\mathscr{E}^n \cup \mathscr{X}$. We are going to find $(\mathfrak{F}P) \in \mathbf{SOF}$ such that $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$. Take a reduct $\mathfrak{S}_0 = (S, L, D)$ of $\mathfrak{S}$. Since $\mathbf{E}^n$ is an elementary class (see [1], p. 73_1), $\mathfrak{M}(\mathscr{E}^n) \underset{\text{iso}}{=} \mathbf{E}^n$ and thus, for some $\mathfrak{F} \in \mathbf{F}$,

$$\mathfrak{S}_0 \simeq \mathfrak{E}^n(\mathfrak{F}),$$

i.e. $\mathfrak{S}_0 = (S, L, D) \simeq (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}) = \mathfrak{E}^n(\mathfrak{F})$. Let

$$\Phi\colon \mathfrak{E}^n(\mathfrak{F}) \to \mathfrak{S}_0 \quad \text{and} \quad \Psi\colon \mathfrak{S}_0 \to \mathfrak{E}^n(\mathfrak{F})$$

be mutually inverse isomorphisms. It now suffices to extend Ψ over the whole $\mathfrak{S}$. Thus we construct a semi-positive cone P of $\mathfrak{F}$ such that $\Psi(B) = B_{\mathfrak{F}P}$.

For every $x \in F$, let $e(x) = (x, 0, \ldots, 0) \in F^n$. Then

$$e(x)+e(y) = e(x+y) \quad \text{and} \quad -e(x) = e(-x).$$

The set $K = \Phi e(F)$ is a line in S. Since

$$e(0) = e(x) \oplus_{\mathfrak{F}} e(-x),$$

we get

$$\Phi e(0) = \Phi e(x) \oplus \Phi e(-x)$$

(because $\oplus$ is definable in terms of $\mathfrak{S}_0$). Thus, by T8,

$$B(\Phi e(x)\Phi e(0)\Phi e(-x)) \quad \text{for every } x \in F;$$

in particular $B(\Phi e(1)\Phi e(0)\Phi e(-1))$.

Define two halflines

$$A = \mathbf{HL}(\Phi e(0)\Phi e(1)) \quad \text{and} \quad C = \mathbf{HL}(\Phi e(0)\Phi e(-1)).$$

By L2

$$A \cup C = K, \quad A \cap C = \{\Phi e(0)\},$$

and

$$\Phi e(-x) \in A \Leftrightarrow \Phi e(x) \in C.$$

Let $P = \{x\colon \Phi e(x) \in A\}$, i.e. $P = e^{-1}\Psi(A)$. Then

$$-P = e^{-1}\Psi(C),$$

and therefore

$$P \cup -P = F, \quad P \cap -P = \{0\},$$

i.e. $(\mathfrak{F}P)$ satisfies (i) and (ii). In turn, by T8,

$$B\left(\Phi e(x)\Phi e\left(\frac{x+y}{2}\right)\Phi e(y)\right) \quad \text{and} \quad B\left(\Phi e(0)\Phi e\left(\frac{x+y}{2}\right)\Phi e(x+y)\right),$$

whence

$$x, y \in P \models\!\Rightarrow \Phi e(x), \Phi e(y) \in A \underset{\text{L1}}{\models\!\Rightarrow} \Phi e\left(\frac{x+y}{2}\right) \in A$$

$$\underset{\text{L1}}{\models\!\Rightarrow} \Phi e(x+y) \in A \vee \Phi e\left(\frac{x+y}{2}\right) = \Phi e(0)$$

$$\models\!\Rightarrow x+y \in P \vee y = -x \underset{\text{(ii)}}{\models\!\Rightarrow} x+y \in P,$$

i.e. $(\mathfrak{F}P)$ satisfies (iii). Evidently $1 \in P$, i.e. (v) is satisfied as well. It remains to show that $\Psi(B) = B_{\mathfrak{F}P}$ (just now A4 is needed). Take $p, q, r \in F^n$ and let $a = \Phi(p)$, $b = \Phi(q)$, $c = \Phi(r)$. Let

$$(1) \qquad \|p-q\| = x \quad \text{and} \quad \|q-r\| = z;$$

then

$$(2) \qquad B_{\mathfrak{F}P}(pqr) \Leftrightarrow \|p-r\| = x+z.$$

Since

$$\|e(x)-e(0)\| = x, \quad \|e(0)-e(-z)\| = z,$$

and

$$\|e(x)-e(-z)\| = x+z,$$

by (1) and (2) we get

$$(1') \qquad e(x)e(0) \underset{\mathfrak{F}}{\equiv} pq \wedge e(0)e(-z) \underset{\mathfrak{F}}{\equiv} qr$$

and

$$(2') \qquad B_{\mathfrak{F}P}(pqr) \Leftrightarrow e(x)e(-z) \underset{\mathfrak{F}}{\equiv} pr.$$

The conditions (1′) and (2′) imply

$$(1'') \qquad \Phi e(x)\Phi e(0) \equiv ab \wedge \Phi e(0)\Phi e(-z) \equiv bc$$

and

$$(2'') \qquad B_{\mathfrak{F}P}(pqr) \Leftrightarrow \Phi e(x)\Phi e(-z) \equiv ac.$$

By (1), $x, z, x+z \in P$, whence

$$\Phi e(x), \Phi e(z) \in A, \quad \Phi e(-z) \in C$$

and therefore, by L2, $B(\Phi e(x)\Phi e(0)\Phi e(-z))$. Thus, applying T15, by (1″) and (2″) we get

$$B_{\mathfrak{F}P}(pqr) \Leftrightarrow B(abc),$$

which completes the proof. ∎

THEOREM 2. $\mathbf{OE}^n \underset{iso}{=} \mathfrak{M}(\mathcal{SOE}^n \cup \{\mathrm{WP}\})$.

Proof. $\subseteq$: If $(\mathfrak{F}P) \in \mathbf{OF}$ then $\mathfrak{E}^n(\mathfrak{F}P) \in \mathbf{OE}^n \subseteq \mathbf{SOE}^n$ and $\mathfrak{E}^n(\mathfrak{F}P)$ satisfies WP.

$\supseteq$: Let $\mathfrak{S}$ be a model of $\mathcal{SOE}^n \cup \{\mathrm{WP}\}$. We are going to find $(\mathfrak{F}P) \in \mathbf{OF}$ such that $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$. By Theorem 1, $\mathbf{SOE}^n$ is an elementary class, thus

$$\mathbf{SOE}^n \underset{iso}{=} \mathfrak{M}(\mathcal{SOE}^n).$$

Since $\mathfrak{S} \in \mathfrak{M}(\mathcal{SOE}^n \cup \{\mathrm{WP}\}) \subseteq \mathfrak{M}(\mathcal{SOE}^n)$, hence $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$ for some $(\mathfrak{F}P) \in \mathbf{SOF}$. So, it suffices to show that the semi-positive cone P is closed under the multiplication:

$$(\text{iv}) \qquad P \cdot P \subseteq P.$$

For any $x \in F$, let

$$e_1(x) = (x, 0, \ldots, 0) \in F^n, \quad e_2(x) = (0, x, 0, \ldots, 0) \in F^n.$$

Suppose that

$$x, y \in P \quad \text{and} \quad x \cdot y \notin P.$$

Then

$$B_{\mathfrak{F}P}(e_1(x)e_1(0)e_1(x\cdot y))$$

and

$$B_{\mathfrak{F}P}(e_2(0)e_2(1)e_2(y)) \vee B_{\mathfrak{F}P}(e_2(0)e_2(y)e_2(1))$$

(see Fig. 1). By WP, there is a point $p \in F^n$ such that

$$L_{\mathfrak{F}}(e_1(x\cdot y)pe_2(y)) \wedge L_{\mathfrak{F}}(e_1(x)e_2(1)p)\,.$$

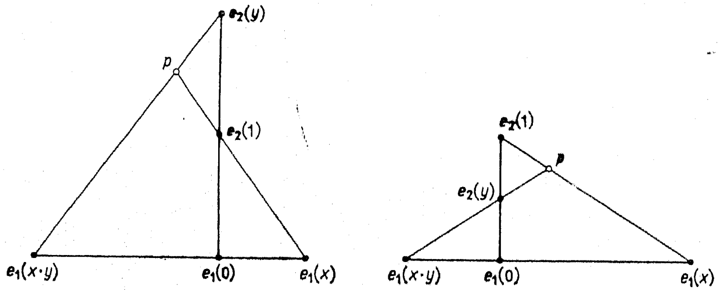

Fig. 1

Thus p is of the form

$$p = (z_1, z_2, 0, \ldots, 0)\,,$$

where

$$z_1 + x\cdot z_2 = x \quad \text{and} \quad (z_1 + x\cdot z_2)\cdot y = x\cdot y^2\,.$$

Hence $x\cdot y = x\cdot y^2$, i.e. $x\cdot y\cdot(1-y) = 0$. Since $x\cdot y \neq 0$, we get $y = 1$ and thus $x\cdot y = x \in P$ in contrary to the assumption. ■

As corollaries of Theorems 1 and 2, we get

THEOREM 1′. $\mathcal{SOE}^n = \mathrm{Cn}(\mathcal{E}^n \cup \mathcal{X})$.

THEOREM 2′. $\mathcal{OE}^n = \mathrm{Cn}(\mathcal{SOE}^n \cup \mathcal{Y})$.

4. We pass to independence models. As was stated in [1] p. 76, each of the axioms A1–A4 is independent of the remaining three together with the whole theory $\mathcal{E}^n$. If, moreover, A3 is replaced by the conjunction of

A3.1. $B(abc) \Rightarrow L(abc)$

and

A3.2. $L(abc) \Rightarrow B(abc) \vee B(bca) \vee B(cab)$,

then each of the five sentences A1, A2, A3.1, A3.2, A4 is independent of the remaining four together with the whole $\mathcal{E}^n$. The independence models are of the form

$$\mathfrak{E}_i^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_i) \quad \text{for} \quad i = 1, 2, 4$$

and

$$\mathfrak{E}_{3j}^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_{3j}) \quad \text{for} \quad j = 1, 2$$

for some $(\mathfrak{F}P) \in$ **SOF**. Thus, we have to define relations B_1, B_2, B_4, B_{31}, and B_{32}, such that B_i (B_{3j}) satisfies all the axioms except Ai (A3.j). Let

$$B_1 = B_{\mathfrak{F}P} - \{(aab)\colon a \neq b\}, \qquad B_{31} = (F^n)^3\,,$$
$$B_2 = L_{\mathfrak{F}}\,, \qquad B_{32} = \varnothing\,.$$

It remains to define B_4. Consider first an arbitrary line K in the space $\mathfrak{E}^n(\mathfrak{F}) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}})$ and let $f\colon K \to K$ be a bijection. Let

$$B^{Kf}(abc) \Leftrightarrow \begin{cases} B_{\mathfrak{F}P}(abc) & \text{if} \quad \{abc\} \not\subset K\,, \\ B_{\mathfrak{F}P}(f(a)f(b)f(c)) & \text{if} \quad \{abc\} \subset K\,. \end{cases}$$

It is easy to prove the following

LEMMA. *$(F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B^{Kf})$ is a model of $\mathcal{E}^n \cup \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}\}$.*

Let us choose now a bijection f as follows. Fix three points $a, b, c \in F^n$ such that

$$\neq (abc) \quad \text{and} \quad B_{\mathfrak{F}P}(abc)\,,$$

and let f satisfy the conditions:

$$f(a) = a\,, \quad f(b) = c\,, \quad f(c) = b\,.$$

Then $B^{Kf}(abc)$ and $\sim B^{Kf}(ab'c')$ if $b', c' \notin K$. But there exist $b', c' \notin K$ such that $abc \underset{\mathfrak{F}}{\equiv} ab'c'$, whence B^{Kf} does not satisfy A4. Thus, by Lemma, setting

$$B_4 = B^{Kf}\,,$$

we get a desired relation.

Finally, let us turn to the axiom A5. It was stated in [1] p. 76 that A.4 is independent of $\mathcal{E}^n \cup \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{A5}\}$. We shall prove even more (comp. [2]): A4 is independent of $\mathcal{E}^n \cup \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{A5}'\}$. To obtain a suitable model, we choose particular $(\mathfrak{F}P)$ and f in $\mathfrak{E}_4^n(\mathfrak{F}P)$. Let $\mathfrak{F} = R$, $P = R^+$; let $e_1(x) = (x, 0, \ldots, 0) \in R^n$ and $K = e_1(R)$. There is a bijection

$$f_0\colon R \to R$$

satisfying the conditions:

$$f_0(0) = 0\,, \quad f_0(\sqrt{2}) = \sqrt{3}\,, \quad f_0(\sqrt{3}) = \sqrt{2}$$

and

$$f_0(x+y) = f_0(x) + f_0(y)\,.$$

Let $f\colon K \to K$ be defined by the formula

$$f(e_1(x)) = e_1(f_0(x))\,,$$

and let

$$B_5 = B^{Kf}\,.$$

The structure $\mathfrak{E}_5^n(\mathfrak{R}R^+) = (R^n, L_{\mathfrak{R}}, D_{\mathfrak{R}}, B_5)$ does not satisfy A4, while it is a m of $\mathscr{E}^n \cup \{A1, A2, A3, A5'\}$. Indeed, it suffices to verify A5′. If $\{abcd\} \not\subset K$ then is obviously satisfied. Let $a, b, c, d \in K$. Then $a = e_1(x)$, $b = e_1(y)$, $c = e_1$ $d = e_1(v)$ for some $x, y, z, v \in R$. Assume

$$B^{Kf}(abd) \wedge B^{Kf}(bcd) \wedge bc \underset{\mathfrak{R}}{\equiv} ad;$$

then $B_{\mathfrak{R}R^+}(f(a)f(b)f(d)) \wedge B_{\mathfrak{R}R^+}(f(b)f(c)f(d)) \wedge ||b-c|| = ||a-d||$, thus

$$[f_0(x) \leqslant f_0(y) \leqslant f_0(z) \leqslant f_0(v) \vee f_0(x) \geqslant f_0(y) \geqslant f_0(z) \geqslant f_0(v)] \wedge |y-z| = |x-v|.$$

In turn

$$|y-z| = |x-v| \models x+z = y+v \vee x+y = z+v$$
$$\models f_0(x)+f_0(z) = f_0(y)+f_0(v) \vee f_0(x)+f_0(y) = f_0(z)+f_0(v),$$

whence $f_0(z) = f_0(v)$ and therefore $c = d$.

Thus $A4 \notin \mathrm{Cn}(A1, A2, A3, A5')$ and so A4 cannot be replaced by A5′.

It is not difficult to check that A4 becomes dependent in the presence of WP. In conclusion

$$\mathcal{O}\mathscr{E}^n = \mathrm{Cn}(\mathscr{E}^n \cup \{A1, A2, A3, WP\}),$$

and so the weak Pasch axiom is the only plane axiom of ordered Euclidean geometry, concerning the betweenness relation.

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Pointed and unpointed shape and pro-homotopy

by

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Abstract. In the paper we consider whether every unpointed shape morphism can be realized as a pointed shape morphism and whether every pointed shape morphism being an unpointed shape equivalence is also a pointed shape equivalence.

1. Introduction. The main pointed shape invariants, i.e., pointed 1-movability, pointed movability, being pointed FANR are at the same time invariants of the unpointed shape theory (see [10] and [12]). However it is not known whether they are hereditary shape invariants. On the way to attack this problem arise the following questions:

QUESTION 1. *Let (X, x) and (Y, y) be pointed continua and let $f\colon X \to Y$ be a shape morphism. Does there exist a morphism $g\colon (X, x) \to (Y, y)$ such that the induced morphism $g'\colon X \to Y$ is equal to f?*

QUESTION 2. *Let (X, x) and (Y, y) be pointed continua and let $f\colon (X, x) \to (Y, y)$ be a shape morphism such that the induced morphism $f'\colon X \to Y$ is an isomorphism. Is f an isomorphism?*

The analogous questions may be considered in pro-homotopy.

In this paper we consider the above questions. We show that in general the answers to Question 1 and Question 2 (in pro-homotopy) are negative. However they can be positively answered in some special cases.

Specially interesting is Question 2 because the negative answer to it would give a weak proper homotopy equivalence not being a proper homotopy equivalence which existence has been asked by T. A. Chapman and L. C. Siebenmann [7].

2. Notations and terminology. By $\boldsymbol{H}(\boldsymbol{H}_0)$ we denoted the homotopy category of (pointed) connected CW complexes.

For any category $\boldsymbol{C}$ we denote by pro-$\boldsymbol{C}$ its pro-category (see [1] and [19]) and by tow$(\boldsymbol{C})$ we denote a full subcategory of pro-$\boldsymbol{C}$ whose objects are towers i.e. inverse sequences in $\boldsymbol{C}$. (see [11]).

By $\boldsymbol{F}$: pro-$\boldsymbol{H}_0 \to$ pro-$\boldsymbol{H}$ we denote the forgetful functor obtained by suppressing base points.