PROPOSITION 5. *If a paracompact space X has a feathering in a locally compact and locally connected space Y, then compactness and connectness is transferred into X onto small layers.*

PROPOSITION 6. *If a paracompact space X has a feathering in a paracompact p-space* $Y \in \mathrm{cl}\, c^n_G$, *then the property* $H^k(Z; G) = 0$ *is a property transferred onto small layers.*

Proof. The space X has feathering $\mathscr{P} = \{P_n: n = 1, 2, \ldots\}$ in βY. Define relations b^k_m on $\mathrm{ext}_{\beta Y}\mathscr{U}^*_X$: $(P', P) \in b^k_m$ iff $P'|X \underset{*}{\succ} P|X$, $\mathrm{cl}_{\beta Y} P' \succ P \wedge P_m$ and for each $u' \in P'$ there exists a $u \in P$, $u' \subset u$, such that the induced homomorphism $H^k(u \cap Y; G) \to H^k(u' \cap Y; G)$ is trivial. Let $\mathscr{B} = \{b^k_m: k \leqslant n, m < \infty\}$. The uniformity $\mathscr{U}^*_X$ is a $\mathscr{B}$-uniformity. We shall verify that for each $\mathscr{B}$-pseudouniformity $\mathscr{U} \subset \mathscr{U}^*_X$ we have $\tilde{H}^k([x]_{\mathscr{U}}; G) = 0$, $k \leqslant n$, $x \in X$. Notice that for each $\mathscr{B}$-pseudouniformity $\mathscr{U}$ a family $\{\mathrm{st}(x, P): P \in \mathrm{ext}_{\beta Y}\mathscr{U}\}$ is a base of neighbourhoods of $[x]_{\mathscr{U}} = [x]_{\mathrm{ext}_{\beta Y}\mathscr{U}}$, $x \in X$. Hence a family $\{\mathrm{st}(x, P|Y): P \in \mathrm{ext}_{\beta Y}\mathscr{U}\}$ is also a neighbourhood base of $[x]_{\mathscr{U}}$. Now, from the definition of the relations b^k_m it follows that for each neighbourhood $u \in P \in \mathrm{ext}_{\beta Y}\mathscr{U}$ of $[x]_{\mathscr{U}}$ there exists a neighbourhood $u' \in P' \in \mathrm{ext}_{\beta Y}\mathscr{U}$, $(P', P) \in b^k_m$, such that $u' \cap Y \subset u \cap Y$ and the induced homomorphism $H^k(u \cap Y; G) \to H^k(u' \cap Y; G)$ is trivial. By Theorem 6.6.2 from [7] it follows that $\tilde{H}^k([x]_{\mathscr{U}}; G) = 0$.

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Accepté par la Rédaction le 5. 9. 1977

The category of abelian Hopf algebras

by

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Abstract. By abelian Hopf algebra we mean a commutative, cocommutative, connected, graded Hopf algebra over a field. In this paper we investigate the category $\mathfrak{H}$ of all abelian Hopf algebras and the full subcategory $\mathfrak{L}$ of $\mathfrak{H}$ consisting of all primitively generated Hopf algebras. In particular we give a complete description of injective objects in categories $\mathfrak{L}$ and $\mathfrak{H}$ and we prove that $\mathrm{gl.dim}\,\mathfrak{L} = 1$ and $\mathrm{gl.dim}\,\mathfrak{H} = 2$.

Introduction. Let K be an arbitrary field. A graded Hopf K-algebra which is commutative, cocommutative and connected will be called an *abelian Hopf algebra* (see [10], [18]). Denote by $\mathscr{H}$ the category of all abelian Hopf algebras. Recall that $\mathscr{H}$ is a locally noetherian Grothendieck category and an object H in $\mathscr{H}$ is noetherian if and only if H is finitely generated as a K-algebra (see [7], [10]). The tensor product $\otimes$ over K is the coproduct in $\mathscr{H}$. Let p be the characteristic of K. If $p = 0$ then $\mathrm{gl.dim}\,\mathscr{H} = 0$ (see [10]). Assume $p \geqslant 2$. In [10] Schoeller showed that $\mathscr{H} = \mathscr{H}^- \times \mathscr{H}^+$ where $\mathscr{H}^-$ is the full subcategory of $\mathscr{H}$ consisting of all Hopf algebras generated by elements of odd degrees and $\mathscr{H}^+$ consists all Hopf algebras which are zero in odd degrees. Furthermore, $\mathrm{gl.dim}\,\mathscr{H}^- = 0$ and $\mathscr{H}^+$ is a product of countably many $\prec$ categories each of which is equivalent to the full subcategory $\mathscr{H}_1$ of $\mathscr{H}^+$ consisting of all Hopf algebras generated by elements of degrees $2p^i$ where $i = 0, 1, 2, \ldots$

Let H be an object in $\mathscr{H}$ and Δ the comultiplication of H. An element x of H will be called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. From Theorem 6.3 in [7] it follows that each subobject of a primitively generated abelian Hopf algebra is also primitively generated. Denote by $\mathscr{L}$ (resp. $\mathscr{L}^-$, $\mathscr{L}^+$, $\mathscr{L}_1$) the full subcategory of $\mathscr{H}$ (resp. $\mathscr{H}^-$, $\mathscr{H}^+$, $\mathscr{H}_1$) consisting of all primitively generated Hopf algebras. Then $\mathscr{L}$ is a locally noetherian Grothendieck category, $\mathscr{L} = \mathscr{L}^- \times \mathscr{L}^+$ and $\mathscr{L}^+$ is a product of countably many categories each of which is equivalent to the category $\mathscr{L}_1$.

Let $\mathscr{K}$-GrMod denote the category of graded K-modules and let

$$P\colon \mathscr{H} \to K\text{-GrMod}$$

be the functor which assigns to each H from $\mathscr{H}_1$ the graded K-module $P(H)$ of all primitive elements of H. Moreover, let

$$Q\colon \mathscr{H}_1 \to K\text{-GrMod}$$

be the functor which assigns to each H from $\mathscr{H}_1$ the quotient graded K-module $I(H)/I(H)^2$ where $I(H)$ is the ideal $\bigoplus_{n=1} H_n$ (see [7]).

From now on K is an arbitrary field of characteristic $p>0$ and $N = \{0, 1, 2, \ldots\}$. We investigate categories $\mathscr{L}_1$ and $\mathscr{H}_1$ using a representation of some special trees in these categories.

In Section 1 we investigate the category $\mathscr{L}_1$. In particular we give a description of projective objects in $\mathscr{L}_1$ and we show that $\mathrm{gl.dim}\,\mathscr{L}_1 = 1$. In Section 2 for an arbitrary field K of characteristic $p>0$ we introduce the category $\mathscr{T}_{\mathfrak{m}}$ of $\mathfrak{m}$-special trees with ballast where $\mathfrak{m}$ is the cardinality of a basis of K over K^p. Section 3 contains definitions of two functors $L\colon \mathscr{T}_{\mathfrak{m}} \to \mathscr{L}_1$ and $H\colon \mathscr{T}_{\mathfrak{m}} \to \mathscr{H}_1$ and their basic properties. In Section 4 we study the structure of injective objects in categories $\mathscr{L}_1$ and $\mathscr{H}_1$. In particular we prove that the functor H (resp. L) gives a one-one correspondence between some $\mathfrak{m}$-special trees $G_n^{\varphi_n}$, $0 \leqslant n \leqslant \infty$, defined in Section 2, and indecomposable injective objects in $\mathscr{H}_1$ (resp. in $\mathscr{L}_1$). Moreover, we show that $\mathrm{gl.dim}\,\mathscr{H}_1 = 2$. In Section 5 we compute endomorphism rings of indecomposable injective objects in $\mathscr{L}_1$.

Some results presented in the paper were proved in [10] for a perfect field (see also [18]).

The paper is a part of the author's doctoral dissertation [16] written at the Institute of Mathematics, N. Copernicus University under supervision of Professor Daniel Simson. Some results of the paper were announced in [15].

I wish to thank heartily Professor Daniel Simson for suggesting the field of research, useful discussions and help in the preparation of this paper.

§ 1. The category $\mathscr{L}_1$.

We recall some notation and definitions. Let R be a commutative ring with identity. An additive category $\mathscr{C}$ (not necessarily with coproducts) is an R-*category* if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an R-module for any X, Y from $\mathscr{C}$ in such a way that the morphism composition is R-bilinear (see [1], [8]). A functor $T\colon \mathscr{C} \to \mathscr{C}'$ between R-categories is an R-*functor* if the natural morphism $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(TX, TY)$ given by $f \to T(f)$ is a homomorphism of R-modules for each X, Y from $\mathscr{C}$. If $\mathscr{C}$ is an R-category and F is a $\mathscr{C}$-*module* (i.e. a covariant functor from $\mathscr{C}$ to abelian groups [1], [12], [13]), then $F(X)$ is in a natural way an R-module for each X in $\mathscr{C}$. Moreover, if $f\colon X \to Y$ is a morphism in $\mathscr{C}$, then $F(f)$ is an R-homomorphism. It follows that the category $\mathscr{C}$-Mod of all $\mathscr{C}$-modules is equivalent to the category of all R-functors from $\mathscr{C}$ to R-Mod (see [1], § 1). If $\mathscr{C}$ is an R-category, then there is a unique R-category structure on $\mathscr{C}$-Mod such that the Yoneda embedding is an R-functor ([14], Proposition 3.1).

A Grothendieck category $\mathscr{A}$ is *perfect* if every object in $\mathscr{A}$ has a projective cover (see [6], [12], [13]). An object M of $\mathscr{A}$ is *serial* if the family of all subobjects of M is linearly ordered by inclusing [17]. $\mathscr{A}$ is said to be *locally serial* if it has a family of serial generators. The *Jacobson radical* of an additive category $\mathscr{C}$ is a two-sided ideal $J(\mathscr{C})$ defined by

$$J(\mathscr{C})(A, B) = \{f \in \mathrm{Hom}_{\mathscr{C}}(A, B);\ 1_A\text{-}gf \text{ has a two-sided inverse for every } g\}$$

(see [8]). Recall also that the Jacobson radical $J(\mathscr{C})$ of an additive category $\mathscr{C}$ is *right T-nilpotent* if for any sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to \ldots$$

with $f_n \in J(\mathscr{C})(A_n, A_{n+1})$, there exists such m that $f_m \ldots f_2 f_1 = 0$ (see [2], [12]).

Now let K be a field of characteristic $p \geqslant 2$. For a natural number n we denote by ${}^nL$ the polynomial Hopf algebra $K[x]$ with $\deg x = 2p^n$ and with the comultiplication given by $\Delta(x) = x \otimes 1 + 1 \otimes x$. The full subcategory of $\mathscr{L}_1$ consisting of all objects ${}^nL$ will be denoted by $\mathscr{K}$.

THEOREM 1.1. (1) $\mathscr{L}_1 \approx \mathscr{K}^{\mathrm{op}}$-Mod *and* $\mathrm{gl.dim}\,\mathscr{L}_1 = 1$.

(2) *$\mathscr{L}_1$ is a locally serial and perfect K-category.*

(3) *If $(K\colon K^p)$ is finite then the endomorphism ring of every noetherian object in $\mathscr{L}_1$ is a finite dimensional K-algebra.*

Proof. (1) In order to prove the equivalence $\mathscr{L}_1 \approx \mathscr{K}^{\mathrm{op}}$-Mod it is sufficient to show that the objects ${}^nL$, $n \in N$, form a set of noetherian projective generators in $\mathscr{L}_1$ (see [9], p. 103).

Fix $n \in N$ and let $u\colon H \to {}^nL$ be an epimorphism in $\mathscr{L}_1$. We prove that u splits. First observe that since H is in $\mathscr{L}_1$, the natural epimorphism $I(H) \to I(H)/I(H)^2 = Q(H)$ of graded K-modules induces the epimorphism $P(H) \to Q(H)$ of graded K-modules. Further, $Q({}^nL)_{2p^n} = Kx$ and $Q({}^nL)_r = 0$ if $r \neq 2p^n$. Then there exists an element y in $P(H)_{2p^n}$ such that $u(y) - x$ belongs to $I({}^nL)^2$. Since $I({}^nL)^2_r = 0$ for $r \leqslant 2p^n$, so $u(y) = x$. We define the morphism $s\colon {}^nL \to H$ by $s(x) = y$. Consequently $us = \mathrm{id}$ and ${}^nL$ is a projective object in $\mathscr{L}_1$. Now, let H belong to $\mathscr{L}_1$ and let H' be a proper subobject of H. Then there exists a homogeneous primitive element z of H which is not contained in H'. Let $\deg z = 2p^r$. Then we have a morphism $g\colon {}^rL \to H$ given by $g(x) = z$, where x is a generator of ${}^rL$, which does not factor through $H' \subset H$. Hence the objects ${}^nL$, $n \in N$, form a set of generators in $\mathscr{L}_1$.

We now show the equality $\mathrm{gl.dim}\,\mathscr{L}_1 = 1$. By Proposition 7.8 in [7] every subobject of ${}^nL$ is isomorphic to ${}^mL$ for a certain $m \geqslant n$. Hence every right ideal in $\mathscr{K}$ has the form $\mathrm{Hom}_{\mathscr{K}}(-, {}^nL)$, and so is projective in $\mathscr{L}_1$. Then by Theorems 7.24 and 7.25 in [4] $\mathrm{gl.dim}\,\mathscr{L}_1 \leqslant 1$. To prove that the equality holds it is now sufficient to observe that the exact sequence

$$0 \to (x^p) \to {}^0L \to {}^0L/(x^p) \to 0$$

is not splitable. Consequently $\mathrm{gl.dim}\,\mathscr{L}_1 = 1$ and (1) is proved.

(2) The fact that $\mathscr{L}_1$ is locally serial follows immediately from Theorem 7.8 in [7]. We now show that $\mathscr{L}_1$ is perfect. By Theorem 5.4 in [13] it is enough to show that the endomorphism ring of every object in $\mathscr{K}$ is left artinian and the Jacobson radical $J(\mathscr{K})$ is right T-nilpotent. But

$$\mathrm{Hom}_{\mathscr{K}}({}^nL, {}^mL) = \begin{cases} K & \text{for } n \geqslant m, \\ 0 & \text{for } n < m \end{cases}$$

so it is sufficient to prove the second part of the last statement. For this purpose consider a sequence

$$^{n_1}L \xrightarrow{f_1} {}^{n_2}L \to \dots \to {}^{n_r}L \xrightarrow{f_r} {}^{n_r+1}L \to \dots$$

where each f_r belongs to $J(\mathscr{K})$. It is not difficult to check that $J(\mathscr{K})({}^nL, {}^mL) \neq 0$ if and only if $n > m$. Assume that each $f_i \neq 0$. Then $n_1 > n_2 > n_3 > \dots$ and we get a contradiction. Consequently $f_m = 0$ for a suitable m and $J(\mathscr{K})$ is right T-nilpotent.

Finally we define a K-category structure on $\mathscr{K}$. Let $n \in N$ and consider the following homomorphism of rings $u_n\colon K \to \mathrm{End}_{\mathscr{K}}({}^nL)$ given by $(u_n(a))(x) = a^{p^n}x$ where ${}^nL = K[x]$ and $a \in K$. Then for each $n, m \in N$, $n \geqslant m$, a K-module structure on $\mathrm{Hom}_{\mathscr{K}}({}^nL, {}^mL)$ is given by the formula

$$u_m(a)f = a \cdot f = f u_n(a)$$

where $f \in \mathrm{Hom}_{\mathscr{K}}({}^nL, {}^mL)$, $a \in K$. It is easy to check that the morphism composition in $\mathscr{K}$ is K-bilinear. Hence by Proposition 3.1 in [14] the category $\mathscr{L}_1 \approx \mathscr{K}^{\mathrm{op}}$-Mod is a K-category.

(3) Let H be a noetherian object in $\mathscr{L}_1$. Then there exists an epimorphism $\bigotimes_{i=1}^{m} {}^{n_i}L \to H$. Consider the following diagram of K-linear spaces

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ 0 \to \mathrm{Hom}_{\mathscr{L}_1}(H, H) \to \mathrm{Hom}_{\mathscr{L}_1}\Big(\bigotimes_{i=1}^{m} {}^{n_i}L, H\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad\qquad \mathrm{Hom}_{\mathscr{L}_1}\Big(\bigotimes_{i=1}^{m} {}^{n_i}L, \bigotimes_{j=1}^{m} {}^{n_j}L\Big) \end{array}$$

with exact row and column. Moreover, it is easy to observe that the natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathscr{L}_1}\Big(\bigotimes_{i=1}^{m} {}^{n_i}L, \bigotimes_{j=1}^{m} {}^{n_j}L\Big) \approx \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \mathrm{Hom}_{\mathscr{L}_1}({}^{n_i}L, {}^{n_j}L)$$

is K-linear. Hence if $(K\colon K^p)$ is finite, then $\mathrm{End}({}^nL)$, $n \in N$, are finite dimensional K-algebras and $\mathrm{End}(H)$ is also a finite dimensional K-algebra. This completes the proof of the theorem.

COROLLARY 1.2. (1) *Every projective object P in $\mathscr{L}_1$ is isomorphic with a coproduct of objects ${}^nL$, $n \in N$, and any two such decompositions of P are isomorphic.*

(2) *If $(K\colon K^p)$ is finite, then every noetherian object H in $\mathscr{L}_1$ is a coproduct of indecomposable objects and any two such decompositions of H are isomorphic.*

Proof. It follows from Corollary 1.4 in [14], Theorem 1.3 on p. 320 in [9], Lemma 7.4 on p. 369 in [9] and the fact that if the endomorphism ring of an indecomposable object is artinian then it is local.

§ 2. $\mathfrak{m}$-special trees. Troughout this section we assume that K is a fixed field of characteristic $p > 0$ and that $(K\colon K^p)$ is a cardinal number $\mathfrak{m}$.

Let $G = (X, U)$ be a directed graph with a set of vertices X and a set of edges U (not necessarily finite). For each $x \in X$, denote by $d_G^+(x)$ (resp. $d_G^-(x)$) the cardinality of the set of edges with initial vertex x (resp. final vertex x) (see [3]). We say that a vertex x is a *node* (resp. *input*, *output*) iff $d_G^+(x) \geqslant 2$ (resp. $d_G^-(x) = 0$, $d_G^+(x) = 0$). Denote by $W(G), Z(G), I(G)$ the sets of all nodes, inputs and outputs of G, respectively. Whenever no confusion arises we shall write simply $d^-(x)$, $d^+(x)$, W, Z, I instead of a $d_G^-(x)$, $d_G^+(x)$, $W(G)$, $Z(G)$, $I(G)$. If for $x, y \in X$ there exists a chain from x to y, then denote by $d(x, y)$ the distance from x to y. A path

$$u = ((x_1, x_2), \dots, (x_{k-1}, x_k))$$

from x_1 to x_k is said to be a *branch* if the following condition is satisfied: for each $1 \leqslant l \leqslant k$, x_l is a node iff $l = 1$ or $l = k$. A graph $G = (X, U)$ is said to be *normal* if for every path $u = ((x_1, x_2), \dots, (x_{k-1}, x_k))$ from x_1 to x_k and $k > 2$, $(x_1, x_k) \notin U$. G is *antisymmetric* if $(y, x) \notin U$ whenever $(x, y) \in U$ (see [3]).

DEFINITION 2.1. An $\mathfrak{m}$-special tree is a connected, normal, antisymmetric graph $G = (X, U)$ without cycles satisfying the following conditions:

(a) $d^-(x) \leqslant 1$ and $d^+(x) \leqslant \mathfrak{m}$ for each $x \in X$,

(b) if $w \in W$ then $d^-(w) = 1$,

(c) for each $x \in X$ there exists a path of finite length from x to a certain $i \in I$.

Denote by $\mathfrak{B}$ the family of all sets of elements of K which are linearly independent over K^p.

DEFINITION 2.2. An $\mathfrak{m}$-special tree with a ballast φ, ψ, $\leqslant$ is a sequence $G = (X, U, \varphi, \psi, \leqslant)$ where (X, U) is an $\mathfrak{m}$-special tree, $\leqslant$ is a well order in I and $\varphi\colon W \to \mathfrak{B}$, $\psi\colon X \to N$ are set mappings satisfying the following conditions:

(a) $d^+(w)$ is the cardinality of the set $\varphi(w)$ for each node $w \in W$,

(b) if there exists a path with an initial vertex x and a final vertex y then $d(x, y) = \psi(x) - \psi(y)$.

Let $G = (X, U, \varphi, \psi, \leqslant)$ be a fixed $\mathfrak{m}$-special tree with ballast φ, ψ, $\leqslant$. For each $n \in N$ we define the set

$$W_n = \{w \in W;\ \psi(w) = n\}.$$

Further, if $W(G) \neq 0$ then by induction on $n \geqslant n_0 = \min\{\psi(w);\ w \in W(G)\}$ we define for each $w \in W_n$ the set $I_w \subset I$ and the element $i_w \in I_w$ as follows. If $w \in W_{n_0}$ then we put

$$I_w = \{i \in I;\ \text{there exists a path from } w \text{ to } i\}$$

and let i_w be the minimal element in I_w. For every $n > n_0$ and $w \in W_n$ put $I_w = I'_w \cup I''_w$ where

$$I'_w = \Big\{i_{w'};\ w' \in \bigcup_{k=n_0}^{n-1} W_k \text{ and there exists a branch from } w \text{ to } w'\Big\},$$

$$I''_w = \{i \in I;\ \text{there exists a branch from } w \text{ to } i\},$$

and let i_w be the minimal element in I_w. It is clear that $d^+(w)$ is the cardinality of the set I_w for each $w \in W$.

We shall use the following notation:

$$\varphi(w) = \{v_{w,i};\ i \in I_w\},\ k_{w,i} = v_{w,i}/v_{w,i_w}$$

for each $w \in W$ and $i \in I_w$.

Moreover, we observe that for each vertex $x \notin I \cup W$ there exists a unique vertex $j(x) \in I \cup W$ such there exists a branch from x to $j(x)$. Then we define a function $\sigma\colon X \to I$ by

$$\sigma(x) = \begin{cases} x, & \text{if } x \in I, \\ i_x, & \text{if } x \in W, \\ j(x), & \text{if } x \notin I \cup W \text{ and } j(x) \text{ is an output such that there exists} \\ & \quad \text{a branch from } x \text{ to } j(x), \\ i_{j(x)}, & \text{if } x \notin I \cup W \text{ and } j(x) \text{ is a node such that there exists} \\ & \quad \text{a branch from } x \text{ to } j(x). \end{cases}$$

DEFINITION 2.3. A morphism $f\colon G = (X, U, \varphi, \psi, \leqslant) \to G' = (X', U', \varphi', \psi', \leqslant')$ of $\mathfrak{m}$-special trees with ballast is a morphism $f\colon (X, U) \to (X', U')$ of directed graphs satisfying the following conditions:

(a) $\psi'(f(x)) = \psi(x)$, $\sigma' f\sigma(x) = \sigma' f(x)$, $d^-(x) = d^-(f(x))$ for $x \in X$,

(b) $v_{w,i} \cdot v'_{f(w)\sigma' f(j)} = v_{w,j} \cdot v'_{f(w),\sigma' f(i)}$ for each $w \in W$ and $i, j \in I_w$.

Observe that, if the morphism f in the definition above satisfies condition (a), then for every $w \in W$, $i \in I_w$ the vertex $f(w)$ is a node in G and $\sigma' f(i) \in I'_{f(w)}$. So, condition (b) is correct.

LEMMA 2.4. *Let* $f\colon G = (X, U, \varphi, \psi, \leqslant) \to G' = (X', U', \varphi', \psi', \leqslant')$, $g\colon G' = (X', U', \varphi', \psi', \leqslant') \to G'' = (X'', U'', \varphi'', \psi'', \leqslant'')$ *be morphisms of* $\mathfrak{m}$-*special trees with ballast. Then* $gf\colon G \to G''$, $\mathrm{id}_G\colon G \to G$ *are morphisms of* $\mathfrak{m}$-*special trees with ballast.*

Proof. The fact that id_G is a morphism of $\mathfrak{m}$-special trees with ballast follows from the equalities $\sigma(i) = i$, $i \in I(G)$. We shall prove that gf satisfies conditions (a), (b) of Definition 2.3.

(a) If $x \in X$, then $\psi'' gf(x) = \psi' f(x) = \psi(x)$, $\sigma'' gf\sigma(x) = \sigma'' g\sigma' f\sigma(x) = \sigma'' g\sigma' f(x) = \sigma'' gf(x)$ and $d^-(gf(x)) = d^-(f(x)) = d^-(x)$.

(b) Let $w \in W$ and $i, j \in I_w$. Then $v_{w,i} \cdot v'_{f(w),\sigma' f(j)} = v_{w,j} \cdot v'_{f(w),\sigma' f(i)}$, $v'_{f(w)\sigma' f(i)} \cdot v''_{gf(w),\sigma'' g\sigma' f(j)} = v'_{f(w),\sigma' f(j)} \cdot v''_{gf(w),\sigma'' g\sigma' f(i)}$. Since $\sigma'' g\sigma' f(j) = \sigma'' gf(j)$, $\sigma'' g\sigma' f(i) = \sigma'' gf(i)$ we have that $v_{w,i} \cdot v''_{gf(w),\sigma'' gf(j)} = v_{w,j} \cdot v''_{gf(w),\sigma'' gf(i)}$. This finishes the proof of the lemma.

$\mathfrak{m}$-special trees with ballast form a category which will be denoted by $\mathcal{T}_{\mathfrak{m}}$.

We now give a method for constructing $\mathfrak{m}$-special trees. Let A be a well-ordered set of cardinality $\mathfrak{m}$. Moreover, assume that $G = (X, U)$ is an $\mathfrak{m}$-special tree such that $Z(G) \neq \emptyset$. It follows from Definition 2.1 that $Z(G)$ contains only one element z.

For each $j \in A$, let $G_j = (X_j, U_j)$ denote a copy of $G = (X, U)$ and let z_j be the unique element of $Z(G_j)$. We define $\mathfrak{m}$-special trees

$$M(G) = (X_M, U_M), \quad N(G) = (X_N, U_N),$$

putting

$$X_M = \Big(\dot{\bigcup_{j \in A}} X_j/\{z_j \sim z_m;\ j, m \in A\}\Big) \cup \{z_M\},$$

$$U_M = \Big(\dot{\bigcup_{j \in A}} U_j\Big) \mathbin{\dot\cup} \{(z_M, w_M)\}$$

where $\dot\cup$ denoted the disjoint union, $w_M = \{z_j;\ j \in A\}$,

$$X_N = X \mathbin{\dot\cup} N$$

and

$$U_N = U \mathbin{\dot\cup} \{(0, z)\} \mathbin{\dot\cup} \{(r+1, r);\ r \in N\}.$$

Observe that $I(M(G)) = \dot{\bigcup_{j \in A}} I(G_j)$, $W(M(G)) = \big(\dot{\bigcup_{j \in A}} W(G_j)\big) \mathbin{\dot\cup} \{w_M\}$, $Z(M(G)) = \{z_M\}$ and $I(N(G)) = I(G)$, $W(N(G)) = W(G)$, $Z(N(G)) = \emptyset$.

If the set $I(G)$ is well ordered by $\leqslant_G$, then we define a well order $\leqslant_{M(G)}$ in $I(M(G))$ as follows: if $i \in I(G_j)$, $i \in I(G_m)$ then

$$i \leqslant_{M(G)} i \quad \text{iff} \quad j < m \text{ or } j = m \text{ and } i \leqslant_{G_j} i.$$

For each $n \in N$ we define by induction on n *standard* $\mathfrak{m}$-*special trees*

$$E_n = (Y_n, T_n), \quad G_n = (X_n, U_n)$$

such that $d^+(w) = m$ for $w \in W$. If $n = 0$, then we put

$$X_0 = \{0, 1\}, \quad U_0 = \{(1, 0)\}, \quad E_0 = N(G_0).$$

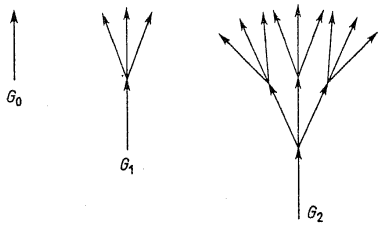
If $n > 0$, we put $G_n = M(G_{n-1})$, $E_n = N(G_n)$.

Now we define the *maximal* $\mathfrak{m}$-*special tree* $E_\infty = G_\infty = (X_\infty, U_\infty)$. Let o be the minimal element of A. For each $n \in N$, we consider the set $Y_n = X_n \mathbin{\dot\cup} N$ and define a function $r_n\colon Y_n \to Y_{n+1}$ by

$r_{n|X_n}$ is the natural inclusion $X_n = (X_n)_0 \hookrightarrow Y_{n+1}$,
$r_n(0) = z_{n+1}$ is the unique element in $Z(G_{n+1})$, $r_n(l) = l-1$ for $l \geqslant 1$.

The injections $r_n\colon Y_n \to Y_{n+1}$ induce in a natural way morphisms $t_n\colon E_n \to E_{n+1}$, $n \in N$, of $\mathfrak{m}$-special trees. The $\mathfrak{m}$-special tree $G_\infty = (X_\infty, U_\infty)$ is defined as follows: $X_\infty = \varinjlim\{Y_n, r_n\}$ and the set U_∞ induced in a natural way by the sets T_n, $n \in N$. Then we have the canonical injections $s_n\colon E_n \to E_\infty$ of $\mathfrak{m}$-special trees, such that $s_{n+1} t_n = s_n$ for each $n \in N$. Furthermore, we observe that the trivial well order in $I(G_0) = \{o\}$ induces well-orders $\leqslant_n$ in $I(E_n) = I(G_n)$, $0 \leqslant n \leqslant \infty$, preserved by the morphisms t_n, s_n.

For example, if $\mathfrak{m} = 3$ then the trees G_0, G_1, G_2 have the following form:



DEFINITION 2.5. A basic ballast of the $\mathfrak{m}$-special trees G_n, E_n, $0 \leqslant n \leqslant \infty$, is such a ballast $\varphi_n, \psi_n, \leqslant_n$ that $\varphi_n(w)$ is a basis of K over K^p for each $w \in W$, $\psi_n(i) = 0$ for $i \in I$ and $\leqslant_n$ is the order defined above.

For $0 \leqslant n \leqslant \infty$, we denote by $G_n^{\varphi_n}$ (resp. $E_n^{(\varphi_n)}$) the tree G_n (resp. E_n) with a basic ballast $\varphi_n, \psi_n, \leqslant_n$.

§ 3. Relations between $\mathfrak{m}$-special trees and abelian Hopf algebras. Let K be a field of characteristic $p > 0$ and let $(K:K^p) = \mathfrak{m}$ as above. The main tool used in our investigation of abelian Hopf algebras are functors

$$L\colon T_{\mathfrak{m}} \to \mathscr{L}_1, \quad H\colon T_{\mathfrak{m}} \to \mathscr{H}_1$$

defined as follows. If $G = (X, U, \varphi, \psi, \leqslant) \in T_{\mathfrak{m}}$ then we put

$$L(G) = \bigotimes_{i \in I} {}^{\psi(i)}L / S(G)$$

where I is the set of all outputs in G, ${}^{\psi(i)}L = K[X_i]$ with $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ and $S(G)$ is the ideal in $\bigotimes_{i \in I} {}^{\psi(i)}L$ generated in the case $Z(G) \neq \emptyset$ by elements

(a) $$X_i^{p^{\psi(w)-\psi(i)}} - k_{w,i} X_{i_w}^{p^{\psi(w)-\psi(i_w)}}, \quad i \in I_w,\ w \in W,$$

(b) $$X_0^{p^{\psi(w)-\psi(0)}}, \quad z \in Z,$$

and in the case $Z(G) = \emptyset$ by elements of type (a) only, where o is the minimal element of I.

If $f\colon G = (X, U, \varphi, \psi, \leqslant) \to G' = (X', U', \varphi', \psi', \leqslant')$ is a morphism in $\mathscr{T}_{\mathfrak{m}}$, then $L(f)\colon L(G) \to L(G')$ is defined by

$$L(f)(Y_i) = (Y'_{\sigma' f(i)})^{p^{\psi' f(i) - \psi' \sigma' f(i)}}$$

where Y_i, Y'_j denote the images of X_i, X'_j by the natural epimorphisms $\bigotimes_{i \in I(G)} {}^{\psi(i)}L \to L(G)$ and $\bigotimes_{j \in I(G')} {}^{\psi'(j)}L \to L(G')$ respectively.

LEMMA 3.1. *$L\colon \mathscr{T}_{\mathfrak{m}} \to \mathscr{L}_1$ is a covariant functor.*

An easy proof is left to the reader.

Next we define the functor $H\colon \mathscr{T}_{\mathfrak{m}} \to \mathscr{H}_1$. For each $r \in N$, let $K[X]^r$ $K[X_0, X_1, \ldots]$ the algebra polynomial on variables X_n, $n \in N$, with $\deg X_n = 2p^{n+r}$ and with the comultiplication Δ given by

$$\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\Delta(X_n) = X_n \otimes 1 + 1 \otimes X_n + \sum_{m=0}^{n-1} \frac{1}{p^{n-m}} [X_m^{p^{n-m}} \otimes 1 + 1 \otimes X_m^{p^{n-m}} - \Delta(X_m)^{p^{n-m}}]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

see [5] p. 542 and [10] p. 139).

If $G = (X, U, \varphi, \psi, \leqslant)$ is an object in $\mathscr{T}_{\mathfrak{m}}$, then we put

$$H(G) = \bigotimes_{i \in I} K[X]^{\psi(i),i} / T(G)$$

where I is the set of all outputs in G, $K[X]^{\psi(i),i} = K[X]^{\psi(i)} = K[X_{0,i}, X_{1,i}, \ldots]$ and $T(G)$ is the ideal of $\bigotimes_{i \in I} K[X]^{\psi(i),i}$ generated in the case $Z(G) \neq \emptyset$ by elements

(a) $$X_{m,i}^{p^{\psi(w)-\psi(i)}} - k_{w,i}^{p^m} X_{m,i_w}^{p^{\psi(w)-\psi(i_w)}}, \quad i \in I_w,\ w \in W,\ m \in N,$$

(b) $$X_{m,0}^{p^{\psi(z)-\psi(o)}}, \quad z \in Z,\ m \in N,$$

and in the case $Z(G) = \emptyset$ by elements of type (a) only where o is the minimal element of I.

If $f\colon G = (X, U, \varphi, \psi, \leqslant) \to G' = (X', U', \varphi', \psi', \leqslant')$ is a morphism in $\mathscr{T}_{\mathfrak{m}}$, then $H(f)\colon H(G) \to H(G')$ is given by

$$H(f)(Y_{m,i}) = (Y'_{m,\sigma' f(i)})^{p^{\psi' f(i) - \psi' \sigma' f(i)}}$$

where $Y_{m,i}$, $Y'_{m,j}$ denote the images of $X_{m,i}$, $X'_{m,j}$ by the natural epimorphisms $\bigotimes_{i \in I(G)} K[X]^{\psi(i),i} \to H(G)$ and $\bigotimes_{j \in I(G')} K[X]^{\psi'(j),j} \to H(G')$ respectively.

LEMMA 3.2. *$H\colon \mathscr{T}_{\mathfrak{m}} \to \mathscr{H}_1$ is a covariant functor.*

Proof. To prove that for every G in $\mathscr{T}_{\mathfrak{m}}$ the graded K-algebra $H(G)$ belongs to $\mathscr{H}_1$ notice that $\Delta(Y_{m,i}^{p^{\psi(w)-\psi(i)}}) = k_{w,i}^{p^m} \Delta(Y_{m,i_w}^{p^{\psi(w)-\psi(i_w)}})$ for each $w \in W(G)$, $i \in I_w$, $m \in N$. This follows from the fact that K is a field of characteristic p and that the coefficients in formulas defining $\Delta(Y_{m,i})$ are integers. An easy proof that the definition of $H(f)$ is correct and that $H(-)$ is a functor is left to the reader.

Now let $G = (X, U, \varphi, \psi, \leqslant)$ be an object in $\mathscr{T}_{\mathfrak{m}}$. For each $n \in N$, we define sets

$$I_n = \{i \in I;\ \psi(i) \leqslant n\}, \quad X_n = \{x \in X;\ \psi(x) \leqslant n\}.$$

By induction on $m \geqslant m_0 = \min\{\psi(i);\ i \in I\}$ we define sets K_m by

$$K_{m_0} = I_{m_0} \quad \text{and} \quad K_m = [(I_m \setminus I_{m-1}) \cup K_{m-1}] \setminus [\bigcup_{w \in W_m} (I_w \setminus \{i_w\})]$$

for $m>m_0$. Observe that the sets K_m can be defined also in the following way. In every set I_m, $m\geqslant m_0$, we define an equivalence relation $\equiv_m$ as follows: for $i,j\in I_m$, $i\equiv_m j$ iff $i=j$ or there exists a chain connecting i and j such that its vertices belong to X_m. Further, in each equivalence class with respect to $\equiv_m$ we have a well order induced by the well order $\leqslant$ in $I(G)$. Then K_m is the set of minimal elements (in the sense $\leqslant$) of the different equivalence classes in I_m.

Observe that $I_w\subset K_m$ for each $w\in W_{m+1}$, $m\geqslant m_0$, and, for each $m\in N$, define a set K'_m by the formula

$$K'_m=\begin{cases}K_m\setminus(\bigcup\limits_{w\in W_{m+1}} I_w), & \text{if } W_{m+1}\neq\emptyset,\\ K_m, & \text{if } W_{m+1}=\emptyset.\end{cases}$$

We shall need the following technical lemma.

LEMMA 3.3. *Let $G=(X,U,\varphi,\psi,\leqslant)$ be an object in $\mathscr{T}_m$. Then*

$$P(H(G))_r=\begin{cases}\bigoplus\limits_{i\in K_m} KY_{0,i}^{p^{m-\psi(i)}}, & \textit{if } r=2p^m \textit{ and } m\geqslant m_0,\\ 0, & \textit{in the opposite case}\end{cases}$$

where $m_0=\min\{\psi(i);\ i\in I(G)\}$ and K_m are the sets defined above.

Proof. For each $m\geqslant m_0$, we set $N_m=\{(r,i)\in N\times I;\ r+\psi(i)=m\}$. From the definition of $H(G)$ it follows that

$$Q(H(G))_r=\begin{cases}\bigoplus\limits_{(r,i)\in N_m} YK_{r,i}, & \text{if } r=2p^m \text{ and } m\geqslant m_0,\\ 0, & \text{in the opposite case}.\end{cases}$$

Let x be a non-zero primitive homogeneous element of $H(G)$. Since $H(G)\in\mathscr{H}_1$, we have $\deg x=2p^n$ for a suitable $n\geqslant m_0$. Therefore there exist elements $i_1,\dots,i_s\in I(G)$, $m\in N$, such that

$$x=\sum_{[j_{r,t}]\in A} a_{[j_{r,t}]}Y_{0,i_1}^{j_{0,1}}\dots Y_{m,i_1}^{j_{m,1}}\dots Y_{0,i_s}^{j_{0,s}}\dots Y_{m,i_s}^{j_{m,s}}$$

where $A=\{[j_{r,t}]\in N^{m+1}\times N^s;\ 0\leqslant r\leqslant m,\ \ 1\leqslant t\leqslant s,\ \sum\limits_{r=0}^{m}\sum\limits_{t=1}^{s} j_{r,t}p^{r+\psi(i_t)}=p^n\}$ and without loss of generality one can assume that

(a) $0\neq Y_{r,i_t}^{j_{r,t}}\neq lY_{r,i_u}^{j_{r,u}}\neq 0$ if $1\leqslant t\neq u\leqslant s$, $0\leqslant r\leqslant m$, $l\in K$,

(b) there exists a matrix $[j_{r,t}]\in A$ with a non-zero row $(j_{m,1},\dots,j_{m,s})$ such that $a_{[j_{r,t}]}\neq 0$,

(c) there exists a matrix $[j'_{r,t}]$ with a non-zero column $(j'_{0,s},\dots,j'_{m,s})$ such that $a_{[j'_{r,t}]}\neq 0$.

Further we define the following sets:

$$A_x=\{[j_{r,t}]\in A;\ a_{[j_{r,t}]}\neq 0\},$$
$$B_x=\{[j_{r,t}]\in A_x;\ \text{there exists such } 1\leqslant t\leqslant s \text{ that } 0<j_{m,t}<p^{n-m-\psi(i_t)}\},$$
$$C_x=A_x\setminus\{[j_{r,t}]_u;\ 1\leqslant u\leqslant s\}$$

where, for $1\leqslant u\leqslant s$, $[j_{r,t}]_u$ denotes such a matrix that $j_{m,u}=p^{n-m-\varphi(i_u)}$ and $j_{r,t}=0$ for $(r,t)\neq(m,u)$. Clearly $B_x\subset C_x$. Then the element x has the expression

$$x=\sum_{u\in D_x} a_{[j_{r,t}]_u}Y_{m,i_u}^{p^{n-m-\psi(i_u)}}+\sum_{j_{r,t}\in C_x} a_{[j_{r,t}]}X^{[j_{r,t}]}$$

where

$$D_x=\{u;\ 1\leqslant u\leqslant s,\ [j_{r,t}]_u\in A_x\},\quad X^{[j_{r,t}]}=\prod_{0\leqslant r\leqslant m}\ \prod_{1\leqslant t\leqslant s} Y_{r,i_t}^{j_{r,t}}.$$

Denote by x' the summand of $\Delta(x)\in H(G)\otimes H(G)$ which belongs to $\bigoplus\limits_{i=1}^{2p^n-1} H(G)_i\otimes H(G)_{2p^n-i}$. Moreover, recall that

$$(*)\qquad \Delta(Y_{r,i})=Y_{r,i}\otimes 1+1\otimes Y_{r,i}-\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}Y_{r-1,i}^{k}\otimes Y_{r-1,i}^{p-k}+\psi_{r,i}$$

$r\geqslant 1$, $i\in I$, where $\psi_{r,i}=0$ if $r=0,1$ and $\psi_{r,i}$ is a polynomial of

$$Y_{0,1}\otimes 1,\dots,Y_{r-2,i}\otimes 1,\ 1\otimes Y_{0,i},\dots,1\otimes Y_{r-2,i} \text{ if } r\geqslant 2.$$

Now we show that $B_x=\emptyset$. Assume to the contrary that $B_x\neq\emptyset$ and consider the following function $v\colon B_x\to N$ given by the formula

$$v([j_{r,t}])=\max\{j_{m,t};\ 1\leqslant t\leqslant s\}$$

where $[j_{r,t}]\in B_x$. Then $v(B_x)$ is a finite subset of N. Let $[\bar{j}_{r,t}]$ be such an element of B_x that $v([\bar{j}_{r,t}])=\bar{j}_{m,t_0}$ is maximal in the set $v(B_x)$. From the definition of B_x it follows that $0<\bar{j}_{m,t_0}<p^{n-m-\psi(i_{t_0})}$. Hence the element x' has the form

$$x'=a_{[\bar{j}_{r,t}]}Y_{m,i_{t_0}}^{\bar{j}_{m,t_0}}\otimes Y_{0,i_1}^{\bar{j}_{0,1}}\dots Y_{m-1,i_{t_0}}^{\bar{j}_{m-1,t_0}}Y_{0,i_{t_0+1}}^{\bar{j}_{0,t_0+1}}\dots Y_{m,i_s}^{\bar{j}_{m,s}}+f,$$

where f contains no monomials of the form

$$dY_{m,i_{t_0}}^{\bar{j}_{m,t_0}}\otimes Y_{0,i_1}^{\bar{j}_{0,1}}\dots Y_{m-1,i_{t_0}}^{\bar{j}_{m-1,t_0}}Y_{0,i_{t_0+1}}^{\bar{j}_{0,t_0+1}}\dots Y_{m,i_s}^{\bar{j}_{m,s}},\qquad d\in K.$$

Then we get a contradiction since x is a primitive element and $a_{[\bar{j}_{r,t}]}\neq 0$.

Now, in order to prove the lemma it is sufficient to show that $m=0$. Suppose $m>0$. Then by (b) and the equality $B_x=\emptyset$ it follows that $[j_{r,t}]_{u_0}\in A_x$ for a suitable $1\leqslant u_0\leqslant s$. Furthermore, by the definition of $H(G)$ it follows that if

$$0\neq Y_{m,i_t}^{p^{n-m-\psi(i_t)}}\neq lY_{m,i_r}^{p^{n-m-\psi(i_r)}}\neq 0 \text{ for } 1\leqslant t\neq r\leqslant s,\ l\in K,$$

then

$$0\neq Y_{m-1,i_t}^{kp^{n-m-\psi(i_t)}}\neq l'Y_{m-1,i_r}^{k'p^{n-m-\psi(i_r)}}\neq 0 \text{ for } 1\leqslant t\neq r\leqslant s,$$

$1 \leqslant k,\ k' \leqslant p-1$, $l' \in K$. Moreover the equality $B_x = \emptyset$ and the formulas $(*)$ imply that the natural polynomial expression of the element $\Delta\big(\sum_{[j_r,t]\in C_x} X^{[j_r,t]}\big)$ contains no monomials of the form

$$e\,Y_{m-1,i_t}^{kp^{n-m-\psi(i_t)}} \otimes Y_{m-1,i_t}^{(p-k)p^{n-m-\psi(i_t)}}, \qquad e \in K.$$

Then it follows that the element x' has the form

$$x' = a_{[j_r,t]u_0}\Big(-\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k} Y_{m-1,i_{u_0}}^{kp^{n-m-\psi(i_{u_0})}} \otimes Y_{m-1,i_{u_0}}^{(p-k)p^{n-m-\psi(i_{u_0})}}\Big)+g,$$

where g contains no monomials of the form

$$e_k\,Y_{m-1,i_{u_0}}^{kp^{n-m-\psi(i_{u_0})}} \otimes Y_{m-1,i_{u_0}}^{(p-k)p^{n-m-\psi(i_{u_0})}}, \qquad 1 \leqslant k \leqslant p-1,\ e_k \in K.$$

This is a contradiction because $a_{[j_r,t]u_0} \neq 0$ and x is primitive. The lemma is proved.

Denote by $E\colon \mathscr{L}_1 \hookrightarrow \mathscr{H}_1$ the natural embedding functor and consider a natural transformation of functors $u\colon EL \to H$ given by

$$u(G)(Y_i) = Y_{0,i}$$

where $G \in \mathscr{T}_{\mathrm{m}}$, $i \in I(G)$.

Corollary 3.4. *Let $P\colon \mathscr{H}_1 \to \mathscr{K}$-GrMod be the functor defined in the introduction. Then the natural transformation $Pu\colon PEL \to PH$ is an equivalence.*

Proof. Easy.

Lemma 3.5. *Let $G = (X, U, \varphi, \psi, \leqslant) \in \mathscr{T}_{\mathrm{m}}$ and let x be a non-zero primitive homogeneous element of $L(G)$ of degree $2p^n$, $n \leqslant m_0 = \min\{\psi(i);\ i \in I(G)\}$. Then*

(1) $$x = \sum_{i\in K_n} a_i Y_i^{p^{n-\psi(i)}},$$

(2) *$x^p \neq 0$ if either $Z(G) = \emptyset$ or $Z(G) = \{z\}$ and $n < \psi(z)-1$.*

Proof. Statement (1) follows from Lemma 3.3 and Corollary 3.4. Suppose $0 \neq x \neq \sum_{i\in K_n} a_i Y_i^{p^{n-\psi(i)}}$ satisfies the assumption of the statement (2). Then we have the equalities

$$\begin{aligned} x^p &= \sum_{i\in K_n} a_i^p Y_i^{p^{n+1-\psi(i)}} = \sum_{w\in W_{n+1}}\big(\sum_{i\in I_w} a_i^p Y_i^{p^{n+1-\psi(i)}}\big) + \sum_{i\in K_n'} a_i^p Y_i^{p^{n+1-\psi(i)}} \\ &= \sum_{w\in W_{n+1}} b_{i_w} Y_{i_w}^{p^{n+1-\psi(i_w)}} + \sum_{i\in K_n'} a_i^p Y_i^{p^{n+1-\psi(i)}}, \end{aligned}$$

where $b_{i_w} = \sum_{i\in I_w} k_{w,i} a_i^p$ for $w \in W_{n+1}$. Observe that

$$\{Y_{i_w}^{p^{n+1-\psi(i_w)}};\ w \in W_{n+1}\} \cup \{Y_i^{p^{n+1-\psi(i)}};\ i \in K_n'\}$$

is the set of vectors from $L(G)_{2p^{n+1}}$ linearly independent over K. If we assume that $x^p = 0$, then $b_{i_w} = 0$ for $w \in W_{n+1}$ and $a_i^p = 0$ whenever $i \in K'$. Since for each $w \in W_{n+1}$ the elements $k_{w,i}$, $i \in I_w$, of K are linearly independent over K^p, we have $a_i = 0$ for $i \in I_w$, $w \in W_{n+1}$. Therefore $x = 0$, which contradicts our assumption and finishes the proof of the lemma.

Recall that an object M in a Grothendieck category $\mathscr{A}$ is *coirreducible* if each two non-zero subobject of M have a non-zero intersection [4], [9].

Proposition 3.6. *For each object G in $\mathscr{T}_{\mathrm{m}}$ the abelian Hopf algebras $L(G)$ and $H(G)$ are coirreducible.*

Proof. G belongs to $\mathscr{T}_{\mathrm{m}}$. By Corollary 3.4 the subobject of $H(G)$ generated by its all primitive elements is isomorphic with $L(G)$. Hence it is sufficient to show that $L(G)$ is coirreducible. For this purpose it is sufficient to show that for each two non-zero homogeneous primitive elements x, y of $L(G)$ the intersection $(x) \cap (y)$ is non-zero, where (x) and (y) denote the abelian Hopf algebras generated as K-algebra, by x and y, respectively. Since $L(G) \in \mathscr{L}_1$, so $\deg x = 2p^n$ and $\deg y = 2p^m$ for a suitable $n, m \in N$. We can assume that $m = n$. For if $m \neq n$ and $m < n$, then $\deg y^{p^{n-m}} = 2p^n$ and $y^{p^{n-m}} \neq 0$ by Lemma 3.5(2). Let

$$x = \sum_{i\in A} a_i Y_i^{p^{n-\psi(i)}}, \qquad y = \sum_{j\in B} b_j Y_j^{p^{n-\psi(j)}},$$

where $A = \{i \in K_n;\ a_i \neq 0\}$, $B = \{j \in K_n;\ b_j \neq 0\}$. If $W(G) = \emptyset$, then clearly $x = ly$ for a certain $l \in K$. Suppose $W(G) \neq \emptyset$. In the case

$$n \geqslant n_0 = \max\{t \in N;\ W_t \neq \emptyset\}$$

the set K_n has only one element and then by Lemma 3.5(1) the elements x and y are linearly dependent. Let $n < n_0$. Since the sets A and B are finite, then for some $s \geqslant n$ and a node $w \in W_s$ there exist paths from w to i and from w to j for each $i \in A$, $j \in B$. Hence we obtain

$$0 \neq x^{p^{s-n}} = c\,Y_{i_w}^{p^{s-\psi(i_w)}} \quad\text{and}\quad 0 \neq y^{p^{s-n}} = d\,Y_{i_w}^{p^{s-\psi(i_w)}}$$

for a certain $c, d \in K$. Consequently $(x) \cap (y)$ is non-zero and the proposition is proved.

§ 4. Injective objects in $\mathscr{L}_1$ and $\mathscr{H}_1$.

Let K be a fixed field of characteristic $p \geqslant 2$ and let $\mathfrak{B}$ be a fixed p-basis of K over K^p (see [19]). If K is non-perfect then denote by J the subset of a free group $\bigoplus_{b\in\mathfrak{B}} Z$ consisting of all elements $\alpha = (\alpha_b) \in \bigoplus_{b\in\mathfrak{B}} Z$ whose components α_b satisfy the condition $0 \leqslant \alpha_b \leqslant p-1$. If K is perfect, then we admit $\mathfrak{B} = \{1\}$ and $J = \{0\}$. Put $B^\alpha = \prod_{b\in\mathfrak{B}} b^{\alpha_b}$ for every $\alpha \in J$. Then the set $\{B^\alpha;\ \alpha \in J\}$ forms a basis of K over K^p (see [11]). Let $\mathfrak{m}$ be the cardinality of J.

For each $0 \leqslant n \leqslant \infty$, let $G_n^{\bar\varphi_n}$ and $E_n^{\bar\varphi_n}$ denote the standard $\mathfrak{m}$-special trees with basic ballast defined in Section 2, where $\bar\varphi_n$ is given by $\bar\varphi_n(w) = \{B^\alpha;\ \alpha \in J\}$. Put

$${}^nI = H(G_n^{\bar\varphi_n}), \quad {}^nT = H(E_n^{\bar\varphi_n}), \quad {}^nR = L(G_n^{\bar\varphi_n}), \quad {}^nF = L(E_n^{\bar\varphi_n}).$$

We say that an object H from $\mathscr{H}_1$ is *reflexive* if H_m is a finite-dimensional K-module for each $m \in N$. Let $\mathscr{H}_{\mathrm{ref}}$ denote the full subcategory of $\mathscr{H}_1$ consisting of all reflexive algebras and let $*: \mathscr{H}_{\text{ref}} \to \mathscr{H}_{\text{ref}}$ be the dualizing functor (see [7] and [10]).

Recall also that ${}^nS = {}^nL/(x^p)$, $n \in N$, is a complete list of non-isomorphic simple objects in $\mathscr{H}_1$.

The main result of this section is the following.

THEOREM 4.1. (1) *${}^nI$ is the injective envelope of ${}^nS$ in $\mathscr{H}_1$ for each $n \in N$.*

(2) *For each $n \in N$, ${}^\infty I$ is the injective envelope of ${}^nL$ in $\mathscr{H}_1$.*

(3) *Every injective indecomposable object in $\mathscr{H}_1$ is isomorphic with a certain ${}^nI$, $0 \leqslant n \leqslant \infty$.*

(4) $\operatorname{gl.dim} \mathscr{H} = \operatorname{gl.dim} \mathscr{H}_1 = 2$.

For a perfect field the theorem was proved by Schoeller in [10].

Before proving the theorem we need some definitions and technical lemmas.

Denote by u_n and v_n, $n \in N$, the natural epimorphisms

$$\bigotimes_{i \in I(G_n)} K[X]^{0,i} \to {}^nI \quad \text{and} \quad \bigotimes_{i \in I(E_n)} K[X]^{0,i} \to {}^nT,$$

respectively. As in Section 3, by $Y_{r,i}$ stand for the images of $X_{r,i}$ by u_n and v_n. Further, for each $n \in N$, let ${}^nC$ be subobject of ${}^0T = K[X]^{0,0}$ generated, as a K-algebra, by elements $X_{0,0}^{p^n}, X_{1,0}^{p^n}, \ldots$ and ${}^nD$ the subobject of ${}^nI$ generated by elements $Y_{0,0}^{p^n}, Y_{1,0}^{p^n}, \ldots$ Observe that the canonical injections $t_n: E_n \to E_{n+1}$ and $s_n: E_n \to E_\infty$, $n \in N$, of $\mathfrak{m}$-special trees, defined in Section 2, induce the injections $H(t_n): {}^nT \to {}^{n+1}T$ and $H(s_n): {}^nT \to {}^\infty T = {}^\infty I$ in $\mathscr{H}_1$. Therefore, for each $0 \leqslant m \leqslant \infty$, $n \in N$, ${}^nC$ is a subobject of ${}^mT$. Suppose $m > 1$ and, for each $n \in N$, $n \leqslant m \leqslant \infty$, denote by w_n^0 the unique node of trees $G_m^{\bar{\varphi}^m}$ and $E_m^{\bar{\varphi}^m}$ satisfying: (i) $\psi_m(w_n^0) = n$ and (ii) there exists a path from w_n^0 to the minimal element o of $I(G_m^{\bar{\varphi}_m})$ and $I(E_m^{\bar{\varphi}_m})$. We put

$${}^nQ = \bigotimes_{i \in I_{w_{n+1}^0} \setminus \{o\}} ({}^nI)^i \quad \text{for} \quad n \in N,$$

$${}^0U = 0 \text{ and } {}^nU = \bigotimes_{i \in I_{w_n^0}} ({}^{n-1}I) \quad \text{for} \quad n > 0,$$

where $({}^nI)^i = {}^nI$. If $m = 1$, then we admit ${}^nQ = 0 = {}^0U$ for all $n \in N$ and ${}^nU = {}^{n-1}I$ for $n > 0$.

LEMMA 4.2. *For each $n \in N$, there exist in $\mathscr{H}_1$ the following exact sequences:*

(a) $$0 \to {}^nD \to {}^nI \xrightarrow{\pi_n} {}^nU \to 0,$$

(b) $$0 \to {}^nC \to {}^nT \xrightarrow{\mu_n} {}^nU \to 0,$$

(c) $$0 \to {}^nT \to {}^{n+1}T \xrightarrow{\eta_n} {}^nQ \to 0.$$

Proof. The case $\mathfrak{m} = 1$ is obvious because ${}^nT = {}^{n+1}T = K[X]^{0,0}$ and ${}^nI = {}^nT/(X_{0,0}^{p^n}, X_{1,0}^{p^n}, \ldots)$. Now assume $\mathfrak{m} > 1$. Let n be a fixed natural number.

First we show that there exist sequences of types (a) and (b). If $n = 0$, then $\pi_0 = 0$ and $\mu_0 = 0$. Suppose $n > 0$. From the definition of G_n we have the equalities

$$I_{w_n^0} = \{i_{w_{n-1}};\ w_{n-1} \in W(G_n)_{n-1}\},$$

$$I(G_n) = \dot{\bigcup_{i \in I_{w_n^0}}} I(G_{n-1,i}):$$

$$W(G_n) = \Big(\dot{\bigcup_{i \in I_{w_n^0}}} W(G_{n-1,i})\Big) \dot{\cup} \{w_n^0\},$$

where $W(G_1)_0 = I(G_1)$, $G_{n-1,i} = G_{n-1}$ for $i \in I_{w_n^0}$ and $\dot{\cup}$ is the disjoint union. Consider the following diagram in $\mathscr{H}_1$ with exact rows

$$\begin{array}{ccccccc} 0 \to K(G_n) & \hookrightarrow & \bigotimes_{i \in I(G_n)} K[X]^{0,i} & \xrightarrow{u_n} & {}^nI \to 0 \\ & & \| & & \\ 0 \to \bigotimes_{i \in I_{w_n^0}} K(G_{n-1,i}) & \hookrightarrow & \bigotimes_{i \in I_{w_n^0}} \bigotimes_{j \in I(G_{n-1,i})} K[X]^{0,j} & \xrightarrow{v_n} & {}^nU \to 0 \end{array}$$

where $v_n = \otimes u_{n-1,i}$, $u_{n-1,i} = u_{n-1}$ and $K(G_n) = \ker u_n$, $K(G_{n-1,i}) = \ker u_{n-1,i}$. We will show that there exists a morphism $\pi_n: {}^nI \to {}^nU$ in $\mathscr{H}_1$ such that $\pi_n u_n = v_n$. First we observe that $K(G_n) \subset T(G_n)$ and $K(G_{n-1,i}) \subset T(G_{n-1,i})$. From the equalities above every generator of the ideal $T(G_n)$ of the form $X_{m,i}^{p^{\psi(w)}} - k_{w,i}^{p^m} X_{m,i_w}^{p^{\psi(w)}}$, $w \neq w_n^0$, $i \in I_w$, $m \in N$, belongs to $T(G_{n-1,j})$ for a suitable $j \in I_{w_n^0}$. Consequently it is contained in $\ker v_n$. Furthermore, observe that if $i \in I_{w_n^0}$ then $i = i_{w_{n-1}}$ for a certain $w_{n-1} \in W(G_n)_{n-1}$. Then, for each $m \in N$,

$$v_n\big(X_{m,i_{w_{n-1}}}^{p^{\psi(w_n^0)}} - k_{w_n,i_{w_{n-1}}}^{p^m} X_{m,i_{w_n^0}}^{p^{\psi(w_n^0)}}\big) = u_{n-1,i_{w_{n-1}}}(X_{m,i_{w_{n-1}}}^{p^n}) - k_{w_n,i_{w_{n-1}}}^{p^m} u_{n-1,0}(X_{m,0}^{p^n}) = 0$$

and clearly $v_n(X_{m,0}^{p^{n+1}}) = u_{n-1,0}(X_{m,0}^{p^{n+1}}) = 0$. Hence $v_n(T(G_n)) = 0$ and there exists a $\pi_n: {}^nI \to {}^nU$ such that $\pi_n u_n = v_n$. We shall prove that π_n is the cokernel of ${}^nD \hookrightarrow {}^nI$. Let $g: {}^nI \to H$ be a morphism in $\mathscr{H}_1$ satisfying $g({}^nD) = 0$. Then $gu_n(T(G_n)) = 0$ and $gu_n(X_{m,i_{w_{n-1}}}^{p^n}) = 0$ for $m \in N$, $w_{n-1} \in W(G_n)_{n-1}$. Hence, for each $i \in I_{w_n^0}$, $gu_n(T(G_{n-1,i})) = 0$ and there exists a unique morphism $g': {}^nU \to H$ in $\mathscr{H}_1$ such that $g'v_n = gu_n$. Since $\pi_n u_n = v_n$ and u_n is an epimorphism, we have $g'\pi_n = g$. Furthermore, g' is unique with this property since π_n is an epimorphism. This finishes the proof (a).

In order to prove (b) observe that we have the exact sequence

$$0 \to {}^{n+1}C \hookrightarrow {}^nT \xrightarrow{p_n} {}^nI \to 0$$

such that $p_n({}^nC) = {}^nD$. Consider the following commutative diagram in $\mathscr{H}_1$:

$$\begin{array}{ccccc} 0 \to {}^nC & \hookrightarrow & {}^nT & \xrightarrow{\mu_n} & {}^nU \\ \downarrow & & \downarrow{\scriptstyle p_n} & & \| \\ 0 \to {}^nD & \hookrightarrow & {}^nI & \xrightarrow{\pi_n} & {}^nU \to 0 \end{array}$$

in which the bottom sequence is exact and $\mu_n = \pi_n p_n$. We prove that μ_n is the cokernel of ${}^nC \hookrightarrow {}^nT$. First observe that μ_n is an epimorphism. Let $f\colon {}^nT \to H'$ be a morphism in $\mathscr{H}_1$ and let $f({}^nC) = 0$. Since ${}^{n+1}C \hookrightarrow {}^nC$, there exists a unique morphism $f'\colon {}^nI \to H'$ such that $f = f'p_n$. Further, we have $f'({}^nD) = 0$ because $p_n({}^nC) = {}^nD$. Therefore, there exists a morphism $f''\colon {}^nU \to H'$ such that $f''\pi_n = f'$. Consequently, $f''\mu_n = f''\pi_n p_n = f'p_n = f$ and (b) is proved.

(c) We recall that $I(E_r) = I(G_r)$, $W(E_r) = W(G_r)$, $r \in N$, and

$$I(E_{n+1}) = \bigcup_{i \in I_{w^0_{n+1}}} I(E_{n,i}),\quad W(E_{n+1}) = \bigcup_{i \in I_{w^0_{n+1}}} W(E_{n,i})$$

where $E_{ni} = E_n$ for $i \in I_{w^0_{n+1}}$. Now consider the following diagram in $\mathscr{H}_1$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & K(E_n) & & K(E_{n+1}) & & \bigotimes_{i \in I_{w^0_{n+1}} \setminus \{o\}} K(G_{n,i}) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigotimes_{j \in I(E_{n,0})} K[X]^{0,j} & \hookrightarrow & \bigotimes_{i \in I_{w^0_{n+1}}} \bigotimes_{j \in I(E_{n,i})} K[X]^{0,j} & \xrightarrow{\pi'_n} & \bigotimes_{i \in I_{w^0_{n+1}} \setminus \{o\}} \bigotimes_{j \in I(E_{n,i})} K[X]^{0,j} & \to 0 \\
 & \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle v_{n+1}} & & \downarrow{\scriptstyle v'_n} & \\
 & {}^nT & \hookrightarrow & {}^{n+1}T & & ({}^nI)^i = {}^nQ & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

with exact columns and an exact middle row, where $v'_n = \otimes u_{n,i}$, $u_{n,i} = u_n$ and $K(E_n) = \ker v_n$, $K(E_{n+1}) = \ker v_{n+1}$, $K(G_{n,i}) = \ker u_{n,i}$ for $i \in I_{w^0_{n+1}} \setminus \{o\}$. Let $\eta'_n = v'_n \pi'_n$. We shall prove that $\eta'_n(K(E_{n+1})) = 0$. Since $K(E_{n+1}) \subset T(E_{n+1})$, so it is sufficient to show that η'_n is zero on each generator of $T(E_{n+1})$. If $w \neq w^0_{n+1}$ and $i_w \notin I(E_{n,0})$, then the generator y of $T(E_{n+1})$ of the form $X^{p^{\varphi(w)}}_{m,j} - k^{p^m}_{w,j} X^{p^{\varphi(w)}}_{m,i_w}$, $j \in I_w$, belongs to a certain $T(G_{n,i})$, $i \in I_{w^0_{n+1}} \setminus \{o\}$ and $\eta'_n(y) = 0$. From the definition of π'_n we conclude that

$$\pi'_n(X^{p^{\varphi(w)}}_{m,j} - k^{p^m}_{w,j} X^{p^{\varphi(w)}}_{m,i_w}) = \pi'_n(X^{p^{\varphi(w)}}_{m,j}) - k^{p^m}_{w,j} \pi'_n(X^{p^{\varphi(w)}}_{m,i_w}) = 0$$

if $i_w \in I(E_{n,0})$, $j \in I_w$, $m \in N$. Moreover,

$$\eta'_n(X^{p^n}_{m,i_{w_n}} - k^{p^m}_{w^0_{n+1}, i_{w_n}} X^{p^n}_{m,0}) = u_{n,i_{w_n}}(X^{p^n}_{m,i_{w_n}}) = 0$$

for $w_n \in W(E_{n+1})_n$, $w_n \neq w^0_n$. Consequently $\eta_n(T(E_{n+1})) = 0$ and there exists a unique morphism $\eta_n\colon {}^{n+1}T \to {}^nQ$ in $\mathscr{H}_1$ such that $\eta'_n = \eta_n v_{n+1}$.

Now we prove that η_n is the cokernel of ${}^nT \hookrightarrow {}^{n+1}T$. Observe that η_n is an epimorphism and $\eta_n({}^nT) = 0$. Let $h\colon {}^{n+1}T \to H''$ be a morphism in $\mathscr{H}_1$ and let $h({}^nT) = 0$. Denote by ξ_n the inclusion

$$\bigotimes_{i \in I_{w^0_{n+1}} \setminus \{o\}} \bigotimes_{j \in I(E_{n,i})} K[X]^{0,j} \hookrightarrow \bigotimes_{j \in I(E_{n+1})} K[X]^{0,j}$$

and put $h' = hv_{n+1}\xi_n$. Fix $i \in I_{w^0_{n+1}} \setminus \{o\}$. From the definition of v_{n+1} and the equality $h({}^nT) = 0$ we obtain

$$\begin{aligned} h'(X^{p^n}_{m,i}) &= hv_{n+1}(X^{p^n}_{m,i}) = h(k^{p^m}_{w_{n+1},i} v_{n+1}(X^{p^n}_{m,0})) \\ &= k^{p^m}_{w_{n+1},i} hv_{n+1}(X^{p^n}_{m,0}) = 0, \quad m \in N. \end{aligned}$$

Furthermore, if $w \in W(E_{n,i}) = W(G_{n,i})$, $j \in I_w$, $m \in N$, then $h'(X^{p^{\psi(w)}}_{m,j} - k^{p^m}_{w,j} X^{p^{\varphi(w)}}_{m,i_w}) = 0$. Hence $h'(\bigotimes_{i \in I_{w^0_{n+1}} \setminus \{o\}} K(G_{n,i})) = 0$ and there exists such a morphism $h''\colon {}^nQ \to H''$ that $h''v'_n = h'$. Consequently $h''\eta_n v_{n+1} = h''v'_n\pi'_n = h'\pi'_n = hv_{n+1}\xi_n\pi'_n = hv_{n+1}$ and $h''\eta_n = h$, because v_{n+1} is an epimorphism. The proof of the lemma is now complete.

Let $j_m\colon {}^{m+1}L = K[X_{m+1}] \to {}^mL = K[X_m]$, $m \in N$, be the monomorphism in $\mathscr{H}_1$ given by $j_m(X_{m+1}) = X^p_m$.

LEMMA 4.3. *Let $m, n \in N$ and $m < n$. The monomorphism j_m induces epimorphisms*

(a) $\zeta_m\colon \mathrm{Hom}({}^mL, {}^nT) \to \mathrm{Hom}({}^{m+1}L, {}^nT)$,

(b) $\Theta_m\colon \mathrm{Hom}({}^mL, {}^nI) \to \mathrm{Hom}({}^{m+1}L, {}^nI)$.

Proof. (a) Let $f\colon {}^mL \to {}^nT$ and $g\colon {}^{m+1}L \to {}^nT$ be two morphisms in $\mathscr{H}_1$. Since ${}^mL$ and ${}^{m+1}L$ are in $\mathscr{L}_1$, we know that f and g are uniquely determined by elements $f(X_m)$ and $g(X_{m+1})$, respectively, which must be primitive. From Lemma 3.5 (1) we get

$$f(X_m) = \sum_{i \in K_m} a_i Y^{p^m}_{0,i},$$
$$g(X_{m+1}) = \sum_{j \in K_{m+1}} b_j Y^{p^{m+1}}_{0,j}$$

where $a_i, b_j \in K$. Suppose $m = 1$. Then $K_m = I = \{o\}$ for each $m \in N$ and $f(X_m) = a_0 X^{p^m}_{0,0}$, $g(X_{m+1}) = b_0 X^{p^{m+1}}_{0,0}$. Hence if $b_0 \in K$ and a_0 is a p'th root of b_0, then we have $\zeta_m(f)(X_{m+1}) = f(j_m(X_{m+1})) = f(X^p_m) = (f(X_m))^p = a^p_0 X^{p^{m+1}}_{0,0} = b_0 X^{p^{m+1}}_{0,0} = g(X_{m+1})$ and $\zeta_m(f) = g$. Consequently, ζ_m is an epimorphism. Suppose now that $m > 1$. It is easy to observe that

$$K_m = \{i_{w_m};\ w_m \in W_m\} = \bigcup_{w_m \in W_m} I_{w_m},$$
$$K_{m+1} = \{i_{w_{m+1}};\ w_{m+1} \in W_{m+1}\} = \bigcup_{w_{m+1} \in W_{m+1}} I_{w_{m+1}}$$

(see Section 2). Then in the above notation we have

$$\begin{aligned}\zeta_m(f)(X_{m+1}) &= f(j_m(X_{m+1})) = f(X_m^p) = f(X_m)^p \\ &= \Big(\sum_{i\in K_m} a_i Y_{0,i}^{p^m}\Big)^p = \sum_{i\in K_m} a_i^p Y_{0,i}^{p^{m+1}} = \sum_{w_{m+1}\in W_{m+1}} \Big(\sum_{i\in I_{w_{m+1}}} a_i^p Y_{0,i}^{p^{m+1}}\Big) \\ &= \sum_{w_{m+1}\in W_{m+1}} \Big(\sum_{i\in I_{w_{m+1}}} a_i^p k_{w_{m+1},i}\Big) Y_{0,i_{w_{m+1}}}^{p^{m+1}}.\end{aligned}$$

Recall that for each $w_{m+1}\in W_{m+1}$ the set $\{k_{w_{m+1},i};\ i\in I_{w_{m+1}}\}$ is a basis of K over K^p. Thus for each $b_{i_{w_{m+1}}}\in K$ there exist elements $a_i\in K$, $i\in I_{w_{m+1}}$, such that

$$b_{i_{w_{m+1}}} = \sum_{i\in I_{w_{m+1}}} a_i^p k_{w_{m+1},i},\quad w_{m+1}\in W_{m+1}.$$

Consequently, if we put

$$f(X_m) = \sum_{i\in K_m} a_i Y_{0,i}^{p^m},$$

then $\zeta_m(f) = g$ and ζ_m is an epimorphism. Condition (a) is proved.

In order to prove (b) consider the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}({}^mL, {}^nT) & \xrightarrow{\zeta_m} & \mathrm{Hom}({}^{m+1}L, {}^nT) \\ \downarrow & & \downarrow \\ \mathrm{Hom}({}^mL, {}^nR) & \longrightarrow & \mathrm{Hom}({}^{m+1}L, {}^nR) \end{array}$$

induced by the canonical epimorphism ${}^nT\to{}^nR$. Since ${}^mL$ and ${}^{m+1}L$ are projective objects in $\mathscr{L}_1$ (Theorem 1.1), and ${}^nT$ and ${}^nR$ belong to $\mathscr{L}_1$ we know that the vertical maps are epimorphisms. Moreover, since the subobject of ${}^nI$ generated by all primitive elements is isomorphic with ${}^nR$, then $\mathrm{Hom}({}^mL, {}^nI) = \mathrm{Hom}({}^mL, {}^nR)$ and $\mathrm{Hom}({}^{m+1}L, {}^nI) = \mathrm{Hom}({}^{m+1}L, {}^nR)$. Hence (b) follows. This finishes the proof of the lemma.

Proof of statement (1) in the theorem. First we prove that ${}^nI$, $n\in N$, are injective objects in $\mathscr{H}_1$. For this aim, by 1.6(a) in [10], it is sufficient to show that $\mathrm{Ext}^1({}^mS, {}^nI) = 0 = \mathrm{Ext}^1({}^mL, {}^nI)$ for $m, n\in N$. We apply the induction on $0\leqslant n<\infty$. If $n = 0$, then applying the arguments from p. 140 in [10], one can show that there exists an isomorphism $f\colon ({}^0L)^*\to{}^0I$ and hence ${}^0I$ is an injective object in $\mathscr{H}_1$. Now assume that $n>0$ and the induction hypothesis holds for $0\leqslant k\leqslant n-1$. Denote by $g\colon {}^0L\to{}^nL$ and $h\colon {}^0I\to{}^nD$ the isomorphisms of Hopf algebras (of degrees p^n) given by $g(X_0) = X_n$ and $h(Y_{r,0}) = Y_{r,0}^{p^n}$, $r\in N$. Then the composite map $f' = hf(g^*)\colon ({}^nL)^*\to{}^nD$ is of zero degree and it is a map in $\mathscr{H}_1$. Let m be a fixed natural number. By Lemma 4.2(a) we have an exact sequence

$$0\to({}^nL)^*\to{}^nI\xrightarrow{\pi_n}{}^nU\to 0$$

which induces the exact sequence

$$\mathrm{Ext}^1({}^mL, ({}^nL)^*)\to\mathrm{Ext}^1({}^mL, {}^nI)\to\mathrm{Ext}^1({}^mL, {}^nU).$$

By Lemma 2.2 in [10], $\mathrm{Ext}^1({}^mL, ({}^nL)^*) = 0$. Moreover, we know that $\mathscr{H}_1$ is a locally noetherian Grothendieck category, and so, by Proposition 6.51 in [4] and by the inductive assumption ${}^nU$ is an injective object in $\mathscr{H}_1$, as the coproduct of copies of ${}^{n-1}I$. Consequently $\mathrm{Ext}^1({}^mL, {}^nU) = 0$ and $\mathrm{Ext}^1({}^mL, {}^nI) = 0$ for $m\in N$. Further the exact sequence

$$0\to{}^{m+1}L\xrightarrow{j_m}{}^mL\to{}^mS\to 0$$

induces the exact sequence

$$0\to\mathrm{Hom}({}^mS, {}^nI)\to\mathrm{Hom}({}^mL, {}^nI)\xrightarrow{\Theta_m}\mathrm{Hom}({}^{m+1}L, {}^nI)\to\mathrm{Ext}^1({}^mS, {}^nI)\to 0$$

since we have proved that $\mathrm{Ext}^1({}^mL, {}^nI) = 0$. But, by Lemma 4.3 Θ_m is an epimorphism, and so $\mathrm{Ext}^1({}^mS, {}^nI) = 0$. Thus ${}^nI$ is injective. Finally, by Proposition 3.6, the object ${}^nI$ is coirreducible for each $n\in N$, and so the natural monomorphism $i_n\colon {}^nS\to{}^nI$ given by $i_n(Y_n) = Y_{0,0}^{p^n}$, where Y_n denotes a generator of ${}^nS$, is essential. Statement (1) of the theorem is proved.

Before proving statements (2) and (4) of the theorem we need one additional technical lemma. Put

$${}^{r,n}Q = {}^nU\otimes\Big(\bigotimes_{k=n}^{r}{}^nQ\Big),\qquad r, n\in N,\ r\geqslant n,$$

$${}^{\infty,n}Q = {}^nU\otimes\Big(\bigotimes_{k=n}^{\infty}{}^nQ\Big),\qquad n\in N.$$

Further, let $\lambda_{n,r}\colon {}^{r,n}Q\to{}^{r+1,n}Q$ be the inclusions ${}^{r,n}Q\hookrightarrow{}^{r,n}Q\otimes{}^{r+1}Q = {}^{r+1,n}Q$, $r, n\in N$, $r\geqslant n$. Then ${}^{\infty,n}Q = \varinjlim_{r\geqslant n}\{{}^{r,n}Q, \lambda_{n,r}\}$ for each $n\in N$.

LEMMA 4.4. *For each $n\in N$, there exists an exact sequence*

$$0\to{}^nC\to{}^{\infty}I\to{}^{\infty,n}Q\to 0.$$

Proof. Let n be a fixed natural number. Since ${}^{\infty}I = \bigcup_{r>n}{}^rT$, it is sufficient to show that there exist commutative diagrams

$$(r)\qquad\begin{array}{ccccccccc} 0 & \to & {}^nC & \hookrightarrow & {}^rT & \xrightarrow{\pi_{r,n}} & {}^{r-1,n}Q & \to & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\lambda_{n,r-1}} & & \\ 0 & \to & {}^nC & \to & {}^{r+1}T & \xrightarrow{\pi_{r+1,n}} & {}^{r,n}Q & \to & 0 \end{array}$$

with exact rows, where ${}^{n-1,n}Q = {}^nU$ and $\lambda_{n,n-1}$ is the inclusion ${}^nU\to{}^nU\otimes{}^nQ$. We apply induction on $r\geqslant n$. Let $r = n$. Observe that there exists a commutative diagram in $\mathscr{H}_1$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & K & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
0 & \to & {}^{n}C & \hookrightarrow & {}^{n}T & \xrightarrow{\pi_{n,n}} & {}^{n}U & \to & 0 \\
 & & \| & & \cap\downarrow & & \downarrow{\scriptstyle\gamma} & & \\
0 & \to & {}^{n}C & \hookrightarrow & {}^{n+1}T & \xrightarrow{\alpha} & A & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\eta_n} & & \downarrow{\scriptstyle\sigma} & & \\
 & & 0 & & {}^{n}Q & & B & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where η_n, $\pi_{n,n} = \mu_n$ are the morphisms from Lemma 4.2, $(A, \alpha) = \mathrm{Coker}({}^{n}C \hookrightarrow {}^{n+1}T)$, $(K, \beta) = \ker\gamma$, $(B, \sigma) = \mathrm{Coker}\gamma$. Then, by the Snake Lemma ([9], p. 230) we conclude that $K = 0$ and ${}^{n}Q \approx B$. Thus we get the exact sequence

$$0 \to {}^{n}U \to A \to {}^{n}Q \to 0$$

which splits because ${}^{n}U$ is injective in $\mathscr{H}_1$. Consequently, in the diagram above we can replace A by ${}^{n,n}Q$, γ by $\lambda_{n,n}$ and α by such a morphism $\pi_{n+1,n}$ that the diagram (n) is commutative and has exact rows. Now assume that $m > n$ and that we have commutative diagrams (r) for $n \leqslant r < m$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & K' & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\beta'} & & \\
0 & \to & {}^{n}C & \hookrightarrow & {}^{m}T & \xrightarrow{\pi_{m,n}} & {}^{m-1,n}Q & \to & 0 \\
 & & \| & & \cap\downarrow & & \downarrow{\scriptstyle\gamma'} & & \\
0 & \to & {}^{n}C & \hookrightarrow & {}^{m+1}T & \xrightarrow{\alpha'} & A' & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\eta_m} & & \downarrow{\scriptstyle\sigma'} & & \\
 & & 0 & & {}^{m}Q & & B' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, and $(A', \alpha') = \mathrm{Coker}({}^{n}C \hookrightarrow {}^{m+1}T)$, $(K', \beta') = \ker\gamma'$, $(B', \sigma') = \mathrm{Coker}\gamma'$. As above, one can prove that there exists a diagram of the form (m). This finishes the proof of the lemma.

Proof of statement (2) of the theorem. Let m be a fixed natural number. Since ${}^{\infty 0}Q$ is injective, by Lemma 4.4 we have the exact sequence

$$\mathrm{Ext}^1({}^{m}L, {}^{0}C) \to \mathrm{Ext}^1({}^{m}L, {}^{\infty}I) \to 0 .$$

Further, as on pp. 143 and 144 in [10] one can show that $\mathrm{Ext}^1({}^{m}L, {}^{0}C) = 0$ and therefore $\mathrm{Ext}^1({}^{m}L, {}^{\infty}I) = 0$ for $m \in N$. We now consider the exact sequence

$$0 \to \mathrm{Hom}({}^{m}S, {}^{\infty}I) \to \mathrm{Hom}({}^{m}L, {}^{\infty}I) \xrightarrow{\Theta_\infty} \mathrm{Hom}({}^{m+1}L, {}^{\infty}I) \to \mathrm{Ext}^1({}^{m}S, {}^{\infty}I) \to \mathrm{Ext}^1({}^{m}L, {}^{\infty}I) \to \ldots$$

induced by the exact sequence

$$0 \to {}^{m+1}L \xrightarrow{j_m} {}^{m}L \to {}^{m}S \to 0 .$$

Since ${}^{m}L$ is a noetherian object, we have

$$\mathrm{Hom}({}^{m}L, {}^{\infty}I) = \mathrm{Hom}({}^{m}L, \bigcup_{r>m} {}^{r}T) = \bigcup_{r>m} \mathrm{Hom}({}^{m}L, {}^{r}T) .$$

By Lemma 4.3(a), ζ_r are epimorphisms and therefore Θ_∞ is also an epimorphism. Thus $\mathrm{Ext}^1({}^{m}S, I) = 0$ and by 1.6(a) in [10] we know that ${}^{\infty}I$ is an injective object in $\mathscr{H}_1$. Finally, since ${}^{\infty}I = H(G_\infty^{\varphi_\infty})$ is coirreducible, the natural monomorphisms $\beta_n\colon {}^{n}L \to {}^{\infty}I$, $n \in N$, given by $\beta_n(X_n) = X_{0,0}^{p^n}$ are essential. Statement (2) is proved.

Proof of statement (3). Let Q be an indecomposable injective object in $\mathscr{H}_1$. Then by Proposition 6.36 in [4], Q is coirreducible. Since each homogeneous element of minimal degree $\geqslant 2$ in Q is primitive, Q contains a subobject F which is isomorphic either with ${}^{n}L$ or with ${}^{n}S$ for a certain $n \in N$. Hence there exists a commutative diagram

$$\begin{array}{ccc}
F & \xrightarrow{v} & Q \\
{\scriptstyle t}\downarrow & \swarrow{\scriptstyle u} & \\
{}^{m}I & &
\end{array}$$

with $m = \infty$ provided $F \approx {}^{n}L$ and $m = n$ whenever $F \approx {}^{n}S$, in which t and v are essential monomorphisms. Therefore u is an isomorphism because ${}^{m}I$ is indecomposable. Then statement (3) is proved.

Proof of statement (4). To prove (4) it is sufficient to show that

$$\mathrm{Ext}^i(-, {}^{n}L) = 0 = \mathrm{Ext}(-, {}^{n}S) \quad \text{for} \quad i \geqslant 3,\ n \in N$$

(see 1.6(c) in [10] (p. 136)). Fix $n \in N$ and denote by $\gamma_n\colon {}^{n}C \to {}^{n+1}C$ the morphism given by the formula

$$\gamma_n(X_{r,0}^{p^n}) = \begin{cases} 0 & \text{for} \quad r = 0, \\ X_{r-1,0}^{p^{n+1}} & \text{for} \quad r > 0 . \end{cases}$$

The fact that γ_n is in $\mathscr{H}_1$ follows from [5], p. 544, 545. Now we observe that there exist commutative diagrams in H_1

$$
\text{(n)}\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & {}^{n}H' & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta_n} & & \\
0 & \to & {}^{n}L & \overset{\beta_n}{\hookrightarrow} & {}^{\infty}I & \overset{\tau_n}{\longrightarrow} & {}^{n}H & \to & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \chi_n} & & \\
0 & \to & {}^{n}C & \hookrightarrow & {}^{\infty}I & \overset{\omega_n}{\longrightarrow} & {}^{\infty n}Q & \to & 0 \\
 & & \downarrow{\scriptstyle \gamma_n} & & \downarrow & & \downarrow & & \\
 & & {}^{n+1}C & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

with exact rows and columns, where $({}^nH, \tau_n) = \operatorname{Coker}\beta_n$, $({}^nH', \delta_n) = \ker\chi_n$. By the Snake Lemma we conclude that ${}^nH \approx {}^{n+1}C$. Then, considering the long exact sequences of Ext's induced by the upper row and the right column of the diagram (n) and using the fact that ${}^{\infty}I$ and ${}^{\infty n}Q$ are injective, we obtain equivalences of functors

$$\operatorname{Ext}^i(-, {}^nL) = \operatorname{Ext}^{i-1}(-, {}^nH),\ \operatorname{Ext}^i(-, {}^nH) = \operatorname{Ext}^i(-, {}^{n+1}C).$$

Moreover, the long exact sequence of Ext's induced by the bottom row of $(n+1)$ yields $\operatorname{Ext}^i(-, {}^{n+1}C) = 0$ for $i \geqslant 2$. Hence $\operatorname{Ext}^i(-, {}^nL) = 0$ for $i \geqslant 3$. Furthermore, for each $n \in N$, the exact sequence

$$0 \to {}^{n+1}L \to {}^nL \to {}^nS \to 0$$

induces the exact sequence

$$\operatorname{Ext}^i(-, {}^{n+1}L) \to \operatorname{Ext}^i(-, {}^nL) \to \operatorname{Ext}^i(-, {}^nS) \to \operatorname{Ext}^{i+1}(-, {}^{n+1}L),$$

and we conclude that $\operatorname{Ext}^i(-, {}^nS) = 0$ for all $i \geqslant 3$. Consequently $\operatorname{gl.dim}\mathscr{H} = \operatorname{gl.dim}\mathscr{H}_1 \leqslant 2$. The fact that the equality holds can be proved by using the same type of arguments as in the proof of Theorem 3.3 on p. 145 in [10]. The proof of Theorem 4.1 is complete.

We have the following characterization of indecomposable injective objects in $\mathscr{L}_1$.

COROLLARY 4.5. (1) *For each $n \in N$, ${}^nR$ is the injective envelope of ${}^nS$ in $\mathscr{L}_1$ and ${}^{\infty}R$ is the injective envelope of ${}^nL$.*

(2) *Every indecomposable injective object in $\mathscr{L}_1$ is isomorphic with a certain ${}^nR$, $0 \leqslant n \leqslant \infty$.*

Proof. Observe that the subobject of ${}^nI$ generated by of all primitive elements is isomorphic with ${}^nR$, $0 \leqslant n \leqslant \infty$. Moreover, if $g\colon L \to H$ is a morphism in $\mathscr{H}_1$ and L belongs to $\mathscr{L}_1$ then the image of g also belongs to $\mathscr{L}_1$. Then the corollary follows from the fact that Q' is an indecomposable injective object in $\mathscr{L}_1$ iff Q' is a maximal subobject from $\mathscr{L}_1$ of an indecomposable injective object Q in $\mathscr{H}_1$.

COROLLARY 4.6. (1) *Every injective object Q in $\mathscr{H}_1$ is a coproduct of objects isomorphic with the objects ${}^nI$, $0 \leqslant n \leqslant \infty$, and any two such decompositions of Q are isomorphic.*

(2) *Every injective object L in $\mathscr{L}_1$ is a coproduct of objects isomorphic with the objects ${}^nR$, $0 \leqslant n \leqslant \infty$, and any two such decompositions of L are isomorphic.*

Proof. It is an immediate consequence of Theorem 4.1 and of Theorem 8.11 on p. 377 in [9].

Remark 4.7. From the proof of Theorem 4.1 we conclude that if $\varphi_n, \psi_n, \leqslant_n$ is a basic ballast of the $\mathfrak{m}$-special tree G_n, $0 \leqslant n \leqslant \infty$, then $H(G_n^{\varphi_n})$ (resp. $L(G_n^{\varphi_n})$) is the injective envelope in $\mathscr{H}_1$ (resp. in $\mathscr{L}_1$) of object ${}^nS$ or ${}^nL$, respectively. Then ${}^nI \approx H(G_n^{\varphi_n})$ and ${}^nR \approx L(G_n^{\varphi_n})$, $0 \leqslant n \leqslant \infty$.

§ 5. Endomorphism rings. In this section we give a description of endomorphism rings of all ${}^nR$, $0 \leqslant n \leqslant \infty$. In Section 1 we showed that $\mathscr{L}_1$ is a K-category.

For each $n \in N$, denote by $K^{(n)}$ the K-algebra structure on K given by the Frobenius map $\cdot^{p^n}\colon K \to K$. Moreover, put $K^{(\infty)} = \varinjlim_{n \geqslant 0}\{K^{(n)}, c_n\}$ where $c_n = \cdot^p\colon K^{(n)} \to K^{(n+1)}$. It is easy to observe that K^{∞} is a field.

The main result of this section is the following.

THEOREM 5.1. *For each $0 \leqslant n \leqslant \infty$, the K-algebras $\operatorname{End}({}^nR)$ and $K^{(n)}$ are isomorphic.*

Before proving the theorem we will prove two technical lemmas.

LEMMA 5.2. *Let n be a natural number. Then*

(1) *If $f\colon {}^nF \to {}^nF$ is a morphism in $\mathscr{L}_1$, then $f(Y_0^{p^n}) = aY_0^{p^n}$ for a certain $a \in K$.*

(2) *For each $a \in K$, there exists a unique morphism $f_a\colon {}^nF \to {}^nF$ in $\mathscr{L}_1$ such that $f_a(Y_0^{p^n}) = aY_0^{p^n}$.*

Proof. If either $\mathfrak{m} = 1$ or $n = 0$, then the proof is obvious. Suppose $\mathfrak{m} > 1$ and $n > 0$. Let $\varphi_n(w) = \{k_{w,i};\ i \in I_w\}$ for each $w \in W(E_n)$. Let $f\colon {}^nF \to {}^nF$ be a morphism in $\mathscr{L}_1$. It is uniquely determined by elements $f(Y_i)$, $i \in I(E_n)$. Let

$$f(Y_i) = \sum_{j \in I} a_{ij} Y_j, \quad i \in I.$$

Further, for $2 \leqslant r \leqslant n$ and $w_r \in W_r$ denote by $\overline{w}_r$ the unique node of W_{r-1} such that $i_{\overline{w}_r} = i_{w_r}$. For $r = 1$, $w_1 \in W_1$ we put $\overline{w}_1 = i_{w_1}$.

Now by induction on $1 \leqslant r \leqslant n$ we will prove that

$$(*)\qquad \sum_{\substack{t_{r-1} \in W_{r-1} \\ i_{t_{r-1}} \in I s_r}} a^p_{w-1,t_{r-1}} k_{s_r, i_{t_{r-1}}} = a_{w_r,s_r} k_{w_r, i_{w_{r-1}}}$$

$$(**)\qquad f(Y^{p^r}_{i_{w_r}}) = \sum_{s_r \in W_r} a_{w_r,s_r} Y^{p^r}_{i_{s_r}},$$

for each $w_r, s_r \in W_r$, $i_{w_{r-1}} \in I_{w_r}$, where $W_0 = I$,

$$a_{w_r,s_r} = \sum_{\substack{t_{r-1} \in W_{r-1} \\ i_{t_{r-1}} \in I_{s_r}}} a^p_{w_r,t_{r-1}} k_{s_r, i_{t_{r-1}}}.$$

Let $r=1$. Then for each $w_1\in W_1$, $i\in I_{w_1}$, we have

$$f(Y_i^p)=f(Y_i)^p=\sum_{j\in I}a_{ij}^pY_j^p=\sum_{s_1\in W_1}\Big(\sum_{j\in I_{s_1}}a_{ij}^pY_j^p\Big)=\sum_{s_1\in W_1}\Big(\sum_{j\in I_{s_1}}a_{ij}^pk_{s_1,j}\Big)Y_{j_{s_1}}^p,$$

$$f(k_{w_1,i}Y_{i_{w_1}}^p)=k_{w_1,i}f(Y_{i_{w_1}}^p)=k_{w_1,i}\Big(\sum_{s_1\in W_1}\Big(\sum_{j\in I_{s_1}}a_{i_{w_1},j}^pk_{s_1,j}\Big)Y_{i_{s_1}}^p\Big).$$

The equalities (*) and (**) for $r=1$ follow from the fact that $Y_{i_{s_1}}^p$, $s_1\in W_1$, are linearly independent over K and that $\overline{w}_1=i_{w_1}$ for each $w_1\in W_1$. Now, suppose that (*) and (**) hold for each m, $1\leqslant m<r$, $r\leqslant n$. Then for each $w_r\in W_r$, $i_{w_{r-1}}\in I_{w_r}$, we have

$$f(Y_{i_{w_{r-1}}}^{p^r})=f(Y_{i_{w_{r-1}}}^{p^{r-1}})^p=\Big(\sum_{t_{r-1}\in W_{r-1}}a_{\overline{w}_{r-1},t_{r-1}}Y_{i_{t_{r-1}}}^{p^{r-1}}\Big)^p$$

$$=\sum_{t_{r-1}\in W_{r-1}}a_{\overline{w}_{r-1},t_{r-1}}^pY_{i_{t_{r-1}}}^{p^r}=\sum_{s_r\in W_r}\Big(\sum_{\substack{t_{r-1}\in W_{r-1}\\ i_{t_{r-1}}\in I_{s_r}}}a_{\overline{w}_{r-1},t_{r-1}}^pY_{i_{t_{r-1}}}^{p^r}\Big)$$

$$=\sum_{s_r\in W_r}\Big(\sum_{\substack{t_{r-1}\in W_{r-1}\\ i_{t_{r-1}}\in I_{s_r}}}a_{\overline{w}_{r-1},t_{r-1}}^pk_{s_r,i_{t_{r-1}}}\Big)Y_{i_{s_r}}^{p^r},$$

$$f(k_{w_r,i_{w_{r-1}}}Y_{i_{w_r}}^{p^r})=k_{w_r,i_{w_{r-1}}}f(Y_{i_{w_r}}^{p^r})=k_{w_r,i_{w_{r-1}}}f(Y_{i_{\overline{w}_r}}^{p^r})$$

$$=k_{w_r,i_{w_{r-1}}}\Big(\sum_{s_r\in W_r}\Big(\sum_{\substack{t_{r-1}\in W_{r-1}\\ i_{t_{r-1}}\in I_{s_r}}}a_{\overline{w}_r,t_{r-1}}^pk_{s_r,i_{t_{r-1}}}\Big)Y_{i_{s_r}}^{p^r}\Big).$$

Hence we get (*) and (**) for r, because the elements $Y_{i_{s_r}}^{p^r}$, $s_r\in W_r$, are linearly independent over K. Then the equalities (*) and (**) are proved.

From the definition of the tree E_n it follows that $W_n=\{w_n\}$ and $W_k=\varnothing$ for $k\geqslant n+1$. Therefore, for each $w_{n-1}\in W_{n-1}$, the element $i_{w_{n-1}}$ belongs to I_{w_n}. Hence, for $r=n$, the equalities (*) and (**) have the form

$$\sum_{t_{n-1}\in W_{n-1}}a_{w_{n-1},t_{n-1}}^pk_{w_n,i_{t_{n-1}}}=a_{w_n,w_n}k_{w_n,i_{w_{n-1}}},\qquad w_{n-1}\in W_{n-1}.$$

$$f(Y_0^{p^n})=a_{w_n,w_n}Y_0^{p^n}$$

where $a_{w_n,w_n}=\sum_{t_{n-1}\in W_{n-1}}a_{w_n,t_{n-1}}^pk_{w_n,i_{t_{n-1}}}$, since $i_{w_n}=0$. Thus statement (1) is proved.

(2) Let a be an element of K and put $a_{w_n,w_n}=a$. Then, using the fact that $\{k_{w,j};\ j\in I_w\}$ is a basis of K over K^p, we conclude that there exist elements $a_{w_r,s_r}\in K$, w_r, $s_r\in W_r$, $1\leqslant r\leqslant n$, satisfying the equalities (*) and (**). Moreover, it is easy to observe that for a fixed $w_r\in W_r$ the set $\{s_r\in W_r;\ a_{w_r,s_r}\neq 0\}$ is finite. We define f_a: ${}^nF\to{}^nF$ putting $f_a(Y_i)=\sum_{j\in I}a_{ij}Y_j$, $i\in I$. The correctness of f_a follows from (*) and (**). Furthermore, $f_a(Y_0^{p^n})=aY_0^{p^n}$ since $a_{w_n,w_n}=a$. This completes the proof of the lemma.

Corollary 5.3. *Let n be a natural number. Then*

(1) *If g: ${}^nR\to{}^nR$ is a morphism in $\mathscr{L}_1$, then $g(Y_0^{p^n})=aY_0^{p^n}$ for a certain $a\in K$.*

(2) *For each $a\in K$, there exists a unique morphism g_a: ${}^nR\to{}^nR$ in $\mathscr{L}_1$ such that $g_a(Y_0^{p^n})=aY_0^{p^n}$.*

Proof. First we observe that the natural sequence

$$0\to(Y_0^{p^{n+1}})\to{}^nF\xrightarrow{v_n}{}^nR\to 0$$

is exact. Moreover, $I(E_n)=I(G_n)$ and $W(E_n)=W(G_n)$. Hence for each f: ${}^nF\to{}^nF$ there exists a unique morphism g: ${}^nR\to{}^nR$ such that $v_nf=gv_n$ and similarly for each g': ${}^nR\to{}^nR$ there exists a unique morphism f': ${}^nF\to{}^nF$ such that $v_nf'=g'v_n$. Then the corollary is a consequence of Lemma 5.2.

Lemma 5.4. *Let n be a natural number. Then*

(1) *If h: ${}^0L\to{}^nF$ is a morphism in $\mathscr{L}_1$, then $h(Y_0^{p^n})=aY_0^{p^n}$ for a suitable $a\in K$.*

(2) *For each $a\in K$, there exists a unique morphism h_a: ${}^nF\to{}^nF$ in $\mathscr{L}_1$ such that $h_a(Y_0^{p^n})=aY_0^{p^n}$.*

The proof is similar to proof of Lemma 5.2 and it is left to the reader.

Proof of the theorem. Let n be a fixed natural number. By Corollary 5.3 we have a bijection

$$\omega_n\colon \mathrm{End}({}^nR)\to K^{(n)}$$

which assigns to each $g\in\mathrm{End}({}^nR)$ an element $a\in K$ such that $g(Y_0^{p^n})=aY_0^{p^n}$. We will show that ω_n is an isomorphism of K-algebras. Let $f,g\in\mathrm{End}({}^nR)$. If $f(Y_0^{p^n})=aY_0^{p^n}$, $g(Y_0^{p^n})=bY_0^{p^n}$, then

$$(f+g)(Y_0^{p^n})=\cdot(f\otimes g)\Delta(Y_0^{p^n})=f(Y_0^{p^n})+g(Y_0^{p^n})=(a+b)Y_0^{p^n},$$

$$(f\cdot g)(Y_0^{p^n})=f(bY_0^{p^n})=abY_0^{p^n}.$$

Consequently, $\omega_n(f+g)=\omega_n(f)+\omega_n(g)$, $\omega_n(f\cdot g)=\omega_n(f)\cdot\omega_n(g)$ and clearly $\omega_n(\mathrm{id})=1_{K^{(n)}}$. This proves that ω_n is an isomorphism of fields. We will show that it is also K-linear. Recall that $\mathscr{L}_1\approx\mathscr{K}^{\mathrm{op}}$-Mod and this equivalence is given by the correspondence $H\to\mathrm{Hom}_{\mathscr{L}_1}(-,H)=h_H$, $H\in\mathscr{L}_1$. Moreover, the K-algebra structure on $\mathrm{Hom}_{\mathscr{L}_1}({}^nR,{}^nR)$ is obtained from the K-algebra structure on $\mathrm{Hom}_{\mathscr{K}^{\mathrm{op}}\text{-Mod}}(h_{{}^nR},h_{{}^nR})$ (see Theorem 1.1). Furthermore, by Proposition 3.1 in [14] and Theorem 1.1 we have

$$(a\cdot h_f)({}^mL)(u)=h_f(u(a\cdot 1_{{}^mL}))=fu(a\cdot 1_{{}^mL})$$

for each $f\in\mathrm{End}({}^nR)$, $u\in\mathrm{Hom}({}^mL,{}^nR)$, $m\in N$, $a\in K$. For every $i\in I(G_n)$, denote by α_i the morphism from ${}^0L$ to ${}^nR$ given by $\alpha_i(X)=Y_i$, where X is a generator of ${}^0L$. Then the K-algebra structure on $\mathrm{End}({}^nR)$ is defined as follows: if $f\in\mathrm{End}({}^nR)$, $i\in I(G_n)$, $a\in K$, then

$$(a\cdot f)(Y_i)=\big((a\cdot h_f)({}^0L)\big)(\alpha_i(X))=f\alpha_i(a\cdot 1_{{}^0L})(X)=f\alpha_i(aX)=af(Y_i).$$

Hence, if $f(Y_0^{p^n})=bY_0^{p^n}$, then we have

$$(a\cdot f)(Y_0^{p^n})=((a\cdot f)(Y_0))^{p^n}=(af(Y_0))^{p^n}=a^{p^n}f(Y_0^{p^n})=a^{p^n}bY_0^{p^n}.$$

Consequently, $\omega_n(a\cdot f)=a^{p^n}\cdot\omega_n(f)$ and ω_n is an isomorphism of K-algebras.

We now show that the K-algebras $\mathrm{End}({}^{\infty}R)$ and $K^{(\infty)}$ are isomorphic. This is obvious for $\mathfrak{m} = 1$ because ${}^{\infty}R = {}^{0}L$. Suppose $\mathfrak{m} > 1$. Similarly as above, one can prove that the K-algebras $\mathrm{End}({}^{n}F)$ and $K^{(n)}$, $n \in N$, are isomorphic. Further, we observe that the morphisms $t_n\colon E_n \to E_{n+1}$ and $s_n\colon E_n \to E_{\infty} = G_{\infty}$, $n \in N$, of $\mathfrak{m}$-special trees, defined in Section 2, induce monomorphisms $L(t_n)\colon {}^{n}F \to {}^{n+1}F$ and $L(s_n)\colon {}^{n}F \to {}^{\infty}R$. Moreover, we recall that for each $n \in N$ the set $Z(E_n)$ of all inputs of E_n is empty. Hence from Lemma 3.5(2) the algebras ${}^{n}F$ and ${}^{\infty}R = \varinjlim_{n \geq 0} \{{}^{n}F, s_n\}$, $n \in N$, contain no nilpotents. On the other hand, it is easy to observe that for $n \geq m$ each element of ${}^{n}F/{}^{m}F$ and ${}^{\infty}R/{}^{n}F$ is nilpotent. Then for each $n \in N$

$$\mathrm{Hom}({}^{n}F/{}^{0}F, {}^{n}F) = 0 = \mathrm{Hom}({}^{\infty}R/{}^{n}F^{\infty}, R)\,.$$

From Lemma 5.2 and Lemma 5.4 the exact sequence

$$0 \to {}^{0}F \to {}^{n}F \to {}^{n}F/{}^{0}F \to 0$$

induces the following isomorphisms of K-linear spaces

$$\gamma_n\colon \mathrm{Hom}({}^{n}F, {}^{n}F) \to \mathrm{Hom}({}^{0}F, {}^{n}F), \qquad n \in N\,.$$

Further, from Corollary 4.5(2) the functor $\mathrm{Hom}(-, {}^{\infty}R)$ is exact. Then the exact sequences

$$0 \to {}^{n}F \to {}^{\infty}R \to {}^{\infty}R/{}^{n}F \to 0$$

induce the following isomorphisms of K-linear spaces

$$h_n\colon \mathrm{Hom}({}^{\infty}R, {}^{\infty}R) \to \mathrm{Hom}({}^{n}F, {}^{\infty}R)\,.$$

It is easy to check that the following diagram of K-linear spaces

$$\begin{array}{ccc} K^{(n)} \approx \mathrm{Hom}({}^{n}F, {}^{n}F) & \xrightarrow{\gamma_n} & \mathrm{Hom}({}^{0}F, {}^{n}F) \\ \downarrow{\scriptstyle c_n} & & \downarrow \\ K^{(n+1)} \approx \mathrm{Hom}({}^{n+1}F, {}^{n+1}F) & \xrightarrow{\gamma_{n+1}} & \mathrm{Hom}({}^{0}F, {}^{n+1}F) \end{array}$$

commutes for each $n \in N$. Hence and from the fact that ${}^{0}F$ is a noetherian object in $\mathscr{L}_1$ we obtain the following sequence of K-linear isomorphisms:

$$\mathrm{Hom}({}^{\infty}R, {}^{\infty}R) \xrightarrow{h_0} \mathrm{Hom}({}^{0}F, {}^{\infty}R) = \mathrm{Hom}({}^{0}F, \varinjlim_{n \geq 0} {}^{n}F)$$
$$\approx \varinjlim_{n \geq 0} \mathrm{Hom}({}^{0}F, {}^{n}F) \approx \varinjlim_{n \geq 0} K^{(n)} = K^{(\infty)}\,.$$

In order to prove the theorem it is sufficient to show that the K-linear isomorphism $\omega_{\infty}\colon \mathrm{End}({}^{\infty}R) \to K^{(\infty)}$ is a ring homomorphism. For this purpose we observe that, if $f \in \mathrm{Hom}({}^{\infty}R, {}^{\infty}R)$ and $f({}^{0}F) \subset {}^{m}F$, then for $n \geq m$ there exists such an element $g_n \in \mathrm{Hom}({}^{n}F, {}^{n}F) \subset \mathrm{Hom}({}^{n}F, {}^{\infty}R)$ that $\gamma_n(g_n) = h_0(f) = h_n(f)|_{{}^{0}F}$ and $g_n = h_n(f)$.

Hence, if $f \in \mathrm{Hom}({}^{\infty}R, {}^{\infty}R)$ and $f({}^{0}F) \subset {}^{m}F$, then $f({}^{n}F) \subset {}^{n}F$ for $n \geq m$. Let $f, g \in \mathrm{End}({}^{\infty}R)$ and $f({}^{0}F) \subset {}^{m}F$, $g({}^{0}F) \subset {}^{m}F$. For $n \geq m$ we have

$$h_0(gf) = \gamma_n(h_n(gf)) = \gamma_n(h_n(g)h_n(f))\,.$$

Then ω_{∞} is an isomorphism of K-algebras and the theorem is proved.

In another paper in this series indecomposable projective objects in $\mathscr{H}_1$ and their endomorphism rings will be described under the assumption that $(K:K^p)$ is finite.

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Accepté par la Rédaction le 3. 10. 1977