A class of infinite-dimensional spaces. Part II: An Extension Theorem and the theory of retracts

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Abstract. A class of infinite-dimensional spaces, called C-spaces, is defined.

DEFINITION. A space X has *property* C (is a C-*space*) if for every sequence $\{\mathcal{C}_i\}_{i=1}^{\infty}$ of open covers of X there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of families of open sets such that (i) each family $\mathcal{U}_i$ is pairwise disjoint, (ii) if $U \in \mathcal{U}_i$, the $U \subseteq G$ for some member G of $\mathcal{C}_i$, and (iii) $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a cover of X. The sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is called a C-*refinement* of $\{\mathcal{C}_i\}_{i=1}^{\infty}$.

THEOREM. *A countable-dimensional metric space is a C-space.*

Using the dimension theory developed for this class of spaces in this paper Part I and techniques involving nerves of specially chosen covers the following theorem is proved.

THEOREM. *Let Y be a metrizable locally contractible space. Suppose that X is a metrizable, A a closed subspace of X such that the boundary of A has property C, and $f\colon A \to Y$ a continuous function. Then there exists a neighborhood U of A in X and a continuous extension $F\colon U \to Y$ of the map f. If Y is also contractible, we may take $U = X$.*

COROLLARY. *A contractible locally contractible metrizable C-space is an* ANR (AR).

Examples limiting the conditions of the theorem are referenced.

The problem of when continuous functions defined on closed subspaces can be extended has provided one of the most fruitful areas of research in topology. Among the early important theorems in this regard are Tietze's Extension Theorem and the No Retraction Theorem, the latter equivalent to Brouwer's Fixed-point Theorem. Later developments have included Dugundji's Extension Theorem and the following theorem of Kuratowski and Dugundji [4].

THEOREM. *If Y is a locally n-connected* (LC^n) *metric space, $n \geqslant 0$, X metric, A a closed subset of X with $\dim(X-A) \leqslant n+1$, then every continuous function $f\colon A \to Y$ can be extended to a neighborhood U of X. If Y is also n-connected* (C^n), *we may take $U = X$.*

As can be seen from the foregoing, there is a relationship between dimension and extendability of maps. However, Borsuk's space [2] limits the dimension condition in the theorem above. In particular, we may let Y be locally contractible (LC), let $X-A$ be countable-dimensional, and the conclusion of the theorem may fail.

Nevertheless, the principal result of this paper is that if we retain the local contractibility of Y and instead let A be countable-dimensional, the map f can be extended.

Many of the results upon which this paper depends are found in Addis and Gresham [1]. For convenience, we shall state those needed here. All spaces are assumed to be metric.

DEFINITION. A space X has *property* C (is a C-*space*) if for every sequence $\{\mathcal{C}_i\}_{i=1}^{\infty}$ of open covers of X there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of families of open sets such that

(i) each $\mathcal{U}_i$ is pairwise disjoint,

(ii) if $U \in \mathcal{U}_i$, then $U \subset G$ for some member G of $\mathcal{C}_i$,

(iii) $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a cover of X.

The sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is called a C-*refinement* of $\{\mathcal{C}_i\}_{i=1}^{\infty}$.

THEOREM. *A countable-dimensional metric space is a* C-*space.*

THEOREM. F_σ *subspaces of metric* C-*spaces have property* C.

THEOREM. *A metric space* X *is a* C-*space if and only if* $X \times [0, 1]$ *is a* C-*space. In particular, we shall use the fact that* $X \times [0, 1)$ *is a* C-*space if* X *is.*

A fundamental idea in our proofs is to build a map from the nerve of some carefully chosen cover of a C-space, and then compose this map with the barycentric map from the space into the nerve. The first result is a technical lemma which provides the basic construction when the range space enjoys a strong local connectivity property. We use K to denote a polytope, K^n its n-skeleton, and $|K|$ the space of the polytope with the weak topology.

DEFINITION. A polytope K is said to be a C-*polytope* if $K^0 = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ where each $\mathcal{U}_i$ is a family of vertices such that if σ is a simplex in K, then no two vertices of σ belong to the same $\mathcal{U}_i$.

Finally, if X is a metric space and $p \in X$, $N_\varepsilon(p)$ will denote the open ε-ball centered at p. Further, if $A \subset X$, $\operatorname{Bd} A$ and $\operatorname{Int} A$ are, respectively, the boundary and interior of A.

LEMMA 1. *Let* (Y, d) *be a uniformly* LC *metric space. Let* n *be a fixed positive integer and let* $\{\varepsilon_i\}_{i=1}^{\infty}$ *be a nonincreasing sequence of positive real numbers such that for each* i, $N_{(n+1)\varepsilon_i}(y)$ *is contractible in* $N_{\varepsilon_{i-1}}(y)$, $i \geqslant 2$. *Suppose* K *is a* C-*polytope and a map* $G_0\colon |K^0| \to Y$ *is given with the property that* $d(G_0(G_{i0}), G_0(U_{i_1})) < n\varepsilon_{i_0+1}$ *where* $U_{i_0} \in \mathcal{U}_{i_0}$, $U_{i_1} \in \mathcal{U}_{i_1}$, $i_0 < i_1$, *and* $\langle U_{i_0}, U_{i_1}\rangle$ *is a* 1-*simplex in* K. *Then* G_0 *can be extended to a map* $G\colon |K| \to Y$ *with the property that if* τ *is a simplex in* K, *say* $\tau = \langle U_{i_0}, U_{i_1}, \ldots, U_{i_m}\rangle$, *and* $i_0 < i_1 < \ldots < i_m$, *then*

$$G(\tau) \subset N_{\varepsilon_{i_0}}(G_0(U_{i_0})) .$$

Consequently, for all $\tau \in K$, $\operatorname{diam} G(\tau) < 2\varepsilon_1$.

Proof. For each vertex U_i in $\mathcal{U}_i$ let

$$\psi_{U_i}\colon N_{(n+1)\varepsilon_{i+1}}(G_0(U_i)) \times I \to N_{\varepsilon_i}(G_0(U_i))$$

be a fixed homotopy such that $\psi_{U_i}(x, 0) = x$ and $\psi_{U_i}(x, 1) = G_0(U_i)$. Assume inductively that for all integers $r < m$ a map $G_r\colon |K^r| \to Y$ has been given in such a way that

(1) $$G_r| \, |K^{r-1}| = G_{r-1} ,$$

(2) if $\sigma = \langle U_{i_0}, \ldots, U_{i_r}\rangle \in K^r$, $i_0 < i_1 < \ldots < i_r$, then $G_r(\sigma) \subset N_{\varepsilon_{i_0}}(G_r(U_{i_0}))$,

(3) if $\sigma = \langle U_{i_0}, U_{i_1}, \ldots, U_{i_r}\rangle \in K^r$, G_r is defined on σ by considering σ as the cone of U_{i_0} over $\langle U_{i_1}, \ldots, U_{i_r}\rangle$ and then using $\psi_{U_{i_0}} | G_{r-1}(\langle U_{i_1}, \ldots, U_{i_r}\rangle) \times I$ in the natural way (note that (1) and (2) are used to insure that $\psi_{U_{i_0}}$ is defined on $G_{r-1}(\langle U_{i_1}, \ldots, U_{i_r}\rangle) \times I$).

We note that the induction starts easily for $m = 1$. Given $\langle U_{i_0}, U_{i_1}\rangle$ in K^1, $i_0 < i_1$, we know that $d(G_0(U_{i_0}), G_0(U_{i_1})) < n\varepsilon_{i_0+1}$ so that we can use $\psi_{U_{i_0}}$ and define G_1 on $\langle U_{i_0}, U_{i_1}\rangle$ by

$$G_1(tU_{i_0} + (1-t)U_{i_1}) = \psi_{U_{i_0}}(G_0(U_{i_1}), t) , \qquad t \in [0, 1] .$$

Now let $\tau = \langle U_{i_0}, U_{i_1}, \ldots, U_{i_m}\rangle \in K^m$, and we may, as always, assume $i_0 < i_1 < \ldots < i_m$ since K is a C-polytope. By the inductive assumptions,

$$G_{m-1}(\langle U_{i_1}, \ldots, U_{i_m}\rangle) \subset N_{\varepsilon_{i_1}}(G_0(U_{i_1})) \subset N_{\varepsilon_{i_0+1}}(G_0(U_{i_1})) .$$

Thus

$$\begin{aligned} d(G_0(U_{i_0}), G_{m-1}(\langle U_{i_1}, \ldots, U_{i_m}\rangle)) &\leqslant d(G_0(U_{i_0}), G_0(U_{i_1})) + d(G_0(U_{i_1}), G_{m-1}(\langle U_{i_1}, \ldots, U_{i_m}\rangle)) \\ &< n\varepsilon_{i_0+1} + \varepsilon_{i_0+1} = (n+1)\varepsilon_{i_0+1} . \end{aligned}$$

Therefore we may use $\psi_{U_{i_0}}$ to define G_m on τ according to (3) and by doing so for each τ in K^m, we obtain a map $G_m\colon |K^m| \to Y$ satisfying (1) and (2) as well. By induction, the conclusion now follows. Q.E.D.

The following lemma will be used to ensure that the hypotheses are purely topological.

LEMMA 2 (Dugundji and Michael [5]). *If* (X, d) *is a locally contractible metric space, then there is a compatible metric* ϱ *for* X *such that* $\varrho \geqslant d$ *and* (X, ϱ) *is uniformly locally contractible.*

LEMMA 3 (Hanner [7], p. 352). *A map from a closed subset of a metric space into the space of a polytope has a neighborhood extension.*

LEMMA 4. *Let (Y, d) be a metric LC space, let X be a metrizable C-space, and let $f\colon X \to Y$ be a map. If $\varepsilon > 0$ then there exists a map $h\colon X \times I \to Y$ such that*

(a) $$d(h(x,t), f(x)) < \varepsilon \quad \text{for all } (x, t),$$

(b) $$h(x, 1) = f(x) \quad \text{for all } x,$$

(c) *if $X \times [0, 1)$ is a closed subset of a metric space Z,*

then $h|X \times [0, 1)$ has an extension H to a neighborhood of $X \times [0, 1)$ in Z. (Note that without (c) the conclusion is trivial.)

Proof. By Lemma 2 let ϱ be a compatible metric for Y such that $\varrho \geqslant d$ and (Y, ϱ) is uniformly LC. We now work strictly with ϱ. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers such that

(1) for every y in Y, $N_{3\varepsilon_i}(y)$ is contractible in $N_{\varepsilon_{i-1}}(y)$,

(2) $3\varepsilon_i < \varepsilon_{i-1}$, and

(3) $2\varepsilon_1 < \varepsilon$.

Let $N = \{1, 2, 3, \ldots\}$ and define a function $g\colon N \times N \to N$ by

$$g(a, b) = \tfrac{1}{2}(a+b-1)(a+b-2)+b.$$

This is "Cantor's diagonal function", and has the property that $g(a, b) \geqslant b$. For each $n \geqslant 1$ set $J_n = [0, n/(n+1))$. For each k, let $\mathscr{B}_k$ be an open cover of X such that if $B \in \mathscr{B}_k$, then $\operatorname{diam}(f(B)) \leqslant \varepsilon_k$. Now define a cover $\mathscr{C}_n$ of $X \times [0, 1)$ by

$$\mathscr{C}_n = \bigcup_{k=1}^{\infty} \mathscr{C}_{(n,k)}$$

where $\mathscr{C}_{(n,k)} = \{B \times J_k\colon B \in \mathscr{B}_{g(n,k)}\}$.

Since $X \times [0, 1)$ is a C-space (see above) we let $\{\mathscr{U}_n\}_{n=1}^{\infty}$ be a C-refinement of $\{\mathscr{C}_n\}_{n=1}^{\infty}$ and define

$$\mathscr{U}_{(n,k)} = \{U \in \mathscr{U}_n\colon U \subset G \text{ for some } G \text{ in } \mathscr{C}_{(n,k)}\}.$$

For each $i = 1, 2, \ldots$, set $\mathscr{V}_i = \mathscr{U}_{g^{-1}(i)}$. Then $\mathscr{V}_i$ is a pairwise disjoint family of open subsets of $X \times [0, 1)$ and $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i$ is a cover of $X \times [0, 1)$.

Let $p\colon X \times [0, 1) \to X$ be the projection map. We now observe that if $V_i \in \mathscr{V}_i$, then $\operatorname{diam}(fp(V_i)) \leqslant \varepsilon_i$. Indeed, $V_i \in \mathscr{U}_{(m,n)}$ where $g(m, n) = i$ and so $V_i \subset G$ for some G in $\mathscr{C}_{(m,n)}$. Now $G = B \times J_n$ for some B in $\mathscr{B}_{g(m,n)} = \mathscr{B}_i$ and so

$$fp(V_i) \subset fp(B \times J_n) \subset f(B).$$

Hence $\operatorname{diam}(fp(V_i)) \leqslant \operatorname{diam}(f(B)) \leqslant \varepsilon_i$ by definition of $\mathscr{B}_i$.

Let $b\colon X \times [0, 1) \to |\mathscr{N}(\mathscr{V})|$ be the standard barycentric map into the nerve of the cover $\mathscr{V}$. We now define a map $F_0\colon |\mathscr{N}^0(\mathscr{V})| \to Y$ by setting $F_0(V)$ to be a fixed element of $fp(V)$. (Here we think of V as both a vertex in the nerve and a set in the space.) Now suppose $\langle V_{i_0}, V_{i_1}\rangle$ is a one-simplex in $\mathscr{N}(\mathscr{V})$, $V_{i_0} \in \mathscr{V}_{i_0}$, $V_{i_1} \in \mathscr{V}_{i_1}$, and $i_0 < i_1$. Then $V_{i_0} \cap V_{i_1} \neq \emptyset$ so that $fp(V_{i_0}) \cap fp(V_{i_1}) \neq \emptyset$. Therefore

$$\varrho(F_0(V_{i_0}), F_0(V_{i_1})) \leqslant \varepsilon_{i_0} + \varepsilon_{i_1} < 2\varepsilon_{i_0}.$$

By Lemma 1 (using $n = 2$) there exists a map $F\colon |\mathscr{N}(\mathscr{V})| \to Y$ extending F_0 with the property that if $\tau = \langle V_{i_0}, V_{i_1}, \ldots, V_{i_m}\rangle \in \mathscr{N}(\mathscr{V})$, $i_0 < i_1 < \ldots < i_m$, then

$$F(\tau) \subset N_{\varepsilon_{i_0}}(F_0(V_{i_0})).$$

Now we define $h\colon X \times I \to Y$ by

$$h(x, t) = \begin{cases} Fb(x, t) & \text{if } t < 1, \\ f(x) & \text{if } t = 1. \end{cases}$$

The function h is already continuous at points (x, t), $t < 1$. We verify continuity at the points $(x_0, 1)$. Let $\gamma > 0$ be given. We wish to find a neighborhood $U \times W$ of $(x_0, 1)$ so that $h(U \times W) \subset N_\gamma(h(x_0, 1)) = N_\gamma(f(x_0))$. Indeed, let M be a positive integer such that $\varepsilon_m < \frac{1}{3}\gamma$ for $m \geqslant M$. Let U be a member of $\mathscr{B}_M$ containing x_0. We claim that the neighborhood $U \times (M/(M+1), 1]$ of $(x_0, 1)$ meets the requirements. Suppose $(x, t) \in U \times (M/(M+1), 1]$.

Case I. $t = 1$. Then $h(x, t) = f(x)$. Since $x \in U \in \mathscr{B}_M$, and $x_0 \in U$, $\varrho(f(x), f(x_0)) \leqslant \operatorname{diam} f(U) \leqslant \varepsilon_M < \frac{1}{3}\gamma < \gamma$ so that $h(x, t) \in N_\gamma(h(x_0, 1))$.

Case II. $t < 1$. Suppose $b(x, t) \in \langle V_{i_0}, \ldots, V_{i_k}\rangle$, $i_0 < \ldots < i_k$. Since b is barycentric, $(x, t) \in \bigcap_{j=0}^{k} V_{i_j}$ and in particular $(x, t) \in V_{i_0} = \mathscr{U}_{(m,n)}$ where $g(m, n) = i_0$. By definition of $\mathscr{U}_{(m,n)}$ we have $t < n/(n+1)$, and because $M/(M+1) < t$, we must conclude that $n > M$. By the definition of g, $g(m, n) \geqslant n$ and so $i_0 = g(m, n) > M$. By the property of F,

$$Fb(x, t) \in N_{\varepsilon_{i_0}}(f_0(V_{i_0})).$$

Therefore

$$\begin{aligned} &\varrho(Fb(x, t), h(x_0, 1) \\ &\quad \leqslant \varrho(Fb(x, t), F_0(V_{i_0})) + \varrho(F_0(V_{i_0}), f(x)) + \varrho(f(x), f(x_0)) \\ &\quad \leqslant \varepsilon_{i_0} + \varepsilon_{i_0} + \varepsilon_M < 3\varepsilon_M < \gamma. \end{aligned}$$

Thus h is continuous.

By its definition, h satisfies (b) of the conclusion. We now show that h satisfies (a) and (c).

To show (a), it suffices to show $\varrho(h(x, t), f(x)) < \varepsilon$ since $d \leqslant \varrho$. If $t = 1$, there is nothing to prove. Assume that $t < 1$ and that $b(x, t) \in \langle V_{i_0}, \ldots, V_{i_m}\rangle$, $i_0 < \ldots < i_m$. Then $(x, t) \in V_{i_0}$ and

$$\begin{aligned} \varrho(Fb(x, t), f(x)) &\leqslant \varrho(Fb(x, t), F_0(V_{i_0})) + \varrho(F_0(V_{i_0}), f(x)) \\ &\leqslant \varepsilon_{i_0} + \varepsilon_{i_0} = 2\varepsilon_{i_0} \leqslant 2\varepsilon_1 < \varepsilon. \end{aligned}$$

To show (c) assume that $X\times[0,1)$ is a closed subset of a metric space Z. From the proof above we have the commutative diagram

$$\begin{tikzcd} & \mathcal{N}(\mathcal{V}) \arrow[dr, "F"] & \\ X\times[0,1) \arrow[ur, "b"] \arrow[rr, "h|X\times[0,1)"'] & & Y \end{tikzcd}$$

By Lemma 3 there is a neighborhood U of $X\times[0,1)$ in Z and an extension $\bar{b}\colon U\to\mathcal{N}(\mathcal{V})$ of b. Then $F\bar{b}$ extends $h|X\times[0,1)$ to U. Q.E.D.

We now state and prove the main result of this paper.

THEOREM. *Let Y be a metrizable* LC *space. Suppose that X is a metrizable space, A a closed subspace of X such that* $\operatorname{Bd}A$ *has property C, and $f\colon A\to Y$ a map. Then there exists a neighborhood U of A in X and an extension $F\colon U\to Y$ of f. If Y is also contractible, we may take $U=X$.*

Proof. It suffices to show that $f|\operatorname{Bd}A$ has an extension $\bar{f}$ defined on a neighborhood W of $\operatorname{Bd}A$ in X. Then $U=W\cup\operatorname{Int}A$ is a neighborhood of A and we define $F\colon U\to Y$ by

$$F(u)=\begin{cases} f(u) & \text{if}\quad u\in\operatorname{Int}A\,,\\ \bar{f}(u) & \text{if}\quad u\in W-\operatorname{Int}A\,.\end{cases}$$

F is then a continuous extension of f to the neighborhood U of A. Without loss of generality, we may assume that A is a C-space and prove the theorem for this case.

Let d be a metric for Y. Using $\varepsilon=1$ we can find a map $h\colon A\times I\to Y$ satisfying (a), (b), and (c) of Lemma 4. Now consider $A\times[0,1)$ as a closed subset of $X\times[0,1)$. Then there is a neighborhood W of $A\times[0,1)$ in $X\times[0,1)$ and a map $H\colon W\to Y$ which extends $h|A\times[0,1)$. Let $i\colon X\to X\times[0,1)$ be the imbedding $i(x)=(x,0)$ and set $U=i^{-1}(W)$. Then U is a neighborhood of A in X. We shall, eventually, define $F\colon U\to Y$ extending f but we must proceed carefully to insure continuity of F.

Let ϱ be a metric for X and metrize $X\times[0,1)$ by

$$\sigma\big((x,s),(y,t)\big)=\varrho(x,y)+|s-t|\,.$$

We define a subset V of W as follows:

$$V=\{(u,t)\in W\colon \text{for some } (x,t') \text{ in } A\times[0,1)$$
$$\sigma\big((u,t),(x,t')\big)<1-t \text{ and } d\big(H(u,t),H(x,t')\big)<1-t\}\,.$$

Then V contains $A\times[0,1)$ and is open. To see the latter assertion, fix $(u,t)\in V$. Let $(x,t')\in A\times[0,1)$ be as in the definition of V. Set

$$\alpha=\sigma\big((u,t),(x,t')\big)\,,$$
$$\beta=d\big(H(u,t),H(x,t')\big)\,,$$
$$\varepsilon=(1-t)-\alpha\,,$$
$$\gamma=(1-t)-\beta\,.$$

Find $\delta>0$ so that $H(N_\delta(u,t))\subset N_{\gamma/2}(H(u,t))$. Let $\Gamma=\min\{\varepsilon/2,\delta,\gamma/2\}$. Then we can routinely verify that $W\cap N_\Gamma(u,t)\subset V$.

Now let $s_n=1-1/2^{n-1}$ for each $n\geqslant 1$. By the compactness of $[s_n,s_{n+1}]$ we can find an open neighborhood U_n of A in X such that

$$A\times[s_n,s_{n+1}]\subset U_n\times[s_n,s_{n+1}]\subset V\,.$$

We may also assume that $U_n\supset\overline{U}_{n+1}$ for all n.

By Urysohn's Lemma there exist maps $e_n\colon U\to I$ such that

$$e_n(U-U_n)=\{0\}\quad\text{and}\quad e_n(\overline{U}_{n+1})=\{1\}\,.$$

Define $e(u)=\sum_{n=1}^{\infty}2^{-n}e_n(u)$. Then $e\colon U\to I$ is a map with the following properties:

$$(1)\qquad e(u)=1\quad\text{if } u\in A,.$$

$$(2)\qquad e(u)=0\quad\text{if } u\notin U_1\,,$$

$$(3)\qquad (u,e(u))\in V\quad\text{if } u\in U_1-A\,.$$

(1) follows because $A\subset\overline{U}_{n+1}$ for all n. To verify (2), observe that if $x\notin U_1$ then $x\notin U_n$ for all n so that $e_n(x)=0$ for all n. It will require somewhat more work to demonstrate (3). First we must note that $\bigcap_{n=1}^{\infty}U_n=A$. Indeed, if x is in U_n for all n, then (x,t) belongs to V for all t in $[0,1)$. Thus $\sigma\big((x,t),A\times[0,1)\big)<1-t$ for all $t<1$ whence $\varrho(x,A)=0$ and $x\in A$ since A is closed. Therefore if $u\in U_1-A$ then $u\in U_i-U_{i+1}$ for some i. Here are two key facts:

$$(4)\qquad \text{if } n<i \quad\text{then}\quad u\in\overline{U}_{n+1} \text{ and } e_n(u)=1;$$

$$(5)\qquad \text{if } n>i \quad\text{then}\quad u\notin U_n \text{ and } e_n(u)=0.$$

Thus

$$e(u)=(1/2+1/4+\ldots+1/2^{i-1})+(1/2^i)e_i(u)$$
$$=(1-1/2^{i-1})+(1/2^i)e_i(u)$$

and

$$e(u)\in[1-1/2^{i-1},(1-1/2^{i-1})+1/2^i]=[s_i,s_{i+1}]\,.$$

It now follows that $(u,e(u))\in U_i\times[s_i,s_{i+1}]\subset V$.

This work has all been done in order to define $F\colon U\to Y$ in a continuous manner. We define F by

$$F(u)=\begin{cases} H(u,e(u)) & \text{if } u\in U-A\,,\\ f(u) & \text{if } u\in A\,.\end{cases}$$

F is certainly continuous at the interior points of A and at the points of the open set $U-A$. We need only show continuity at the points of the boundary of A. It suffices to show that if $\{u_n\}_{n=1}^{\infty}$ is a sequence in U_1-A such that $u_n\to a\in A$, then

$F(u_n) \to F(a)(=f(a))$. For each n the point $(u_n, e(u_n)) \in V$ so that there exists (x_n, t_n') in $A \times [0, 1)$ with

$$\sigma((u_n, e(u_n)), (x_n, t_n')) < 1 - e(u_n)$$

and

$$d(H(u_n, e(u_n)), H(x_n, t_n')) < 1 - e(u_n).$$

Now $u_n \to a \in A$ so that $e(u_n) \to e(a) = 1$ and

$$(6) \qquad (u_n, e(u_n)) \to (a, 1),$$

$$(7) \qquad \sigma((u_n, e(u_n)), (x_n, t_n')) \to 0,$$

$$(8) \qquad d(H(u_n, e(u_n)), H(x_n, t_n')) \to 0.$$

By (6), (7), and the definition of σ we have

$$(9) \qquad (x_n, t_n') \to (a, 1).$$

Hence

$$(10) \qquad H(x_n, t_n') = h(x_n, t_n') \to h(a, 1) = f(a).$$

From (8) and (10) we have

$$F(u_n) = H(u_n, e(u_n)) \to \lim H(x_n, t_n') = f(a).$$

If Y is also contractible, then we may extend F to all of X as follows. Let V be an open neighborhood of A with $A \subset V \subset \overline{V} \subset U$. By Urysohn's Lemma let $g: X \to [0, 1]$ be a map such that $g(A) = \{0\}$ and $g(X - V) = \{1\}$. Let H: $Y \times [0, 1] \to Y$ be a contraction of Y to a point p and define $\bar{F}$: $X \to Y$ by

$$\bar{F}(x) = \begin{cases} p & \text{if } x \notin V, \\ H(F(x), g(x)) & \text{if } x \in V. \end{cases}$$

$\bar{F}$ is the promised extension of F to X. Q.E.D.

COROLLARY. *Since a countable-dimensional metric space X has property C, the theorem provides an extension for f in the case where A is finite-dimensional or countably infinite-dimensional.*

COROLLARY. *A locally contractible* (*contractible*) *metrizable C-space is an* ANR (AR).

Previously William Haver [8] has proved a rather special case of this corollary with the condition that X be a countable union of finite-dimensional compacta. See also [9].

A final note is that the metrizability conditions cannot be omitted from both X and Y in the theorem. Saalfrank [11] has given an example of a compact Hausdorff LC space of dimension 1 such that X is not a neighborhood retract of a Tychonoff cube in which it can be embedded.

References

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Accepté par la Rédaction le 27. 5. 1977