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Some combinatorial properties of ultrafilters

by

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Abstract. Three unrelated combinatorial results are proved: (1) A result relating non-regular ultrafilters to weakly normal ultrafilters; (2) A partitioning property for indecomposable ultrafilters over singular cardinals and (3) A large cardinal-type result for inaccessible cardinals carrying indecomposable ultrafilters.

0. Introduction. Our notation and terminology follows that of the more recent set-theoretic literature. In particular $\alpha, \beta, \gamma, \ldots$ are variables for ordinals while $\varkappa, \lambda, \mu, \ldots$ are reserved for cardinals. The notation $|x|$ refers to the cardinality of the set x and so on. An ultrafilter over a cardinal is always assumed to be uniform.

0.1. Definition. An ultrafilter D over $\varkappa$ is (λ, μ)-*regular* if $\lambda \leqslant \mu$ and there is a set $S \subseteq D$ of power μ such that

$$T \subseteq S \quad \text{and} \quad \lambda \leqslant |T| \rightarrow \bigcap T = 0 .$$

D is μ-*regular* if it is (ω, μ)-regular. D is *regular* if it is $\varkappa$-regular.

This concept is due to Keisler. It measures the "width" of an ultrafilter. It is diametrically opposite to the notion of completeness of ultrafilters. It is a well-known fact that the existence of suitably complete ultrafilters implies the existence of normal ultrafilters. In the case of simply non-regular ultrafilters we have to replace the condition of normality by a weaker one:

0.2. Definition. An ultrafilter D over $\varkappa$ is weakly normal if every pressing down function is bounded by a constant $<\varkappa$, i.e. if $f\colon \varkappa \to \varkappa$ s.t. $f < \mathrm{id} \pmod D$, then there is a $\xi < \varkappa$ s.t. $f \leqslant \xi \pmod D$.

Kanamori [3] was the first to show that suitably non-regular ultrafilters have weakly normal ultrafilters below them in the Rudin–Keisler order.

0.3. Definition. Given two ultrafilters D, U over $\varkappa$ say $D \leqslant_{\mathrm{RK}} U$ if there is a function $f\colon \varkappa \to \varkappa$ s.t. $f_*(U) = D$; i.e. for all $x \subseteq \varkappa$:

$$x \in D \leftrightarrow f^{-1}(x) \in U .$$

Given $f, g\colon \varkappa \to \varkappa$ say $f \leqslant_{\mathrm{RK}} g \pmod D$ if there is a function $\varphi\colon \varkappa \to \varkappa$ s.t. $f = \varphi \circ g$

(mod D). This is called the *Rudin–Keisler ordering on functions from $\varkappa$ to $\varkappa$* (mod D). There is another ordering on this set of functions:

Set $f \leqslant g$ (mod D) if $\{\alpha|\ f(\alpha) \leqslant g(\alpha)\} \in D$. We will refer to this ordering as the standard ordering on functions from $\varkappa$ to $\varkappa$ (mod D).

0.4. THEOREM (Kanamori [3]). *Suppose D is an ultrafilter over a regular cardinal $\varkappa$ with no weakly normal ultrafilters below it in the Rudin–Keisler order. Then*

(a) *If $\varkappa = \tau^+$, then D is (τ, τ^+)-regular.*

(b) (*Ketonen independently*) *D is λ-regular for all $\lambda < \varkappa$.*

Ketonen [5] used this to obtain the existence of $o^{\#}$ from a suitably non-regular ultrafilter. The motivation of this paper is to obtain combinatorial properties which hopefully can then be applies to obtain large cardinality results.

0.5. DEFINITION. An ultrafilter D over $\varkappa$ is a closed point if it extends the closed unbounded filter. A function $f\colon \varkappa \to \varkappa$ is a closed function (mod D) if $f_*(D)$ is a closed point.

0.6. DEFINITION. Given two ultrafilters D, U over $\varkappa$, their product ultrafilter is the set

$$\big\{X \subset \varkappa \times \varkappa|\ \{\alpha|\ \{B|\ (\alpha, \beta) \in X\} \in U\big\} \in D\big\}.$$

For more on products and closed points, see Ketonen [4].

The following concept is the strongest form of non-regularity we will use in this paper.

0.7. DEFINITION. An ultrafilter D over $\varkappa$ is *ϱ-indecomposable* if for any function $f\colon \varkappa \to \lambda$ $(\lambda < \varkappa)$ there is a set C such that $f^{-1}(C) \in D$ and $|C| = \varrho$. D is *indecomposable* if it is ω-indecomposable.

For more on indecomposable ultrafilters, see Prikry [6] and Silver [7].

0.8. THEOREM (Silver [7]). *Suppose $\varkappa$ is a strong limit cardinal and D an indecomposable ultrafilter over $\varkappa$. Then there is a function $\varphi\colon \varkappa \to \omega$ such that for all functions f on $\varkappa$ with range of power $< \varkappa$ we have $f \leqslant_{\mathrm{RK}} \varphi$.*

Thus, this φ induces a "maximal" partitioning of $\varkappa$ into ω pieces (mod D).

We shall also investigate indecomposable ultrafilters over singular cardinals. This is of interest since f.ex. in the constructible universe the first cardinal where uniform ultrafilters may fail to be regular is the singular cardinal ω_ω (Jensen [2]). It is known (Prikry [6]) that if there are any non-regular ultrafilters over ω_ω in L, they must be λ-indecomposable for some $\lambda < \omega_\omega$.

1. Non-regular ultrafilters. The main result of this section is the following:

1.1. THEOREM. *Suppose D is an ultrafilter over a regular cardinal $\varkappa$ with no weakly normal ultrafilters below D in the Rudin–Keisler order. Then $D \times D$ is regular.*

Given an ultrafilter D, let $\Gamma(D)$ stand for the set of all D-equivalence classes of closed functions, i.e.

$$\Gamma(D) = \{[\varphi]|\ \varphi_*(D) \text{ is a closed point}\}.$$

1.2. THEOREM. *If D is a non-regular ultrafilter over a regular carinal $\varkappa$, then every non-maximal element of $\Gamma(D)$ has a successor in $\Gamma(D)$ in the standard ordering* (mod D).

Thus, the set $\Gamma(D)$ has some well-foundedness properties. This result confirms again the analogy between non-regular ultrafilters and countably complete ultrafilters.

We shall now commence on the proofs of Theorems 1.1 and 1.2. From now on, assume that D is a non-regular ultrafilter over a regular cardinal $\varkappa$.

1.3. DEFINITION. For $C \subseteq \varkappa$ closed unbounded, let k^C be the function

$$k^C\alpha = \sup(C \cap \alpha).$$

The following lemma is a reformulation of a result of Kanamori [3]. He used it to prove that any non-regular ultrafilter has closed points below it in the Rudin–Keisler order.

1.4. LEMMA. *If $f\colon \varkappa \to \varkappa$ is unbounded* (mod D), *then there is a closed unbounded $C_0 \subseteq \varkappa$ s.t. for all closed unbounded $C \subseteq C_0$:*

$$k^C f = k^{C_0} f \,(\mathrm{mod}\, D).$$

Proof. If not, then there is a decreasing sequence of closed unbounded sets $\{C_\xi|\ \xi < \varkappa\}$ s.t. for all $\xi < \varkappa$:

$$X_\xi = \{\alpha|\ k^{C_{\xi+1}} f(\alpha) < k^{C_\xi} f(\alpha)\} \in D.$$

But then $\{X_\xi|\ \xi < \varkappa\}$ would form a regularizing family for D. ■

Observe that in this case $k^{C_0} f$ is a closed function. For partial functions $f, g\colon \varkappa \to \varkappa$ define

$$f \leqslant g \leftrightarrow \forall \alpha \in \mathrm{dom} f \cap \mathrm{dom}\, g\colon f(\alpha) \leqslant g(\alpha),$$

$$f \leqslant_{\mathrm{RK}} g \leftrightarrow \forall \alpha, \beta\colon g(\alpha) = g(\beta) \text{ and } \alpha \in \mathrm{dom} f \to \beta \in \mathrm{dom} f \text{ and } f(\alpha) = f(\beta),$$

$$f \prec g \leftrightarrow f \leqslant_{\mathrm{RK}} g \text{ and } \forall \alpha \in \mathrm{dom} f \cap \mathrm{dom}\, g\colon f(\alpha) < g(\alpha),$$

$$f < g \leftrightarrow \forall \alpha \in \mathrm{dom} f \cap \mathrm{dom}\, g\colon f(\alpha) < g(\alpha).$$

Kanamori [3] has independently proved a version of the following result.

1.6. LEMMA. *If there are no weakly normal ultrafilters below D in the Rudin–Keisler order, then there are unbounded* (mod D) *partial functions $h_\alpha\colon \varkappa \to \varkappa$ $(\alpha < \varkappa)$, ordinals $r_\alpha < \varkappa$ $(\alpha < \varkappa)$ s.t. for all $\alpha < \beta$:*

$$\mathrm{dom}\, h_\alpha \in D$$

and if

$$X = \{\delta|\ h_\alpha(\delta) \geqslant r_\beta\},$$

then

$$h_\beta | X \prec h_\alpha | X.$$

Here $f|X$ means the function f restricted to the set X.

Proof. By assumption, for every unbounded (mod D) function $h\colon \varkappa \to \varkappa$ there is a $g \prec h$ s.t. g is unbounded. Construct by induction the partial functions $h_\alpha\colon \varkappa \to \varkappa$, ordinals, $r_\alpha < \varkappa$ $(\alpha < \varkappa)$ satisfying the statements of this lemma.

Successor stage: For any α, let $r_{\alpha+1} = r_\alpha$, $h_{\alpha+1} \prec h_\alpha$ an unbounded partial function.

Limit stage: Suppose λ is a limit $<\varkappa$ and we are given h_α, r_α for $\alpha<\lambda$ satisfying the statements of the lemma. Let

$$\bar{r} = \sup\{r_\alpha|\ \alpha<\lambda\}$$

and

$$\bar{h}_\alpha = h_\alpha|\{\delta|\ h_\alpha(\delta) \geqslant \bar{r}\}.$$

Then $\bar{h}_\beta \prec \bar{h}_\alpha$ for $\alpha<\beta<\lambda$. Given any $\eta<\varkappa$, let

$$r(\eta) = \max\{\bar{r}, \sup_{\alpha<\lambda} \bar{h}_\alpha''\eta+1\}$$

$$h_{\alpha\eta} = \bar{h}_\alpha|\{\delta|h_\alpha(\delta) \geqslant r(\eta)\}.$$

First Case. For every η there is a $\eta'>\eta$ s.t.

$$\bigcup\{h_{\alpha\eta}^{-1}(h_{\alpha\eta}''\eta')|\ \alpha<\lambda\} \in D.$$

Then we can pick a strictly increasing sequence $\{\eta_\xi|\ \xi<\varkappa\}$ s.t. for all $\xi<\varkappa$

$$X_\xi = \bigcup\{h_{\alpha\eta_\xi}^{-1}(h_{\alpha\eta_\xi}''\eta_{\xi+1})|\ \alpha<\lambda\} \in D.$$

But then $\{X_\xi|\ \xi<\varkappa\}$ is a regularizing family for D: If $\xi_1<\xi_2<\ldots$ and $\theta \in X_{\xi_i}$ for all $i<\omega$, then there are $\alpha_i<\lambda$ s.t.

$$(*) \qquad h_{\alpha_i\eta_{\xi_i}}(\theta) \in h_{\alpha_i\eta_{\xi_i}}''\eta_{\xi+1} \quad (i<\omega).$$

In particular, for all $i<\omega$:

$$\theta \in \operatorname{dom} h_{\alpha_i\eta_{\xi_i}}.$$

Therefore

$$h_{\alpha_i}(\theta) \geqslant r(\eta_{\xi_i}).$$

By $(*)$, for any $i>0$, there is a $\bar{\eta}<\eta_{\xi_i}$ s.t.

$$h_{\alpha_{i-1}}(\theta) = h_{\alpha_{i-1}}(\bar{\eta}) < r(\eta_{\xi_i}).$$

Therefore, $\bar{h}_{\alpha_i}(\theta) > \bar{h}_{\alpha_{i-1}}(\theta)$. Since the $\bar{h}_\alpha$ form a $\prec$-decreasing sequence, we obtain strictly decreasing sequence $\alpha_1>\alpha_2>\ldots$, a contradiction.

Since we are assuming that D is non-regular, we must therefore have:

Second Case. There is a η_0 s.t. for all $\eta>\eta_0$:

$$\bigcup\{h_{\alpha\eta_0}^{-1}(h_{\alpha\eta_0}''(\eta))|\ \alpha<\lambda\} \notin D.$$

Set $r_\lambda = r(\eta_0)$ and

$$\bar{\bar{h}}_\alpha = h_{\alpha\eta_0} \quad (\alpha<\lambda).$$

Hence, for every $\eta<\varkappa$:

$$(**) \qquad \bigcup\{\bar{\bar{h}}_\alpha^{-1}(\bar{\bar{h}}_\alpha''\eta)|\ \alpha<\lambda\} \notin D.$$

Define an equivalence relation $\sim$ on $\varkappa$ via:

$$\eta \sim \eta' \leftrightarrow \exists\alpha<\lambda\colon \bar{\bar{h}}_\alpha(\eta) = \bar{\bar{h}}_\alpha(\eta').$$

Define:

$$\varphi(\eta) = \min\{\bar{\bar{h}}_\alpha(\eta)|\ \alpha<\lambda\}.$$

Then $\varphi \prec \bar{\bar{h}}_\alpha$ for all $\alpha<\lambda$ and φ is unbounded (mod D): If not, by the regularity of $\varkappa$ one can find $<\varkappa$ $\sim$-equivalence classes whose union is in D; a contradiction with $(**)$. Hence, we can choose a $h_\lambda \prec \varphi$. ■

Proof of Theorem 1.1. Let D be a non-regular ultrafilter over a regular cardinal $\varkappa$ with no weakly normal ultrafilters below D in the Rudin–Keisler order. To show that $U = D \times D$ is regular we will apply the proof of Lemma 1.6. First of all, note that the definitions of $\leqslant$, $\prec$ and $\leqslant_{\mathrm{RK}}$ really do not depend on the indexing set of the ultrafilter. For example, we will define for two functions $\varphi, \psi\colon \varkappa\times\varkappa \to \varkappa$:

$$\varphi \leqslant \psi \ (\operatorname{mod} U) \leftrightarrow \{i|\ \varphi(i) \leqslant \psi(i)\} \in U.$$

Thus, when we apply our previous results we can continue with our usual notation. By way of contradiction, assume that U is non-regular.

Let f denote the projection $\varkappa\times\varkappa \to \varkappa$ to the first coordinate and g denote the projection $\varkappa\times\varkappa \to \varkappa$ to the second coordinate. Then f and g are unbounded (mod U) and for all h:

$$h \prec g \to h \text{ bounded or } f \leqslant h \ (\operatorname{mod} U).$$

By the method of proof of Lemma 1.6 we can then get partial functions h_α $(\alpha<\varkappa)$ satisfying all the statements of the lemma and the additional property

$$h_\alpha(i) > f(i) \quad (i \in \operatorname{dom} h_\alpha,\ \alpha<\varkappa).$$

Then for all $\alpha<\varkappa$

$$X_\alpha = \{i|\ h_\alpha(i) \text{ defined and } f(i)>r_\alpha\} \in U.$$

$\{X_\alpha|\ \alpha<\varkappa\}$ is then a regularizing family: If $i \in X_\alpha \cap X_\beta$, then $h_\alpha(i) \geqslant r_\beta$ so

$$h_\alpha(i) > h_\beta(i) \quad (\alpha<\beta).$$

Hence, by the well-foundedness of ordinals, every infinite intersection of the X_α's is empty. ■

To prove Theorem 1.2, we shall need an improved version of Lemma 1.4:

Lemma 1.7. *Suppose g, f_α $(\alpha<\lambda<\varkappa)$ are partial closed functions defined a.e.* (mod D) *s.t. for all $\alpha<\beta<\lambda$:*

$$g<f_\alpha \quad \textit{and} \quad f_\alpha>f_\beta.$$

For any closed unbounded C, define a partial function h^C by:

$$h^C = \min\{k^C f_\alpha|\ k^C f_\alpha > g\},$$

i.e.

$$h^C(\gamma) = \min\{k^C f_\alpha(\gamma)|\ \gamma\in \operatorname{dom} f_\alpha \cap \operatorname{dom} g \text{ and } k^C f_\alpha(\gamma) > g(\gamma)\}.$$

Then there is a closed unbounded set C_0 s.t. for all closed unbounded $C\subseteq C_0$: $h^C = h^{C_0}$ a.e. (mod D) *and h^{C_0} is a closed function* (mod D).

Proof. It is easy to see that for any such C_0, h^{C_0} is a closed function. If a C_0 of the type described does not exist, we can find a decreasing sequence $\{C_\xi|\ \xi<\varkappa\}$ of closed unbounded subsets of $\varkappa$ s.t. for all $\xi<\varkappa$

$$X_\xi = \{\alpha|\ h^{C_{\xi+1}}(\alpha) = h^{C_\xi}(\alpha) \text{ and } \alpha\in\operatorname{dom} h^{C_{\xi+1}} \cap \operatorname{dom} h^{C_\xi}\}\in D.$$

For a closed unbounded set C, define a partial function t^C by setting:

$$t^C(\gamma) = \text{least } \alpha<\lambda \text{ s.t. } k^C f_\alpha(\gamma) = h^C(\gamma).$$

Note that the functions h^C have the following important property: If $C\subseteq E\subseteq\varkappa$ are closed unbounded and $h^C(\gamma)\neq h^E(\gamma)$, then either:

$$h^C(\gamma)<h^E(\gamma) \quad\text{or}\quad t^C(\gamma)<t^E(\gamma).$$

Moreover, for any $C\subseteq E$:

$$t^C\leqslant t^E.$$

We can now prove that $\{X_\xi|\ \xi<\varkappa\}$ is a regularizing family for D: If not, some infinite intersection is non-empty. For the sake of notational convenience, assume that there is a $\theta<\varkappa$ s.t.

$$\theta\in\bigcap\{X_i|\ i<\omega\}.$$

First of all, for all $i<j<\omega$: Since $C_j\subseteq C_i$, $t^{C_i}(\theta)\geqslant t^{C_{i+1}}(\theta)\geqslant t^{C_j}(\theta)$. If $t^{C_i}(\theta) = t^{C_j}(\theta)$, then $t^{C_i}(\theta) = t^{C_{i+1}}(\theta)$ and therefore $h^{C_j}(\theta)\leqslant h^{C_{i+1}}(\theta)<h^{C_i}(\theta)$.

Therefore, for all $i<j<\omega$: Either

$$h^{C_i}(\theta)>h^{C_j}(\theta) \quad\text{or}\quad t^{C_i}(\theta)>t^{C_j}(\theta).$$

But this is clearly impossible by Ramsey's theorem. ■

Proof of Theorem 1.2. By way of contradiction, assume that D is non-regular and that $[g]$ is a non-maximal element of $\Gamma(D)$ with no successor in $\Gamma(D)$ in the standard ordering (mod D).

Construct inductively a decreasing sequence of closed unbounded sets $C_\alpha\subseteq\varkappa$ $(\alpha<\varkappa)$ and partial closed functions h_α $(\alpha<\varkappa)$ s.t.

$$\operatorname{dom} h_\alpha\in D,\quad h_\alpha>g,$$
$$\operatorname{range} h_\alpha\subseteq C_\alpha\quad(\alpha<\varkappa)$$

and for $\alpha<\beta<\varkappa$:

$$h_\beta<h_\alpha \quad\text{on}\quad \{\delta|\ h_\alpha(\delta)\in C_\beta\}.$$

Successor stage: Pick $h_{\alpha+1}$ s.t. $g<h_{\alpha+1}<h_\alpha$ and the range of $h_{\alpha+1}$ is $\subseteq C_\alpha$ where $C_{\alpha+1} = C_\alpha$.

Limit stage. Suppose we are given $\{h_\alpha|\ \alpha<\lambda\}$ and $\{C_\alpha|\ \alpha<\lambda\}$ satisfying the above statements and $\lambda<\varkappa$ is a limit ordinal. Let $\bar{C} = \bigcap\{C_\alpha|\ \alpha<\lambda\}$ and

$$\bar{h}_\alpha = h_\alpha|\ h_\alpha^{-1}(\bar{C})\quad(\alpha<\lambda).$$

Applying Lemma 1.7 to the $\bar{h}_\alpha$ we get a closed function h and a closed unbounded set $C\subseteq\bar{C}$ s.t. $g<h$ and

$$h(\delta)\leqslant\bar{h}_\alpha(\delta) \quad\text{if}\quad k^C\bar{h}_\alpha(\delta)>g(\delta)\quad(\delta<\varkappa).$$

Now let C_λ be the set of limit points of C and select a closed (partial) function h_λ s.t.

$$g<h_\lambda<h$$

and

$$\operatorname{range}(h_\lambda)\subseteq C_\lambda.$$

Given the sequences $(h_\alpha|\ \alpha<\varkappa)$, $(C_\alpha|\ \alpha<\varkappa)$ let E be the diagonal intersection of C_α's:

$$E = \{\alpha|\ \forall\beta<\alpha\colon \alpha\in C_\beta\}.$$

Let

$$N_\alpha = h_\alpha|h_\alpha^{-1}(E).$$

Then the functions N_α $(\alpha<\varkappa)$ are partial closed functions s.t.

$$\operatorname{dom} N_\alpha\in D,\quad N_\alpha>g\quad(\alpha<\varkappa)$$

and for $\alpha<\beta<\varkappa$:

$$N_\beta<N_\alpha \quad\text{on}\quad \{\delta|\ h_\alpha(\delta)\geqslant\beta\}\supseteq\{\delta|\ g(\delta)\geqslant\beta\}\cap\operatorname{dom} N_\alpha.$$

We can now go on to produce a regularizing family for D as in the proof of Theorem 1.1. ■

OPEN PROBLEM. Does the existence of a non-regular ultrafilter imply the existence of weakly normal ultrafilters?

2. Indecomposable ultrafilters over singular cardinals. In this section, we shall derive some partitioning properties for indecomposable ultrafilters over singular cardinals of cofinality ω. These results have natural generalizations to singular cardinals of other cofinalities and ultrafilters which are indecomposable from some point on. However, for the sake of convenience we shall stay with case cofinality ω.

From now on, assume that D is an indecomposable ultrafilter over a strong limit cardinal λ. By the result of Silver (Theorem 0.8) we have a "maximal" function $\varphi\colon\lambda\to\omega$.

Let U be the ultrafilter $\varphi_*(D)$ and F the filter generated by the sets

$$\varphi^{-1}(S)\quad(S\in U).$$

2.1. PROPOSITION. *If $\gamma<\lambda$ and $\langle X_\alpha|\ \alpha<\gamma\rangle$ is a sequence of subsets of λ, then there is an $X\in D$ and sets $S_\alpha\subseteq\omega$ $(\alpha<\gamma)$ such that*

$$X_\alpha\cap X=\varphi^{-1}(S_\alpha)\cap X\quad(\alpha<\gamma).$$

Proof. Define a new function ψ by setting

$$\psi(\gamma)=\{\alpha|\ \gamma\in X_\alpha\}\quad(\gamma<\lambda).$$

We can find an $X\in D$ so that

$$\psi\leqslant_{\mathrm{RK}}\varphi\quad\text{on } X$$

i.e.,

$$\alpha,\beta\in X\wedge\varphi(\alpha)=\varphi(\beta)\to\psi(\alpha)=\psi(\beta)\to\forall\delta<\gamma:\ \alpha\in X_\delta\leftrightarrow\beta\in X_\delta.\ \blacksquare$$

2.2. COROLLARY. *If $\gamma<\lambda$ and $\langle X_\alpha|\ \alpha<\gamma\rangle$ is a sequence of subsets of D, then there is an $X\in D$ and sets $G_\alpha\in F$ $(\alpha<\gamma)$ such that*

$$X\cap G_\alpha\subseteq X_\alpha.$$

From now on, assume that $\mathrm{cof}(\lambda)=\omega$ and $\{\lambda_i|\ i<\omega\}$ is a fixed cofinal sequence of cardinals in λ.

2.3. PROPOSITION. *If $X_\alpha\in D$ for $\alpha<\lambda$, then there is an $X\in D$ such that for any $\alpha<\lambda$ there is a $G_\alpha\in F$ with*

$$X\cap G_\alpha\subseteq X_\alpha.$$

Proof. By Corollary 2.2, for any $i<\omega$ there is an $X_i\in D$ and sets $G_\alpha^i\in F$ $(\alpha<\lambda_i)$ such that

$$X_i\cap G_\alpha^i\subseteq X_\alpha\quad(\alpha<\lambda_i).$$

Applying Corollary 2.2 to the X_i, we find an $X\in D$ such that for any $i<\omega$ there is a $G_i\in F$ with

$$X\cap G_i\cap G_\alpha^i\subseteq X_\alpha\quad(\alpha<\lambda_i,\ i<\omega).\ \blacksquare$$

2.4. PROPOSITION. *If $\beta_0<\lambda$ and $F\colon[\lambda]^2\to\beta_0$, then there is a set $X\in D$, countable set $S\subseteq\beta_0$, and an unbounded function $t\colon\omega\to\omega$ (mod U) such that for $\alpha,\beta\in X$*

$$\varphi(\alpha)\leqslant t(\varphi(\beta))\to F(\{\alpha,\beta\})\in S.$$

Proof. First of all, there is a set $Y\in D$ such that for any $\alpha<\lambda$ there is a $G_\alpha\in F$ and a countable set $S_\alpha\subseteq\beta_0$ with

$$F_\alpha''(Y\cap G_\alpha)\subseteq S_\alpha\quad(\alpha<\lambda),$$

where

$$F_\alpha(\beta)=F(\{\alpha,\beta\}).$$

By indecomposability, there is a set $Z\subseteq Y$, $Z\in D$ and sets $G_i'\in F$ $(i<\omega)$, countable $S\subseteq\beta_0$ such that for $\alpha\in Z$

$$F_\alpha''(Y\cap G'_{\varphi(\alpha)})\subseteq S.$$

We can without a loss of generality assume that

$$G_1'\supseteq G_2'\supseteq G_3'\supseteq\ldots,\quad\text{and}\quad\bigcap G_k'=0.$$

We have obtained: For $\alpha,\beta\in Z$:

$$\beta\in G'_{\varphi(\alpha)}\to F(\{\alpha,\beta\})\in S.$$

Now, let

$$u(\gamma)=\max\{k|\ \gamma\in G_k'\}.$$

We know that $u=t\circ\varphi$ for some $t\colon\omega\to\omega$ on a set $X\in D$, $X\subseteq Z$. Then for $\alpha,\beta\in X$

$$\varphi(\alpha)\leqslant t(\varphi(\beta))\to F(\{\alpha,\beta\})\in S.\ \blacksquare$$

2.5. PROPOSITION. *There is a fixed unbounded function $t\colon\omega\to\omega$ (mod U) s.t. for any $F\colon[\lambda]^2\to\varrho$, $\varrho<\lambda$ there is a set $X\in D$ and a countable set $S\subseteq\varrho$ s.t.*

$$\varphi(\alpha)\leqslant t(\varphi(\beta))\to F(\{\alpha,\beta\})\in S.$$

Proof. First of all, by ω-regularity of $\cup$, we can assume that ϱ is fixed. If no such t exists, then for every t we would have a "counterexample" $F_t\colon[\lambda]^2\to\varrho$. Let $F=(F_t|t\colon\omega\to\omega$ unbounded (mod D)). Then F is a partitioning of $[\lambda]^2$ into $<\lambda$ pieces and clearly contradicts Proposition 2.4. $\blacksquare$

A simple iteration of the above arguments and another ω-regularity argument then yield the main result of this section:

2.6. THEOREM. *There is a fixed unbounded function $p\colon\omega\to\omega$ s.t. for any $n<\omega$, partitioning F of $[\lambda]^n$ into $<\lambda$ pieces there is a set $X\in D$ and a countable set S s.t. for all $\alpha_1,\ldots,\alpha_n\in X$:*

$$\forall i<n\colon p(\varphi(\alpha_i))\leqslant\varphi(\alpha_{i+1})\to F(\{\alpha_1,\ldots,\alpha_n\})\in S.$$

Note that this does not per se say anything "new" about λ: It is well known that given such an F one can always find a set $X\subseteq\lambda$ of card λ with a property of the kind stated above.

Assuming GCH, it is an easy matter to extend this result to pressing down partitionings on the weakly normal indecomposable ultrafilters of Prikry [6].

OPEN QUESTION. Are there any indecomposable ultrafilters over ω_ω in L?

3. Indecomposable ultrafilters over inaccessible cardinals. Throughout this section we shall assume that D is a fixed weakly normal indecomposable ultrafilter over a strongly inaccessible cardinal $\varkappa$. Let $\varphi\colon\varkappa\to\omega$ stand for the maximal function of Theorem 0.8 and $U=\varphi_*(D)$. Our main result is then the following:

3.1. THEOREM. *If $\varkappa$ is weakly compact, then $\varkappa$ is Ramsey. In fact, given any $\varkappa$-field $S\subseteq P\varkappa$ of subsets of $\varkappa$ of cardinality $\varkappa$, there are $\varkappa$-complete ultrafilters $U_i\subseteq S$ s.t. D is the U-sum of the U_i's: For $X\in S$,*

$$X\in D\leftrightarrow\{i|\ X\in U_i\}\in U.$$

We shall start out with a lemma:

3.2. LEMMA. *If $\varkappa$ is weakly compact, then for every sequence $X_\alpha \subseteq \alpha$ $(\alpha<\varkappa)$, there are sets*

$$T_i \subseteq \varkappa \quad (i<\omega) \quad s.t. \quad \{\alpha|\ X_\alpha = T_{\varphi(\alpha)} \cap \alpha\} \in D.$$

Thus, in particular, $\varkappa$ is ineffable in the sense of Jensen [2].

Proof. Let $f_\xi(\alpha) = X_\alpha \cap \xi$ $(\alpha, \xi<\varkappa)$. We can find g_ξ, $X_\xi \in D$ s.t.

$$f_\xi = g_\xi \circ \varphi \quad \text{on } X_\xi \quad (\xi<\varkappa).$$

Hence, for $\xi<\eta<\varkappa$

$$C_{\xi\eta} = \{i|\ g_\xi(i) = g_\eta(i) \cap \xi\} \in U.$$

By weak compactness, there is a set $A \subseteq \varkappa$ of power $\varkappa$ and a set $C \in U$ s.t. for $\xi<\eta$, $\xi, \eta \in A$

$$C_{\xi\eta} = C \in U.$$

For $i \in C$, let

$$T_i = \bigcup \{g_\xi(i)|\ \xi \in A\}. \ ■$$

We can now proceed in a manner entirely analogous to Silver [7]. Consider the model

$$N = \Pi_D^*\langle V, \in\rangle = \{[f]_D|\ |\mathrm{range}(f)| \leqslant \omega\}.$$

Let $\varkappa^* = [\varkappa]_D$ and define an "ultrafilter" M over $\varkappa^*$ as follows:

$$[X]_D \in M \leftrightarrow \{\alpha|\ \alpha \in X(\alpha)\} \in D.$$

Following Silver [7], we can now prove:

3.3. LEMMA. *If $\varkappa$ is weakly compact, then M is an "N-ultrafilter": For all $a, b, c, d, f \in N$*

(I) $a \in M \to N \models a \subseteq \varkappa^*$ *and* $|a| = \varkappa^*$.

(II) $N \models b \cup c = \varkappa^* \to b \in M$ *or* $c \in M$.

(III) $N \models b \cap c = 0 \to b \notin M$ *or* $c \notin M$.

(IV) *If* $N \models a \in \varkappa^*$, *$f$ a function on a and* $\forall b$: $N \models b \in a \to f(b) \in M$, *then*:

$$\bigcap \{f(b)|\ b \in a\} \in M.$$

(V) *If* $N \models f$ *a function on $\varkappa^*$ and* $\{x|\ f(x)<x\}^N \in M$, *then* $\exists y$: $N \models y<\varkappa^*$ *and if* $N \models a = f^{-1}(y)$, *then* $a \in M$.

(VI) *If* $N \models f$ *is a function on $\varkappa^*$, then $\exists c \in N$ s.t. for all d*:

$$N \models d \in c \leftrightarrow N \models d<\varkappa^* \text{ and } f(d) \in M.$$

Proof. For example (VI). Following Silver [7], it suffices to show that there is a canonical isomorphism

$$H\colon (\Pi_D^* V) \cap P(\varkappa^*) \xrightarrow{\sim} \Pi_D P(\alpha).$$

But this is clearly given by Lemma 3.2. ■

3.4. THEOREM. *Assume D is an indecomposable ultrafilter over a weakly compact cardinal $\varkappa$. Then*:

(a) *If $S \subseteq P(\varkappa)$ is any $\varkappa$-field of subsets of $\varkappa$ of power $\varkappa$, then there are $\varkappa$-complete ultrafilters $U_i \subseteq S$ s.t.*

$$X \in D \cap S \leftrightarrow \{i|\ X \in U_i\} \in U.$$

(b) *If* $F\colon [\varkappa]^{<\omega} \to \lambda$, $\lambda<\varkappa$, *then there is a set $X \in D$ such that*

$$|F''[X]^{<\omega}| \leqslant \omega.$$

(c) *$\varkappa$ is Ramsey.*

Proof. (a) Applying property (VI) of Lemma 3.3 to $[S \cap D]$ we get an equivalence class $[\langle U_i|\ i<\omega\rangle]_u$ s.t. for all X:

$$X \in D \cap S \leftrightarrow \{i|\ X \in U_i\} \in U.$$

Furthermore, the U_i are clearly $\varkappa$-complete a.e. (mod U). (b), (c) follow directly from the methods of Ketonen [5]. ■

OPEN QUESTION. Is $\varkappa$ measurable in the above situation?

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