we have $\partial\beta = h(\varkappa)$; in other words $h(\varkappa)$ is homologous to zero. Thus $h_*(p_*(H_2(Q))) = 0$. Since the diagram

$$\begin{array}{ccc} H_2(Q) & \xrightarrow{g_*} & H_2(Y) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_2(Q_1) & \xrightarrow{h_*} & H_2(Y) \end{array} \quad \text{is commutative}$$

and g_* is an isomorphism, f induces the trivial morphism on the two-dimensional homology group of Y. Since Y has the same homologies as the two-dimensional sphere, the Lefschetz number $\Lambda(f) = 1$, then by 3.3, f has a fixed point, which completes the proof.

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On the decidability of the theory of linear orderings with generalized quantifiers

by

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Abstract. $LO(Q_0, Q_1, \ldots, Q_m)$ be the theory of linear orderings with the additional quantifiers $Q_0, \ldots, Q_m$. Under various hypotheses on set theory it is proved that $LO(Q_0, \ldots, Q_m)$ is always decidable. This generalizes the result of the author for $LO(Q_1)$. The proof uses methods from Leonhard and Läuchli. The theorems can be generalized to arbitrary finite sets of regular cardinality quantifiers.

A. Ehrenfeucht proved in [1] that the elementary theory LO of linear orderings is decidable. In [4] H. Läuchli and J. Leonhard established the same result using games. Let us extend the elementary language of linear order by adding the generalized quantifiers $Q_0, Q_1, \ldots, Q_m$ to it.

We interpret the quantifier Q_k as: "there exist at least ω_k-many". Generalized quantifiers were introduced by A. Mostowski [6].

Let $LO(Q_0, \ldots, Q_m)$ be the theory of linear orderings with these additional quantifiers. Then we will prove that $LO(Q_0, Q_1, \ldots, Q_m)$ is decidable. This generalizes the result of H. P. Tuschik [9] for $LO(Q_1)$. As a corollary we infer that $LO(Q_i: i<\omega)$ is decidable.

§ 1. Let L be the first order language with identity and one binary predicate $<$. $L^m(Q)$ arises from L by adding the quantifiers $Q_0, \ldots, Q_m$. LO is the following theory:

(1) $\neg x<x$,

(2) $x<y \wedge y<z \rightarrow x<z$,

(3) $x = y \vee x<y \vee y<x$.

We use some definitions from [4] and [9]: $x<y(\mathrm{mod}\,A)$ denotes the order relation of an ordered set A, $|A|$ denoted the field of A. B is said to be a *segment* of A if B is a substructure of A and if $x<y(\mathrm{mod}\,B)$ and $x<z<y(\mathrm{mod}\,A)$ implies $z \in B$. Some special segments are the open interval $(x, y) = \{z \in |A|\colon x<z<y(\mathrm{mod}\,A)\}$, the left-open and right closed interval $(x, y] = \{z \in |A|\colon x<z\leqslant y(\mathrm{mod}\,A)\}$, the left-closed and right-open interval $[x, y) = \{z \in |A|\colon x\leqslant z<y(\mathrm{mod}\,A)\}$ and the closed interval $[x, y] = \{z \in |A|\colon x\leqslant z\leqslant y(\mathrm{mod}\,A)\}$. A map $f\colon A \rightarrow B$ of an ordered set A

into an ordered set B is called *monotonic* if for all $x, y \in A$: $x \leqslant y$ implies $f(x) \leqslant f(y)$. An ordered set $\mathfrak{A}$ is said to be a *splitting* of A if $|\mathfrak{A}|$ is a set of (non-empty) segments of A which partitions A, and if $B < C (\mathrm{mod}\, \mathfrak{A})$ iff $x < y (\mathrm{mod}\, A)$ for all $x \in |B|$ and $y \in |C|$.

The elements of $\mathfrak{A}$ are called the *parts of the splitting*. Suppose $\mathfrak{A}$ is a splitting of A; then the canonical map $f\colon A \to \mathfrak{A}$ is monotonic and onto.

Suppose $f\colon A \to B$ is monotonic and onto, then $\{f^{-1}(b)\colon b \in |B|\}$ is a splitting of A. For $i < \omega$, ω_i and ω_i^* denote the order type of the ordinals smaller than ω_i and the inverse ordering of ω_i, respectively. $+$, $\cdot$ denote the sum and product of ordered sets (order types).

Our general set theory will be ZFC. However, then there are sentences H of $L^m(Q)$ which are independent of ZFC; more exactly: we may assume that the sentence H has a model or does not have a model; both hypotheses are consistent with ZFC. Therefore we will extend ZFC. S_{km} and $H(i,j)$ denote, respectively, the following sentences of set theory: "k is the least natural number such that there is a dense ordered field K of cardinality ω_k which has at least ω_m-many 1-types over K" and "There is a dense ordered field K of cardinality ω_i such that the set of 1-types over K has cardinality ω_j". Suppose f is a map from $\{0, 1, \ldots, k\}$ into $\{0, 1, \ldots, m\}$ with $f(k) = m$, then let $T_{km}(f)$ be $S_{km} \wedge \bigwedge_{i<k} H(i, f(i)) \wedge \neg H(i, f(i)+1)$. If S is the successor-function, $S(i) = i+1$, then clearly $\mathrm{ZFC}+\mathrm{GCH} \vdash T_{m-1,\, m}(S)$. Assume $\mathrm{ZFC}+T_{km}(f)$ is consistent. From now on we will work in $\mathrm{ZFC}+T_{km}(f)$.

For all $i \leqslant k$ there are dense ordered fields K_i of cardinality ω_i such that there are $\omega_{f(i)}$-many 1-types over K_i. Clearly, there are also $\omega_{f(i)}$-many 1-types over the interval $(0, 1)$ of K_i. Let N_i be an elementary extension of K_i which realizes all 1-types of K_i over $(0, 1)$. Let $X_i \subseteq N_i$ be such that every 1-type over $(0, 1)$ has a realization in X_i and any two different elements of X_i realize different types. The subring of N_i generated by $K_i \cup (X_i \cap (0, 1))$ is denoted by P_i. Then K_i is dense in P_i. Now regard P_i as an ordered vector space over K_i and let B_i be a basis of P_i. The axiom $T_{km}(f)$ implies that there are one-to-one mappings σ_{if}, $\sigma_{if}\colon \omega_{f(i)} \to B_i$. From σ_{if} we get the sets $\sigma_{if}(j, n) = \{x \in P_i\colon$ there exists some limit ordinal $\alpha < \omega_j$ such that $x - \sigma_{if}(\omega_{j-1}+\alpha+n) \in K_i\}$, where $i \leqslant j \leqslant f(i)$ and $n < \omega$, $\omega_{-1} = 0$. As subsets of P_i the sets $\sigma_{if}(j, n)$ are ordered sets. Assume $\mathscr{F} = \langle F_i, \ldots, F_{f(i)} \rangle$ and for $i \leqslant j \leqslant f(i)$ let $F_j = \langle F_0^j, \ldots, F_{n_j}^j \rangle$ be a sequence of ordered sets (order types).

Then substitute in the ordered set $\bigcup_{i \leqslant j \leqslant f(i)} \bigcup_{n \leqslant n_j} \sigma_{if}(j, n)$ for every $x \in \sigma_{if}(j, n)$ the ordered set (order type) F_n^j. The resulting ordered set (order type) is $\sigma_{if}(\mathscr{F})$.

The operation $\sigma_{if}(\mathscr{F})$ enables us to form ordered sets of cardinality $\omega_{f(i)}$ which have for all $i \leqslant j \leqslant f(i)$ dense subsets of cardinality ω_j. σ_{if} is a generalization of the usual shuffling operation on ordered sets.

Now, for every natural number n, we shall define the equivalence relations $A \overset{n}{\sim} B$ between ordered sets and $\langle A, a \rangle \overset{n}{\sim} \langle B, b \rangle$ where A and B are ordered sets and $a \in A$, $b \in B$. We say that $X \subseteq A$ is the *set of realizations* of $\langle B, b \rangle_n$ in A, $X = \langle B, b \rangle_n(A)$, if

(1) every $a \in A$ such that $\langle A, a \rangle \overset{n}{\sim} \langle B, b \rangle$ is an element of X,

(2) for all $x \in X$ $\langle A, x \rangle \overset{n}{\sim} \langle B, b \rangle$.

We call the equivalence classes $\langle B, b \rangle_n$ of $\overset{n}{\sim}$ types and say that $\langle B, b \rangle_n$ occurs in A iff $\langle B, b \rangle_n(A)$ is non-empty. $\langle B, b \rangle_n$ occurs ω_j-many in A iff $\langle B, b \rangle_n(A)$ has cardinality ω_j. We define $\overset{n}{\sim}$ by induction.

Definition 1. (1) For every two ordered sets A, B (possibly empty) and any elements $a \in A$ and $b \in B$

$$A \overset{0}{\sim} B \quad \text{and} \quad \langle A, a \rangle \overset{0}{\sim} \langle B, b \rangle .$$

(2) $A \overset{n+1}{\sim} B$ iff for every $\langle X, x \rangle_n$

(i) $\langle X, x \rangle_n(A) \neq \varnothing$ iff $\langle X, x \rangle_n(B) \neq \varnothing$,

(ii) for every $j < m$ $\mathrm{card}\langle X, x \rangle_n(A) = \omega_j$ iff $\mathrm{card}\langle X, x \rangle_n(B) = \omega_j$,

(iii) $\mathrm{card}\langle X, x \rangle_n(A) \geqslant \omega_m$ iff $\mathrm{card}\langle X, x \rangle_n(B) \geqslant \omega_m$.

(3) $\langle A, a \rangle \overset{n+1}{\sim} \langle B, b \rangle$ iff

(i) $A^{<a} \overset{n+1}{\sim} B^{<b}$, and

(ii) $A^{>a} \overset{n+1}{\sim} B^{>b}$,

where $C^{<c} = \{x \in |C|\colon x < c\}$ and $C^{>c} = \{x \in |C|\colon x > c\}$.

The equivalence relation $\overset{n}{\sim}$ corresponds to the game for the generalized quantifiers $Q_0, \ldots, Q_m$ in case of orderings. For the connection between generalized quantifiers and games we refer to [5] and [10].

Lemma 1. (i) *$A \underset{L^m(Q)}{\equiv} B$ (elementary equivalence with respect to $L^m(Q)$) iff for every $n < \omega$ $A \overset{n}{\sim} B$.*

(ii) *$A \overset{n}{\sim} B$ implies that for all sentences H of $L^m(Q)$ which contain at most n quantifiers $A \vDash H$ iff $B \vDash H$.*

Proof. This is proved by induction as in the elementary case [2] and [4].

Lemma 2. *For each n there are only finitely many equivalence classes of $\overset{n}{\sim}$.*

Proof. This follows easily from the definition.

Lemma 3. *If ordered sets A, B admit isomorphic splittings such that corresponding parts are $\overset{n}{\sim}$-equivalent, then $A \overset{n}{\sim} B$.*

Proof. The lemma is trivial for $n = 0$. Assume then that it is true for j and that we are given isomorphic splittings $\mathfrak{A}$, $\mathfrak{B}$ of A, B with corresponding parts $\overset{j}{\sim}$-equivalent. Let $a \in A$; then $a \in C \in \mathfrak{A}$, and let D be the image of C by the isomorphism of $\mathfrak{A} \cong \mathfrak{B}$, and $b \in D$ such that $\langle C, a \rangle \overset{j}{\sim} \langle D, b \rangle$. $A^{<a}$ and $B^{<b}$ have isomorphic splittings which are $\overset{j}{\sim}$-equivalent, by the induction hypothesis $A^{<a} \overset{j}{\sim} B^{<b}$. Similarly we get $A^{>a} \overset{j}{\sim} B^{>b}$; hence $\langle A, a \rangle \overset{j}{\sim} \langle B, b \rangle$. This implies that some type $\langle X, x \rangle_j$ occurs in A iff it occurs in B. If some type $\langle X, x \rangle_j$ occurs ω_i-many times, $i \leqslant m$, in A, then one part of the splitting contains ω_i-many realizations of $\langle X, x \rangle_j$ or there are ω_i-many parts of the splitting which have at least one realization of $\langle X, x \rangle_j$. In both cases the same holds for B; thus $A \overset{j+1}{\sim} B$ by definition. Q.E.D.

COROLLARY 4. *If ordered sets, A, B admit order-isomorphic splittings such that corresponding parts are elementarily equivalent in $L^m(Q)$, then $A \underset{L^m(Q)}{\equiv} B$.*

COROLLARY 5. *The relations $\underset{L^m(Q)}{\equiv}$ and, for every n, $\overset{n}{\sim}$ are compatible with the operations $A+B$, $A\cdot\omega$, $A\cdot\omega^*$, $A\cdot\omega_i$ and $A\cdot\omega_i^*$, where $i\leqslant m$.*

LEMMA 6. *If ordered sets A, B admit $\overset{n}{\sim}$-equivalent splittings $\mathfrak{A}, \mathfrak{B}$ such that $X \overset{n}{\sim} Y$ for all $X\in\mathfrak{A}$ and $Y\in\mathfrak{B}$, then $A \overset{n}{\sim} B$.*

The proof for Lemma 6 is analogous to the proof of Lemma 3.

Let A be an ordered set and $\mathfrak{A}$ a splitting of A. Then for an ordered set B we say that the n-type of B occurs ω_i-densely in $\mathfrak{A}$ if for all $Y, Z\in\mathfrak{A}$, $Y<Z$, there are at least ω_i-many $X\in\mathfrak{A}$ such that $Y<X<Z$ and $B \overset{n}{\sim} X$.

LEMMA 7. *If ordered sets A, B admit splittings $\mathfrak{A}, \mathfrak{B}$, respectively, such that*

(1) $\mathfrak{A}$ and $\mathfrak{B}$ have no least and no greatest element,

(2) if some n-type X occurs in $\mathfrak{A}$ or $\mathfrak{B}$, then the n-type of X occurs ω_0-densely in $\mathfrak{A}$ and $\mathfrak{B}$,

(3) for all $i\leqslant m$ and for every ordered set X: if the n-type of X occurs at least ω_i-many times in $\mathfrak{A}$ or in $\mathfrak{B}$, then the n-type of X occurs ω_i-densely in $\mathfrak{A}$ and in $\mathfrak{B}$,

then $A \overset{n}{\sim} B$.

Proof. By induction. Clearly, the lemma holds for $n = 0$. Assume then that the lemma is true for j and let A, B be two ordered sets which have splittings $\mathfrak{A}, \mathfrak{B}$, respectively, such that the conditions (1)-(3) of the lemma are satisfied for $j+1$. For $a\in A$ let X be the part of $\mathfrak{A}$ which contains a, $a\in X$. Choose $Y\in\mathfrak{B}$ such that $X \overset{j+1}{\sim} Y$ and some $b\in Y$ with (i) $\langle X, a\rangle \overset{j}{\sim} \langle Y, b\rangle$. $\mathfrak{A}^{<X}$ and $\mathfrak{B}^{<Y}$ form splittings of $A^{<X} = \{x\in A\colon x\in C<X,\ C\in\mathfrak{A}\}$ and $B^{<Y} = \{x\in B\colon x\in C<Y,\ C\in\mathfrak{B}\}$, respectively, which satisfy the conditions (1)-(3) of the lemma for j. By the induction hypothesis (ii) $A^{<X} \overset{j}{\sim} B^{<Y}$. $A^{<X}$ and $X^{<a}$ partition $A^{<a}$, and similarly $B^{<Y}$ and $Y^{<b}$ partition $B^{<b}$.

By (i), (ii) and Corollary 5 we get $A^{<a} \overset{j}{\sim} B^{<b}$ and in the same way also $A^{>a} \overset{j}{\sim} B^{>b}$, i.e. $\langle A, a\rangle \overset{j}{\sim} \langle B, b\rangle$. If there are at least ω_i-many, $i\leqslant m$, realizations of the j-type $\langle X, x\rangle_j$ in $\mathfrak{A}$, then ω_i-many realizations belong to one part of the splitting or at least ω_i-many parts contain at least one realization of $\langle X, x\rangle_j$. It is easy to see that from the assumptions about the splittings $\mathfrak{A}, \mathfrak{B}$ it follow that the same holds for $\mathfrak{B}$. Thus $A \overset{j+1}{\sim} B$.

§ 2. Let M_{kmf} be the smallest class of ordered sets which contains $\underline{1}$ and is closed under $\alpha+\beta$, $\alpha\cdot\omega_j$ and $\alpha\cdot\omega_j^*$ for $j\leqslant m$, and $\sigma_{if}(\mathscr{F})$ where $i\leqslant k$ and $\mathscr{F} = \langle F_i, \ldots, F_{f(i)}\rangle$ and $F_i, \ldots, F_{f(i)}$ are finite sequences of elements of M_{kmf} (not all of $F_i, \ldots, F_{f(i)}$ are the empty sequence). $\underline{1}$ is the unique ordered set with the universe $\{0\}$. An ordered set A is said to be *n-good* if it is $\overset{n}{\sim}$-equivalent to a certain $\alpha\in M_{kmf}$.

LEMMA 8. *For every $n<\omega$, for every $i\leqslant k$, and for every $j\leqslant m$, the class of n-good ordered sets is closed under the operations $\alpha+\beta$, $\alpha\cdot\omega_j$, $\alpha\cdot\omega_j^*$, and $\sigma_{if}(\mathscr{F})$.*

Proof. By Corollary 5 and Lemma 3.

B is a bounded segment of an ordered set A if B is a segment of some closed interval $[x, y]$ of A. A subset $X\subseteq A$ is called *n-homogeneous* if, for all $a, b, c, d\in X$, $a<b$ and $c<d$ imply $(a, b) \overset{n}{\sim} (c, d)$.

LEMMA 9. *Let A be an ordered set with a least element. Assume A has a cofinal subset of type ω_i, $i\leqslant m$. Then for every natural number n there exists a subset $X_n\subseteq A$ which is n-homogeneous and has order type ω_i.*

Proof. The lemma follows from the Ramsey Theorem for additive colourings [8]. We give Shelah's proof here.

Let $X\subseteq A$ be cofinal and of order type ω_i. Define: For $x, y\in X$, $x\sim y$ if there is a $z\in X$, $x, y<z$, such that $(x, z) \overset{n}{\sim} (y, z)$; clearly this implies that for any $z'\in X$, $z<z'$, $(x, z') \overset{n}{\sim} (y, z')$ (by Corollary 5). It is easy to verify that $\sim$ is an equivalence relation with finitely many equivalence classes. So there is at least one equivalence class I, which is an unbounded subset of X. Let x_0 be the first element of I. Furthermore, define an equivalence relation $\approx$ on I: $x\approx y$ iff $(x_0, x) \overset{n}{\sim} (x_0, y)$. There are only finitely many equivalence classes of $\approx$ on I. So there is at least one equivalence class $J\subseteq I$ which is unbounded. Then we define $X_n\subseteq J$, $X_n = \{x_\alpha\colon \alpha<\omega_i\}$, by induction:

(1) x_0 is already defined,

(2) x_α is the least element x of J such that $(x_0, x) \overset{n}{\sim} (x_\beta, x)$ for all $\beta<\alpha$.

Since I and J are well-orderings of type ω_i, X_n is well defined and has order type ω_i. Now X_n is the desired set, because, for every $x_\alpha, x_\beta\in X_n$, $\alpha<\beta$, $(x_\alpha, x_\beta) \overset{n}{\sim} (x_0, x_\beta)$ by the definition of x_β and $(x_0, x_1) \overset{n}{\sim} (x_0, x_\beta)$, since $x_1\approx x_\beta$; hence X_n is n-homogeneous.

LEMMA 10. *Let n be a natural number and let A, B be ordered sets. Assume there are n-homogeneous subsets $X\subseteq A$ and $Y\subseteq B$ such that*

(1) X and Y have order type ω_i, where $i\leqslant m$, and are cofinal in A and B, respectively,

(2) the least element of X is the least element of A and the least element of Y is the least element of B,

(3) $(x_0, x_1) \overset{n}{\sim} (y_0, y_1)$ where x_0 is the least element of X and x_1 is the successor of x_0 in X and the same properties hold for y_0, y_1 and Y.

Then $A \overset{n}{\sim} B$.

Proof. The lemma is true for $n = 0$. Assume then that it is true for j and that we have two ordered sets A and B and subsets $X\subseteq A$ and $Y\subseteq B$ such that the hypotheses of the lemma are satisfied for $j+1$. Assume that $X = \{x_\alpha\colon \alpha<\omega_i\}$ and $Y = \{y_\alpha\colon \alpha<\omega_i\}$ and that both sets are enumerated in their natural ordering. Let

$a \in A$, then we will show that $\langle A, a\rangle_j$ is realized in B. There is an x_β with $a \leqslant x_\beta$. Suppose $x_\alpha < a$; then clearly there is a $b \in (y_\alpha, y_\beta]$ such that $(x_\alpha, a) \overset{j}{\sim} (y_\alpha, b)$ and $(a, x_\beta) \overset{j}{\sim} (b, y_\beta)$ since the conditions (2) and (3) imply $(x_\alpha, x_\beta) \overset{j+1}{\sim} (y_\alpha, y_\beta)$ for all $\alpha, \beta < \omega_i$. By the induction hypothesis it follows that $A^{\geqslant x_\beta} \overset{j}{\sim} B^{\geqslant y_\beta}$; hence $\langle A, a\rangle \overset{j}{\sim} \langle B, b\rangle$. If $\langle A, a\rangle_j$ occurs at least ω_l-many times in A, $l \leqslant m$, then either there is an x_α such that ω_l-many realizations of $\langle A, a\rangle_j$ are smaller than x_α or the set of realizations is cofinal. By similar considerations as above we infer, that $\langle A, a\rangle_j$ has at least ω_l-many realizations also in B. Q.E.D.

Suppose A is an ordered set and X is a cofinal n-homogeneous subset of A with order type ω_j, $j \leqslant m$. For every ordinal $\alpha < \omega_j$ we define $A_\alpha = \{x \in A\colon x \leqslant x_\alpha$ and for all $\beta < \alpha$ $x_\beta < x\}$ and $A_\alpha^<$ is A_α without the greatest element. Clearly, $\{A_\alpha\colon \alpha < \omega_j\}$ defines a splitting on A of order type ω_j and in every part A there exists a greatest element, namely x_α. Then let $B(A)$ be an ordered set with the properties:

(1) there is a splitting on $B(A)$ of order type ω_j $\{B_\alpha\colon \alpha < \omega_j\}$,

(2) $A_0 \overset{n}{\sim} B_0$,

(3) every part B_α has a greatest element y_α, $Y = \{y_\alpha\colon \alpha < \omega_j\}$,

(4) for every $0 < \alpha < \omega_j$: if α is a successor ordinal, then $B_\alpha^< \overset{n}{\sim} A_1^<$; if α is a limit ordinal, then B_α admits a splitting of order type ω^* $\{B_{\alpha i}\colon i < \omega\}$ and each part of the splitting has a greatest element, and each $B_{\alpha i}^<$ is $\overset{n}{\sim}$-equivalent to $A_1^<$.

LEMMA 11. *If A and $B(A)$ have the properties described above, then*

(i) *for all $\alpha < \beta < \omega_j$ $(y_\alpha, y_\beta) \overset{n}{\sim} (x_\alpha, x_\beta)$ and*

(ii) $A \overset{n}{\sim} B(A)$.

Proof. By induction on n. The lemma is trivial for $n = 0$. Assume then that it is true for n and that there are sets A, $B(A)$, X and Y such that the properties are satisfied for $n+1$.

We prove (i) by induction on δ, $\beta = \alpha + \delta$, $\delta > 0$.

1. Case. $\delta = \gamma + 1$. Then $(y_\alpha, y_{\alpha+\delta}) = (y_\alpha, y_{\alpha+\gamma}) + \{y_{\alpha+\gamma}\} + (y_{\alpha+\gamma}, y_{\alpha+\gamma+1})$ and the lemma follows immediately from Lemma 3, the $(n+1)$-homogeneity of X and the induction hypothesis on γ.

2. Case. δ is a limit ordinal. Let $b \in (y_\alpha, y_{\alpha+\delta})$.

2.1. $b \in (y_\alpha, y_{\alpha+\gamma})$ for a certain $\gamma < \delta$. By the induction hypothesis on γ $(y_\alpha, y_{\alpha+\gamma}) \overset{n+1}{\sim} (x_\alpha, x_{\alpha+\gamma})$. Then choose an $a \in (x_\alpha, x_{\alpha+\gamma})$ such that $(y_\alpha, b) \overset{n}{\sim} (x_\alpha, a)$ and $(b, y_{\alpha+\gamma}) \overset{n}{\sim} (a, x_{\alpha+\gamma})$. By the induction hypothesis on n we have $(y_{\alpha+\gamma}, y_{\alpha+\delta}) \overset{n}{\sim} (x_{\alpha+\gamma}, x_{\alpha+\delta})$; hence $(b, y_{\alpha+\delta}) \overset{n}{\sim} (a, x_{\alpha+\delta})$, i.e. a realizes the n-type $\langle (y_\alpha, y_{\alpha+\delta}), b\rangle_n$ in $(x_\alpha, x_{\alpha+\delta})$.

2.2. $b \in B_{\alpha+\delta}$, $b \neq y_{\alpha+\delta}$. Then from the construction of $B(A)$ it follows that $b \in B_{\alpha+\delta,i}$, $i < \omega$, and by the induction hypothesis on $i < \delta$: $B_{\alpha+\delta,i} + \ldots + B_{\alpha+\delta,1} + B_{\alpha+\delta,0}^< \overset{n+1}{\sim} (x_{\alpha+\delta}, x_{\alpha+\delta+i})$ (X is $(n+1)$-homogeneous). Then choose a $c \in (x_{\alpha+\delta}, x_{\alpha+\delta+1})$

such that $(x_{\alpha+\delta}, c) \overset{n}{\sim} B_{\alpha+\delta,i}^{<b}$ and $(c, x_{\alpha+\delta+1}) \overset{n}{\sim} B_{\alpha+\delta,i}^{>b}$. Since $\{B_{\alpha k}\colon k < \omega\}$ has order type ω^*, we infer from the induction hypothesis on n that

$$\bigcup_{\lambda<\delta} (y_\alpha, y_{\alpha+\lambda}) \cup \bigcup_{k>i} B_{\alpha+\delta,k} \overset{n}{\sim} (x_\alpha, x_{\alpha+\delta}) .$$

Now, as above, c and b realize the same n-type. Since $(x_\alpha, x_{\alpha+\delta}) \overset{n+1}{\sim} (x_\alpha, x_{\alpha+\delta+i})$, we can find an $a \in (x_\alpha, x_{\alpha+\delta})$ which realizes the n-type of c and b.

Thus we have proved that every n-type, which is realized in $(y_\alpha, y_{\alpha+\delta})$ is realized also in $(x_\alpha, x_{\alpha+\delta})$. In the same way we prove that every n-type which is realized in $(x_\alpha, x_{\alpha+\delta})$ is realized in $(y_\alpha, y_{\alpha+\delta})$.

From the proof above it easily follows that some n-type $\langle X, x\rangle_n$ is realized at least ω_i-many times, $i \leqslant m$, in $(x_\alpha, x_{\alpha+\delta})$ iff it is realized at least ω_i-many times in $(y_\alpha, y_{\alpha+\delta})$, hence (i) is valid for $n+1$.

(i) implies that Y is also $(n+1)$-homogeneous; thus (ii) is a consequence of Lemma 10. Q.E.D.

LEMMA 12. *Let n be a natural number and let A be an ordered set of cardinality ω_j, $j \leqslant m$. If every bounded segment of A is n-good, then A is n-good.*

Proof. We may assume that A has a least element (the proof is similar in the other cases: if A has a greatest element, then we use an analogue of Lemma 9 for the inverse ordering; if A has no least and no greatest element, then we partition $A = B + C + D$, where B has a greatest and D has a least element, C is bounded).

Clearly, there is a cofinal well-ordering $A' \subseteq A$. If A' is finite, then A is n-good by Lemma 3.

We can assume that A' has order type ω_i, $i \leqslant j$. By Lemma 9 there is an n-homogeneous subset $X \subseteq A$ of order type ω_i. Let B be an ordered set with the properties:

(1) there is a splitting of order type ω_i on B $\{B_\alpha\colon \alpha < \omega_i\}$,

(2) $A_0 = B_0$,

(3) for every $0 < \alpha < \omega_i$: if α is a successor ordinal, then B_α is isomorphic to A_1; if α is a limit ordinal, then B_α admits a splitting of order type ω^* $\{B_{\alpha k}\colon k < \omega\}$ and each part $B_{\alpha k}$ is isomorphic to A_1.

Then B satisfies the conditions about $B(A)$ in Lemma 11. Thus we can conclude that $A \overset{n}{\sim} B$. It is immediately seen that B admits a splitting of order type $1 + \omega + (\omega^* + \omega) \cdot \omega_i$ with every part different from the least isomorphic to A_1, i.e. $A \overset{n}{\sim} A_0 + A_1 \omega + A_1 (\omega^* + \omega) \cdot \omega_i$, where A_0 and A_1 are bounded and n-good. Then A is n-good by Lemma 8. Q.E.D.

LEMMA 13. *Let n be a fixed natural number, then every ordered set is n-good.*

Proof. By the Löwenheim–Skolem theorem for $L^m(Q)$ we may assume that A is an ordered set of cardinality ω_j, $j \leqslant m$ (Lemma 1(a) implies that elementary substructures are $\overset{n}{\sim}$-equivalent).

We define an equivalence relation $\approx$ on A:

$x \approx y$ iff every segment of the closed interval $[x, y]$ is n-good.

1. $\approx$ *is an equivalence relation with the splitting property.*

Proof. By the definition of $\approx$ and Lemma 8.

2. *Every equivalence class C has the property that each segment of C is n-good.*

Proof. By Lemma 12.

3. *Let $\mathfrak{A}$ be the splitting determined by $\approx$. $\mathfrak{A}$ has order type $\underline{1}$, $|\mathfrak{A}| = \{A\}$.*

Proof. Assume that $\mathfrak{A}$ has at least two elements C, D and $C<D$. Between any two elements $C, D \in |\mathfrak{A}|$ there is an element $E \in |\mathfrak{A}|$, $C<E<D$.

(Proof. Suppose there is no element between C and D. Let $x \in C$ and $y \in D$ and let G be a segment of $[x, y]$. If G belongs to C or D, then G is n-good by 2. In the other case C and D form a splitting for G, the parts $G \cap C$ and $G \cap D$ of the splitting are n-good; by Lemma 8 G is n-good. Thus $x \approx y$, which contradicts $x \in C$, $y \in D$ and $C<D$.)

Let $F_i(C, D) = \langle F_0^i, \ldots, F_{n_i}^i\rangle$ be a sequence of ordered sets such that every n-type F_k^i occurs at least ω_i-many times between C and D and every n-type which occurs at least ω_i-many times between C and D is $\overset{n}{\sim}$-equivalent to a certain F_k^i. Assume that we have chosen C and D so that $n_0, \ldots, n_j$ are minimal. Let $x \in C$ and $y \in D$. Consider a segment B of the closed interval $[x, y]$ of A. We prove that B is n-good. This is true if B is a segment of a certain $E \in |\mathfrak{A}|$. Otherwise B is of the form $B_1 + \bigcup \mathfrak{B} + B_2$, where $\mathfrak{B}$ is some segment of (C, D) mod $\mathfrak{A}$ and B_1, B_2 are (possibly empty) segments of some classes $C_1', D_1' \in |\mathfrak{A}|$.

$\bigcup \mathfrak{B}$ denotes the corresponding segment of A. By the minimality of $n_0, \ldots, n_j$ $F_i(C', D') = F_i(C, D)$ for all $C', D' \in |\mathfrak{B}|$ with $C'<D'$ mod $\mathfrak{A}$ and $i \leqslant i$. By 2, every F_l^i, $i \leqslant j$ and $0 \leqslant l \leqslant n_i$, is $\overset{n}{\sim}$-equivalent to some element of M_{kmf}; thus we can assume $F_l^i \in M_{kmf}$.

Now let i be the greatest natural number such that there is some n-type E which occurs ω_i-many times in $\mathfrak{B}$, but not ω_{i+1}-many times. Then $\mathfrak{B}$ has a dense subset of cardinality ω_i. Assume $\mathfrak{B}$ has no least or greatest element. Then by Lemma 7 $\bigcup \mathfrak{B} \overset{n}{\sim} \sigma_{if}(\mathscr{F})$, $\mathscr{F} = \langle F_i(C, D), \ldots, F_{f(i)}(C, D)\rangle$; hence B is n-good. If $\mathfrak{B}$ has a least or greatest element, then omit the endpoints of $\mathfrak{B}$ and argue as before (the segment without endpoints has no least or greatest element, since in $\mathfrak{A}$ no element has an immediate successor or predecessor). We used also the fact that "+" is compatible with the relation $\overset{n}{\sim}$ (Lemma 8). We have proved that every segment of the closed interval $[x, y]$ is n-good. Hence $x \approx y$ which contradicts the assumption $C<D$.

Thus there is only one equivalence class A which is $\overset{n}{\sim}$-equivalent to some element of M_{kmf} (by 2).

THEOREM 1. *Every sentence $H \in L^m(Q)$ which has an ordered set as a model has a model in M_{kmf}.*

Proof. Let A be an ordered set which is a model of H, and let n be the number of quantifiers in H. By Lemma 13, $A \overset{n}{\sim} \alpha$ for a certain $\alpha \in M_{kmf}$, and by Lemma 1(b) $\alpha \models H$.

§ 3. In proving the decidability of certain classes of orderings in $L^m(Q)$ some additional difficulties arise, since the logic of $L^m(Q)$ is not axiomatizable. Thus we will prove decidability results by methods which differ from those used in [4] and [9]. In § 1 we proved the existence of mappings σ_{if}. It is clear that we can choose σ_{if} so that certain parts of σ_{if} are effectively calculable. Though we do not want to specify this part of σ_{if} (e.g. $\sigma_{if}(\omega)$ should be effectively calculable), it will be clear from the proof how this has to be done. Greek letters $\alpha, \beta, \ldots$ will denote terms which represent the elements of M_{kmf}. $\beta < \cdot \alpha$ denotes that α is one of the terms $\beta+\gamma$, $\gamma+\beta$, $\beta\cdot\omega_i$, $\beta\cdot\omega_i^*$, $i \leqslant m$, or $\sigma_{if}(\mathscr{F})$ with $\mathscr{F} = \langle F_i, \ldots, F_{f(i)}\rangle$ and for a certain l $\beta \in |F_l|$. The set $\{\beta\colon \beta < \cdot \alpha\}$ is finite for each $\alpha \in M_{kmf}$. An element $a \in \alpha$ is called *effectively calculable* if it is given in a constructive manner, i.e. there is an effective procedure which produces a. All constants used in the following should be effectively calculable. The equivalence relation $\overset{n}{\sim}$ (Definition 1) is expressible in the language $L^m(Q)$. From this we get the following lemma:

LEMMA 14. *There is a unary recursive function $g(n)$ such that for all A, all $a \in A$, and for all $n<\omega$:*

(i) *there is some sentence $H \in L^m(Q)$ such that for all B: $A \overset{n}{\sim} B$ iff $B \models H$, and H has at most $g(n)$ quantifiers,*

(ii) *there is some formula $H_1(x) \in L^m(Q)$ with at most $g(n)$-many quantifiers such that for all B and all $b \in B$: $\langle A, a\rangle \overset{n}{\sim} \langle B, b\rangle$ iff $B \models H_1(b)$.*

THEOREM 2. (i) *For every term $\alpha \in M_{kmf}$ and every sentence $H(a_1, \ldots, a_j)$, $H(x_1, \ldots, x_j) \in L^m(Q)$, there is an effective decision procedure which decides "$\alpha \models H(a_1, \ldots, a_j)$".*

(ii) *If $\alpha \models \exists x H(x, a_1, \ldots, a_j)$, then there is an effective procedure which produces a certain $b \in \alpha$ such that $\alpha \models H(b, a_1, \ldots, a_j)$.*

Proof. By induction on α. If $\alpha = \underline{1}$, the theorem follows at once. Let α be given. Then we prove the theorem by induction on H. If H is atomic, then the theorem follows immediately. In case H has the form $H_1 \wedge H_2$, $H_1 \vee H_2$, or $\neg H_1$, Theorem (i) is a consequence of the induction hypothesis on H_1 and H_2.

1. Case $H(a_1, \ldots, a_j) = \exists x H'(x, a_1, \ldots, a_j)$.

1.1. $\alpha = \beta+\gamma$. Suppose H' has n quantifiers. We assume $a_1, \ldots, a_j$ are ordered as follows: $a_1 < a_2 < \ldots < a_j$. Let A be the interval (a_i, a_{i+1}) in β. By Lemma 14(ii) there are finitely many $H_1'(x), \ldots, H_s'(x)$, so that for every $a \in A$ $A \models H_r'(a)$ for a certain $H_r'(x)$, $1 \leqslant r \leqslant s$, and for every $H_r'(x)$ $A \models \exists x H_r'(x)$; moreover, condition (ii) of Lemma 14 is satisfied and $H_1'(x), \ldots, H_s'(x)$ are obtained constructively. If $a, b \in A$ and $A \models H_r'(a)$ and $A \models H_r'(b)$, then by Lemma 14(ii), Lemma 3 and Lemma 1(ii) a and b satisfy exactly the same formulae with at most n quantifiers and the constants $a_1, \ldots, a_j$ in α. By the induction hypothesis on β there are effective procedures which produce b_r, $1 \leqslant r \leqslant s$, such that $A \models H_r'(b_r)$. In this way we get for every interval (a_i, a_{i+1}) effectively calculable elements b_r^i, and similarly for $\beta^{<a_1}$ and $\gamma^{>a_j}$ there are effectively calculable elements b_r^0 and b_r^{j+1}, respectively. Now, the set

$\{b_r^i: 0 \leqslant i \leqslant j+1, 1 \leqslant r \leqslant s_i\}$ is finite and for every b_r^i there is (by the induction hypothesis) a decision procedure for "$\alpha \models H'(b_r^i, a_1, \ldots, a_j)$". From the construction it follows that: if $\alpha \models \exists x H'(x, a_1, \ldots, a_j)$, then for a certain b_r^i $\alpha \models H'(b_r^i, a_1, \ldots, a_j)$; thus we have a decision procedure for "$\alpha \models \exists x H'(x, a_1, \ldots, a_j)$".

1.2. $\alpha = \beta \cdot \omega$. Clearly, $\beta \cdot \omega$ is isomorphic to $\beta \cdot m_1 + \beta \cdot \omega$. Then we can assume that all constants $a_1, \ldots, a_j$ occur in $\beta \cdot m_1$. Now, by the definition of α, α admits a splitting of type ω and each part of the splitting is isomorphic to β. In the same way as in case 1.1, there are effective procedures which produce elements b_r^i, $1 \leqslant i \leqslant m_1 + 2^n$ and $0 \leqslant r \leqslant s_i$, and b_r^i belongs to the ith part of the splitting. From Lemma 6 it follows that, for every element $b \in \alpha$ which belongs to some part m_2, $m_1 + 2^n < m_2$, there is a certain b_r^i with $\langle \alpha, b \rangle \overset{n}{\sim} \langle \alpha, b_r^i \rangle$. Thus, as in case 1.1, we get a decision procedure.

1.3. $\alpha = \beta \cdot \omega_i$, $1 \leqslant i \leqslant m$. Analogue of case 1.2.

1.4. $\alpha = \sigma_{if}(\mathscr{F})$, $\mathscr{F} = \langle F_i, \ldots, F_{f(i)} \rangle$. By the definition of α, α admits a splitting of type σ_{if}. Regard those parts which contain some constant of $a_1, \ldots, a_j$. Since these constants are effectively calculable, these parts are effectively given. As before, case 1.1, there are effective procedures which produce elements b_r^i. Now, for every $\beta \in |F_i| \cup \ldots \cup |F_{f(i)}|$ we choose effectively finitely many parts of the splitting, all of type β, such that between any two constants of $a_1, \ldots, a_j$, say a_i and a_{i+1}, there is an effectively chosen part (if a_i and a_{i+1} do not belong to the same part). Moreover, we choose effectively two parts of type β which are greater and smaller, respectively, than any of the parts which contain some of the constants $a_1, \ldots, a_j$. For all these parts we have also procedures which produce elements b_r^i. From Lemma 7 it follows that: for every $b \in \sigma_{if}(\mathscr{F})$ there is a certain b_r^i such that $\langle \sigma_{if}(\mathscr{F}), b \rangle \overset{n}{\sim} \langle \sigma_{if}(\mathscr{F}), b_r^i \rangle$. As in case 1.1 we get a decision procedure.

2. Case $H(a_1, \ldots, a_j) = Q_i x H'(x, a_1, \ldots, a_j)$, $0 \leqslant i \leqslant m$.

For simplicity let $i = 1$. Then by the definition of α there is a splitting on α. For $\alpha \models Q_1 x H'(x, a_1, \ldots, a_j)$ to be valid, there are two possibilities:

(1) there are uncountably many parts of the splitting which contain an element b with $\alpha \models H'(b, a_1, \ldots, a_j)$, or

(2) some part contains uncountably many elements b with $\alpha \models H'(b, a_1, \ldots, a_j)$.

ad (1): This is without sense if the splitting of α has only countably many parts. Thus α must have the form $\beta \cdot \omega_i$ or $\sigma_{if}(\mathscr{F})$, $1 \leqslant i \leqslant m$. Let α be $\beta \cdot \omega_1$. In the decision procedure for $\alpha \models \exists x H'(x, a_1, \ldots, a_j)$ (case 1) we had to decide $\alpha \models H'(b_r^i, a_1, \ldots, a_j)$. By Lemma 6, it follows that uncountably many parts contain a certain b with $\alpha \models H'(b, a_1, \ldots, a_j)$ iff $\alpha \models H'(b_r^i, a_1, \ldots, a_j)$ for some b_r^i, $i = m_1 + 2^n$. However, the right side of this equivalence is decidable by the induction hypothesis.

The other cases for α are similar.

ad (2): In case 1 we decided $\alpha \models \exists x H'(x, a_1, \ldots, a_j)$ by deciding $\alpha \models H'(b_r^i, a_1, \ldots, a_j)$ for effectively calculable elements b_r^i. Every b_r^i was determined by an effective procedure which produced an element such that $\beta \models H_r^i(b_r^i)$. Then from case 1.1 if follows that in one part β there are uncountably many elements b such that $\alpha \models H'(b, a_1, \ldots, a_j)$ iff for a certain $H_r^i(x)$ $\beta \models Q_1 x H_r^i(x)$. However, the latter proposition is decidable.

Thus we have proved (i) of Theorem. (ii) follows automatically from case 1.1. Q.E.D.

THEOREM 3. *The set of all sentences $H \in L^m(Q)$ such that for all terms $\alpha \in M_{kmf}$ $\alpha \models H$ is recursive.*

Proof. This follows from Theorem 2 and the following fact. By the definition of M_{kmf}, M_{kmf} is generated by $\underline{1}$ with the operations "+", "$\cdot \omega_i$", "$\cdot \omega_i^*$" and "$\sigma_{if}(\mathscr{F})$", $0 \leqslant i \leqslant m$. By Lemma 3 we infer that for every n $\overset{n}{\sim}$ is an equivalence relation between ordered sets which is compatible with the operations above, i.e. $\overset{n}{\sim}$ is a congruence relation with respect to these operations. Then the factor structure $M_{kmf}/\overset{n}{\sim}$ is well-defined. Clearly, $M_{kmf}/\overset{n}{\sim}$ is generated by the equivalence class of $\underline{1}$.

Let H be given. Suppose H has n quantifiers. To decide whether $\alpha \models H$ for all $\alpha \in M_{kmf}$, we have only to decide whether for all $A \in M_{kmf}/\overset{n}{\sim}$ there is a certain $\alpha \in M_{kmf}$ such that $A = \{\beta \in M_{kmf}: \beta \overset{n}{\sim} \alpha\}$ and $\alpha \models H$. Lemma 2 implies that $M_{kmf}/\overset{n}{\sim}$ is finite. Thus $M_{kmf}/\overset{n}{\sim}$ is finite and generated by the equivalence class of $\underline{1}$. Hence we have only to decide for finitely many terms $\alpha_1, \ldots, \alpha_r$ whether $\alpha_i \models H$ or not and $\alpha_1, \ldots, \alpha_r$ can be constructed effectively. If for all i, $1 \leqslant i \leqslant r$, $\alpha_i \models H$, then H is valid in all terms $\alpha \in M_{kmf}$. Now we have a decision method which answers the question: "Is H valid in every term $\alpha \in M_{kmf}$?". Since this decision method is uniform, this is the same as saying that the set of all sentences which are valid in all terms $\alpha \in M_{kmf}$ is recursive. Q.E.D.

Now we get the decidability of many classes of orderings in the language $L^m(Q)$.

Let $\mathrm{LO}_i(k, m, f)$ be the set of all sentences of $L^m(Q)$ which are true in all orderings with cardinality greater than or equal to ω_i, and let $\mathrm{LO}(k, m, f)$ be the set of all sentences of $L^m(Q)$ which are valid in all orderings. $\mathrm{DLO}(k, m, f)$ and $\mathrm{DLO}_i(k, m, f)$ are defined similarly with the restriction, that we only regard dense linear orderings.

All the results we have proved depend on the hypothesis $T_{km}(f)$ (see § 1). Let $\mathrm{LO}(Q_0, \ldots, Q_m)$ be the set of all sentences of $L^m(Q)$ which belong to all $\mathrm{LO}(k, m, f)$ for all k and f (there are only finitely many such k and f).

THEOREM 4. (i) $\mathrm{LO}(k, m, f)$, $\mathrm{DLO}(k, m, f)$ *and for every* i, $0 \leqslant i \leqslant m$, $\mathrm{LO}_i(k, m, f)$ *and* $\mathrm{DLO}_i(k, m, f)$ *are all decidable.*

(ii) $\mathrm{LO}(Q_0, \ldots, Q_m)$ *is decidable.*

(iii) $\mathrm{LO}(Q_i: i < \omega)$ *is decidable.*

Proof. Theorem 1 implies that the set of sentences which are valid in all orderings is exactly the set of sentences which are valid in all terms $\alpha \in M_{kmf}$. Then the decidability of $\mathrm{LO}(k, m, f)$ is an immediate consequence of Theorem 3. The other theories of (i) are finite extensions of $\mathrm{LO}(k, m, f)$, hence also decidable. If m is fixed, then there are only finitely many different k and f, hence $LO(Q_0, \ldots, Q_m)$ is decidable. $LO(Q_i\colon i<\omega)$ is decidable, because the decision methods for $LO(Q_0, \ldots, Q_m)$ are uniform in m.

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Prime and coprime modules

by

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Abstract. The results of this paper generalize the notion of prime ideal. Consequently, there is defined prime module and its dual, coprime module. Similarly as the Jacobson radical is defined, we introduce the notion of prime radical. Besides the essentials of the calculus of prime and coprime modules, the main purpose of the paper is to show how these notions are related to the general theory of preradicals as a tool for structural investigation of rings and modules.

1. Introduction. In the following, R is an associative ring with unit and R-mod stands for the category of unital left R-modules. R is called *left V-ring* if all simple modules are injective. Further, R is said to be *left* (*right*) *duo-ring* if all left (right) ideals are two-sided. As usually, $E(M)$ will denote the injective hull of a module M. A submodule N of a modu!: M is called *characteristic* if $f(N)\subseteq N$ for each $f\in \mathrm{Hom}(M, M)$.

Recall that a preradical r for R-mod is a subfunctor of the identity functor. We shall say that r is

idempotent if $r(r(M)) = r(M)$ for every $M\in R$-mod,

a *radical* if $r(M/r(M)) = 0$ for every $M\in R$-mod,

hereditary if $r(N) = N\cap r(M)$ for every $N, M\in R$-mod, $N\subseteq M$,

superhereditary if it is hereditary and the class $\mathcal{T}_r$ of all r-torsion modules is closed under direct products,

cohereditary if $r(M/N) = (r(M)+N)/N$ for all $N, M\in R$-mod such that $N\subseteq M$ (in this case, $r(M) = r(R)M$ for all $M\in R$-mod),

splitting if $r(M)$ is a direct summand for each $M\in R$-mod.

Let r be an arbitrary preradical. For each $M\in R$-mod we define $\bar{r}(M) = \sum N$, $N\subseteq M$ and $r(N) = N$, and $\tilde{r}(M) = \bigcap P$, $P\subseteq M$ and $r(M/P) = 0$. It is easy to see that $\bar{r}$ is the largest idempotent preradical contained in r and $\tilde{r}$ is the least radical containing r. The definition of inclusion, sum and intersection of preradical is obvious. Further, we define $r^1 = r_1 = r$ and $r^{n+1}(M) = r(r^n(M))$, $r_{n+1}(M)/r_n(M) = r(M/r_n(M))$ for every module M.

If I is a two-sided ideal then we define a cohereditary radical r and a superhereditary preradical s corresponding to I by $r(M) = IM$ and

$$s(M) = \{m\in M \mid Im = 0\},$$