Properties of the covering type and a factorization theorem

by

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Abstract. The paper contains a factorization theorem implying the possibility of factorizing a map $f\colon X\to M$, where M is a metric space such that f is equal to a composition $X\overset{h}{\to} M'\to M$, where $h\colon X\overset{\text{onto}}{\to} M'$ satisfies some additional conditions for $h^{-1}hx$, $x\in X$, and a metric space M' preserves a prescribed countable number of properties of the space X. As corollaries, some results concerning p-paracompact spaces can be obtained.

1. Preliminaries. The maps considered in this paper are assumed to be (uniformly) continuous. We use the notion of uniformity in the covering sense. A family which satisfies all the axioms of uniformity except the axiom of separation is called a *pseudouniformity*. Symbols $P\succ Q$, $P\underset{*}{\succ} Q$ mean, respectively, that P is a refinement or a star-refinement of Q. Some symbols and notations are taken from [5].

If X is a completely regular space, then by $\mathcal{U}_X^*$ we denote the greatest uniformity inducing the topology of the space X. Each pseudouniformity $\mathcal{U}\subset\mathcal{U}_X^*$ is said to be *compatible with the topology of the space* X.

There exists a functor h (see [5]) from the category of pseudouniform spaces onto the category of uniform spaces such that for each pseudouniform space $(X,\mathcal{U})$ there exists a uniform map $h\colon (X,\mathcal{U})\to(hX,h\mathcal{U})$ satisfying two conditions:

(a) $h^{-1}h\mathcal{U}=\mathcal{U}$, where $h^{-1}h\mathcal{U}=\{h^{-1}Q\colon Q\in h\mathcal{U}\}$,

(b) for each uniform map $f\colon (X,\mathcal{U})\to(Y,\mathcal{V})$ into a uniform space, there exists a unique uniform map $g\colon (hX,h\mathcal{U})\to(Y,\mathcal{V})$ such that $f=gh$.

The functor h can be obtained in the following way: for each pseudouniform space $(X,\mathcal{U})$ the set hX is obtained by a decomposition of the set X onto layers $[x]_{\mathcal{U}}=\bigcap\{\mathrm{st}(x,P)\colon P\in\mathcal{U}\}$, $hX=\{[x]_{\mathcal{U}}\colon x\in X\}$, $h\colon x\mapsto[x]_{\mathcal{U}}$ and the uniformity $h\mathcal{U}=\{P^*\colon P\in\mathcal{U}\}$, where $P^*=\{h^*u\colon u\in P\}$, $h^*u=hX-h(X-u)$. Proofs of the above remarks are given in [5].

A uniform feathering of a space X in a space $Y\supset X$ is a countable family $\mathcal{P}$ of coverings of X consisting of open sets in Y such that $\mathcal{P}|X\subset\mathcal{U}_X^*$ and $[x]_{\mathcal{P}}=\bigcap\{\mathrm{st}(x,P)\colon P\in\mathcal{P}\subset X$, for each $x\in X$.

2. Properties of a covering type. Let $\mathcal{A}$ be a countable family of relations defined on $\mathcal{U}_X^*$. A pseudouniformity $\mathcal{U}\subset\mathcal{U}_X^*$ is said to be an $\mathcal{A}$-*pseudouniformity* iff for each

$a \in \mathscr{A}$ and each $P \in \mathscr{U}$ there exists a $P' \in \mathscr{U}$ such that $(P', P) \in a$ and $(P', P) \in a$ implies $P' \underset{*}{\succ} P$.

A topological property A of a space X is said to be of the *covering-countable type* (shortly, of type A) iff there exists a countable family $\mathscr{A}$ of relations defined on $\mathscr{U}_X^*$ such that

(a) the greatest uniformity $\mathscr{U}_X^*$ is the $\mathscr{A}$-uniformity,

(b) for each $\mathscr{A}$-pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ with a countable base, the space hX with the topology induced by the uniformity h has the property A.

Let X be a subspace of a space Y. For each pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ denote by $\mathrm{ext}_Y \mathscr{U}$ the set of all the extensions, open in Y, of open coverings belonging to $\mathscr{U}$. Let $\mathscr{B}$ be a countable family of relations defined on the family $\mathrm{ext}_Y \mathscr{U}_X^*$.

A pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ is said to be a *$\mathscr{B}$-pseudouniformity* iff for each $b \in \mathscr{B}$ and $P \in \mathrm{ext}_Y \mathscr{U}$ there exists a $P' \in \mathrm{ext}_Y \mathscr{U}$ such that $(P', P) \in b$ and $(P', P) \in b$ implies $P'|X \underset{*}{\succ} P|X$ and $\mathrm{cl}_Y P' \succ P$.

We say that a topological property B is *transferred onto small layers around the space X form Y* iff there exists a countable set $\mathscr{B}$ of relations defined on $\mathrm{ext}_Y \mathscr{U}_X^*$ such that $\mathscr{U}_X^*$ is a $\mathscr{B}$-uniformity and for each $\mathscr{B}$-pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ a set $[x]_{\mathrm{ext}_Y \mathscr{U}} = \bigcap \{\mathrm{st}(x, P) \colon P \in \mathrm{ext}_Y \mathscr{U}\}$, $x \in X$, has the property B. If, in addition, for each $x \in X$, $[x]_{\mathrm{ext}_Y \mathscr{U}} \subset X$, then we say that the property B is *transferred onto small layers from Y*.

A map $f\colon X \to M$ is said to be a *B-map* iff, for each $y \in M$, $f^{-1}y$ has the property B.

3. A factorization theorem.

THEOREM. *Given*:

1. *spaces X, Y_0, Y_i, $i = 1, 2, \ldots$ such that $X \subset Y_0 \subset Y_i$, $\mathrm{cl}_{Y_0} X = Y_0$ and Y_0 has a uniform feathering in Y_i, $i = 1, 2, \ldots$,*

2. *a countable family of properties A_i of the covering-countable type which has the space X,*

3. *a countable family of properties B_i which are transferred from Y_i around the space X onto small layers, $i = 1, 2, \ldots$*

Then for each map $f\colon X \to M$ into a metric space M there exist spaces X_0, M' and maps $f_0\colon X_0 \to \tilde{M}$ (where $\tilde{M} \supset M$ is the completion of M), $h\colon X_0 \overset{\text{onto}}{\to} M'$, $g\colon M' \to \tilde{M}$ such that

a) *the space $X_0 \supset X$ is a G_δ subspace in Y_0,*

b) *the space M' is metric and the subspace $h(X) \subset M'$ has all the properties A_i, $i = 1, 2, \ldots$,*

c) *f_0 is the extension of the map f and $f_0 = gh$,*

d) *h is a B_i-map for each $i = 1, 2, \ldots$ and h is a perfect map whenever Y_i is compact for some i.*

Proof. Let $f\colon X \to \tilde{M}$ be a map into the complete metric space $\tilde{M}$, which is the completion of the metric space M. Then $f\colon (X, \mathscr{U}) \to (\tilde{M}, \mathscr{V})$ is a uniform map, where $\mathscr{V}$ is a uniformity induced by the metric on $\tilde{M}$. Let $\mathscr{V}_0 \subset \mathscr{V}$ be a countable

base for $\mathscr{V}$. Put $f^{-1}\mathscr{V}_0 = \{Q_n \colon n = 1, 2, \ldots\}$ $G_n^0 = \bigcup \{v \colon v \in \tilde{Q}_n\}$, where $\tilde{Q}_n$ is once for all a fixed extension, open in Y_0, of the covering Q_n. Let G_n^i be once for all a fixed extension, open in Y_i, of G_n^0, $i = 1, 2, \ldots$ Let $\mathscr{A}_i$, $\mathscr{B}_i$ mean countable sets of relations connected with the properties A_i, B_i. Put $R_i = \mathscr{P}_i \cup \{\{G_n^i\} \colon n = 1, 2, \ldots\}$, where $\mathscr{P}_i$ is a feathering of the space Y_0 in the space Y_i.

Define by induction countable sets $\mathscr{W}_i \subset \mathscr{U}_X^*$, $i = 0, 1, \ldots$ Let $\mathscr{W}_0 = f^{-1}\mathscr{V}_0$ and let us fix once for all for each i and for each $P \in \mathscr{W}_0$ an extension $P(i)$, open in Y_i, of the covering P such that $P(o)$ is the greatest extension of P open in Y_0. Suppose that the countable families $\mathscr{W}_k \subset \mathscr{U}_X^*$, $k \leqslant n$, are given and suppose that extensions $P(i)$, open in Y_i, of coverings $P \in \mathscr{W}_k$, $k \leqslant n$, are fixed. Now, we shall define a countable family $\mathscr{W}_{n+1} \subset \mathscr{U}_X^*$. For each pair $P_1, P_2 \in \bigcup \{\mathscr{W}_k \colon k = 1, \ldots, n\}$ let us choose a countable family $\mathscr{W}(P_1, P_2) \subset \mathscr{U}_X^*$ such that:

1. for each relation $a \in \mathscr{A}_i$, $i = 1, 2, \ldots$ there exists a covering $P \in \mathscr{W}(P_1, P_2)$ such that $(P, P_1 \wedge P_2) \in a$,

2. for each relation $b \in \mathscr{B}_i$ and for each $R \in R_j$, $i, j = 1, 2, \ldots$, there exists a covering $P \in \mathscr{W}(P_1, P_2)$ having an extension $P(j)$ open in Y_j, $P(j) \succ R$, (here, we fix the extension $P(j)$ of P) and such that $(P(j), P_1(j) \wedge P_2(j)) \in b$,

3. there exists a $P \underset{*}{\succ} P_1 \wedge P_2$, $P \in \mathscr{U}_X^*$ such that each centred family $Q \subset P$ is finite (the existence of P follows from a result of Dowker [2]). Put

$$\mathscr{W}_{n+1} = \bigcup \{\mathscr{W}(P_1, P_2) \colon P_1, P_2 \in \bigcup \{\mathscr{W}_k \colon k = 1, \ldots, n\}\}.$$

Moreover, for each $P \in \mathscr{W}_{n+1}$ and for each $i = 0, 1, \ldots$ let us fix once for all an extension $P(i)$, open in Y_i, of P (for the case where the extension has not yet been fixed) such that $P(o)$ is the greatest extension of P open in Y_0.

Put $\mathscr{W} = \bigcup \{\mathscr{W}_n \colon n = 0, 1, \ldots\}$. A countable family $\mathscr{W}$ is a base for some pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ and for each $i = 1, 2, \ldots$ $\mathscr{U}$ is an $\mathscr{A}_i$- and a $\mathscr{B}_i$-pseudouniformity.

Put $X_0 = \bigcap \{\bigcup P(o) \colon P \in \mathscr{W}\}$. Notice that for $P, P' \in \mathscr{W}$ if $P' \underset{*}{\succ} P$ and, for each centred family $Q \subset P'$, Q is finite, then $P'(o) \underset{*}{\succ} P(o)$ (because $P'(o)$, $P(o)$ have been chosen as the greatest extensions open in Y_0 and X is dense in Y_0). Hence a family $\mathscr{W}(o) = \{P(o)|X_0 \colon P \in \mathscr{W}\}$ is a base for some pseudouniformity $\mathscr{U}(o)$ compatible with the topology on X_0 and $\mathscr{U}(o)|X = \mathscr{U}$. For each $i = 1, 2, \ldots$ $\mathscr{U}(o)$ is an $\mathscr{A}_i$- and a $\mathscr{B}_i$-pseudouniformity. Since $\mathrm{cl}_{X_0} X = X_0$ and $f^{-1}\mathscr{V}_0 \subset \mathscr{U}_0|X$, it is possible to define a map $f_0\colon X_0 \to \tilde{M}$ which is a an extension of the map f. We put $f_0 y = \bigcap \{\mathrm{cl}_{\tilde{M}} f(U \cap X) \colon U$ is a neighbourhood in X_0 of $y \in X\}$. It can be verified that $f_0\colon (X_0, \mathscr{U}(o)) \to (\tilde{M}, \mathscr{V})$ is a uniform map.

Let $M' = hX_0$ be a space with a metric induced by the uniformity $h\mathscr{U}(o)$. Define a map $g\colon M' \to \tilde{M}$; $g[x]_{\mathscr{U}(o)} = f_0 x$. Since for each $i = 1, 2, \ldots$ and for each $x \in X_0$ we have $[x]_{\mathscr{U}(o)} = \bigcap \{\mathrm{st}(x, P(i)) \colon P \in \mathscr{U}\}$ (because $[x]_{\mathscr{B}_i} \subset Y_0$), the map $h\colon X_0 \overset{\text{onto}}{\to} \tilde{M}$ is a B_i-map, $i = 1, 2, \ldots$

Notice that for each $i = 1, 2, \ldots$ a family $\mathscr{W}(i) = \{P(i)|X_0 \colon P \in \mathscr{W}\}$ is a feathering of the space X_0 in the space Y_i. For this reason, if one of the spaces Y_i is compact, then a family $\{\mathrm{st}(x,P)\colon P\in\mathscr{U}(o)\}$ is a base of neighbourhood of the compact set $[x]_{\mathscr{U}(0)}$. This implies that $h\colon X_0\to M'$ is a perfect map.

4. Examples of properties of the covering type.

PROPOSITION 1. *If X is a completely regular space, then $\dim X = n$ is a property of the covering-countable type.*

Proof. Define a set $\mathscr{A} = \{a_1, a_2\}$ of relations on $\mathscr{U}_X^*$:

1. $P', P\in a_1$ iff $P'\underset{*}{\succ}P$ and $\operatorname{ord}P'\leqslant n+1$,

2. $(P', P)\in a_2$ iff $P'\underset{*}{\prec}P$ and there is no covering $P''\in\mathscr{U}_X^*$ such that $\operatorname{ord}P''<n$ and $P''\succ P'$.

Since the uniformity $\mathscr{U}_X^*$ has a base consisting of all the locally finite and functionally open coverings of X, we have $\dim\mathscr{U}_X^* = n$ iff $\dim X = n$ (see e.g. [4]). Thus $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. Now, we shall verify the condition (b) of the definition of a property of the covering-countable type. From the construction of the functor h it follows that $\dim\mathscr{U} = \dim h\mathscr{U}$ (see the property (a) of the functor h). But, if the uniformity $h\mathscr{U}$ has a countable base, then $\dim hX = \dim h\mathscr{U}$, where the topology of the space hX is induced by $h\mathscr{U}$ (Nagata [6]).

In a paper of Bokštejn [1] there was introduced a coefficient of cyclicity of a space X in a coefficient group G, $\eta_G(X) = \sup\{n\colon H^n(X; G)\neq 0\}$, where H^* means the Čech cohomology functor.

PROPOSITION 2. *If G is a countable generated group, then $\eta_G(X) = n$, $n\leqslant\infty$, is a property of the covering-countable type for compact spaces X.*

Proof. From the theorem on universal coefficients it follows that, for each covering P which has a finite subcovering, the group $H^*(P; G)$ is countable generated. For each covering $P\in\mathscr{U}_X^*$ let us enumerate generators $g_1, g_2, \ldots$ of the group $H^k(P; G)$. Denote by $i^k_{P',P}$ homomorphism of groups $H^k(P; G)\to H^k(P'; G)$ induced a star refinement $P'\underset{*}{\succ}P$. Let us consider the relations:

1. $(P', P)\in a^k_m$ iff $P'\underset{*}{\succ}P$ and $i^k_{P',P}(g_m) = 0$, $g_m\in H^k(P; G)$,

2. $(P', P)\in a^k$ iff $P'\underset{*}{\succ}P$ and there exists a $g\in H^k(P'; G)$ such that for each $P''\succ P'$, $P''\in\mathscr{U}_X^*$ is $i^k_{P'',P'}(g)\neq 0$.

Put $\mathscr{A} = \{a^k_m\colon m = 1, \ldots, k>n\}\cup\{a^k\colon k\leqslant n\}$. Notice that $\eta_G(X) = n$ is equivalent to $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. On the other hand, for each $\mathscr{A}$-pseudouniformity $\mathscr{U}\subset\mathscr{U}_X^*$, the property (a) of the functor h, $\mathscr{U} = h^{-1}h\mathscr{U}$, implies that $h\mathscr{U}$ is an $\mathscr{A}$-uniformity on the set hX. The topology of the space hX induced by the uniformity $h\mathscr{U}$ is compact. Since a compact space has only a unique uniformity inducing the topology, the condition that $h\mathscr{U}$ is an $\mathscr{A}$-uniformity is equivalent to $\eta_G(hX) = n$.

A space X is cohomologically locally connected in a dimension not greater than n, $n\leqslant\infty$, and in a group of coefficients G, (written; $X\in\mathrm{cl}\,c^n_G$), iff for each neighbourhood U of a point x there exists a neighbourhood $V\subset U$ of x such that the homomorphism of reduced cohomology Alexander–Čech groups $\tilde{H}^k(U; G)\to\tilde{H}^k(V; G)$ induced by the embedding $V\subset U$ is trivial.

PROPOSITION 3. *For each paracompact p-space, $X\in\mathrm{cl}\,c^n_G$ and $H^k(X; G) = A_k$ for $k\leqslant n$ is a property of the covering-countable type.*

Proof. Let $\mathscr{P} = \{P_n\colon n = 1, 2, \ldots\}$ be a feathering of the space X in the Čech–Stone compactification βX. Define relations a^k_n on $\mathscr{U}_X^*$: $(P', P)\in a^k_m$ iff $P'\underset{*}{\succ}P$, $\mathrm{cl}_{\beta X}\tilde{P}'\succ P_m\wedge\tilde{P}$ (where $\tilde{P}$ means the greatest extension, open in βX, of $P\in\mathscr{U}_X^*$) and for each $u'\in P'$ there exists a $u\in P$ such that $u'\subset u$ and the homomorphism $H^k(u; G)\to H^k(u'; G)$ is trivial.

Put $\mathscr{A} = \{a^k_m\colon k\leqslant n,\ m<\infty\}$. Notice that $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. Now, let $\mathscr{U}\subset\mathscr{U}_X^*$ be an $\mathscr{A}$-pseudouniformity with a countable base. The condition $\mathrm{cl}_{\beta X}\tilde{P}'\succ P_m\wedge\tilde{P}$, $P'\underset{*}{\succ}P$, ensure that a family $\{\mathrm{st}(x, P)\colon P\in\mathscr{U}\}$ is a base of neighbourhoods of the set $[x]_{\mathscr{U}}\subset X$, because

$$[x]_{\mathscr{U}} = \bigcap\{\mathrm{st}(x, P)\colon P\in\mathscr{U}\} = \bigcap\{\mathrm{cl}_{\beta X}\mathrm{st}(x, P\colon P\in\mathscr{U}\}$$

and βX is a compact space. This implies that for each neighbourhood $U[x]$ of the set $[x]$ there exists a neighbourhood $V[x]\subset U[x]$ of $[x]$ such that the homomorphism $H^k(U[x]; G)\to\tilde{H}^k(V[x]; G)$ is trivial. Hence, for each $x\in X$ and $k\leqslant n$ we obtain $\tilde{H}^k([x]; G) = 0$ (see, Spanier [7], Theorem 6.6.2).

Now let us consider the space hX with the topology induced by the uniformity $h\mathscr{U}$. Since a family $\{\mathrm{st}(x, P)\colon P\in\mathscr{U}\}$ is a base of neighbourhoods of the set $[x]_{\mathscr{U}}$, the map $h\colon X\to hX$ is perfect and $\tilde{H}^k(k^{-1}hx; G) = 0$, $x\in X$, $k\leqslant n$. From the Vietoris–Begle Theorem (see, Spanier [7], Theorem 6.9.15) the map h induces the isomorphism $H^k(hX; G)\to H^k(X; G)$, $k\leqslant n$. From this we immediately obtain $hX\in\mathrm{cl}\,c^n_G$ and $H^k(hX; G) = H^k(X) = A_k$, $k\leqslant n$.

A set $A\subset X$ is said to be *approximatively n-connected in X* (written n-PC_X) iff for each neighbourhood U of A in X there is a neighbourhood $V\subset U$ of A in X such that each map $f\colon S^n\to V$ is homotopic to a constant map in U. The set A is PC^n iff it is k-PC_X for all $0\leqslant k\leqslant n$, $n\leqslant\infty$. The notion reduces to $X\in\mathrm{LC}^n$ iff for each point $x\in X$ the set $\{x\}$ is PC^n.

PROPOSITION 4. *For each paracompact p-space X, $X\in\mathrm{LC}^n_X$ and $\pi_k(X) = A_k$ for $k\leqslant n$ is a property of the covering-countable type.*

Proof. Let $\mathscr{P} = \{P_n\colon n = 1, 2, \ldots\}$ be a feathering of a space in βX. Define relations a^k_m on $\mathscr{U}_X^*$: $(P', P)\in a^k_m$ iff $P'\underset{*}{\succ}P$, $\mathrm{cl}_{\beta X}\tilde{P}'\succ P_m\wedge\tilde{P}$ (where $\tilde{P}$ means the greatest extension of P open in βX and for each $u'\in P'$ there exists a $u\in P$ such that each map $f\colon S^k\to u'$ is homotopic in u to a constant map. Let $\mathscr{A} = \{a^k_m\colon k\leqslant n,\ m<\infty\}$. The uniformity $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. In the same way as in the previous example it can be verified that the family $\{\mathrm{st}(x, P)\colon P\in\mathscr{U}\}$ is a base of the set $[x]_{\mathscr{U}}$ for each $\mathscr{A}$-pseudouniformity $\mathscr{U}\subset\mathscr{U}_X^*$, and the set $[x]_{\mathscr{U}}$ is PC^n_X and the map $h\colon X\to hX$ onto a metrizable space hX is perfect. From the Dugundji–Vietoris Theorem ([3], Theorem 5.4) we infer that the homomorphism $\pi_k(X)\to\pi_k(hX)$, $k\leqslant n$, is an isomorphism. This implies that $hX\in\mathrm{LC}^n$ and $\pi_k(hX) = A_k$, $k\cdot\in n$.

Now, we shall give two examples of properties of type B.

PROPOSITION 5. *If a paracompact space X has a feathering in a locally compact and locally connected space Y, then compactness and connectness is transferred into X onto small layers.*

PROPOSITION 6. *If a paracompact space X has a feathering in a paracompact p-space $Y \in \mathrm{cl}\, c_G^n$, then the property $H^k(Z; G) = 0$ is a property transferred onto small layers.*

Proof. The space X has feathering $\mathcal{P} = \{P_n: n = 1, 2, \ldots\}$ in βY. Define relations b_m^k on $\mathrm{ext}_{\beta Y}\mathcal{U}_X^*$: $(P', P) \in b_m^k$ iff $P'|X \underset{*}{\succ} P|X$, $\mathrm{cl}_{\beta Y}P' \succ P \wedge P_m$ and for each $u' \in P'$ there exists a $u \in P$, $u' \subset u$, such that the induced homomorphism $H^k(u \cap Y; G) \to H^k(u' \cap Y; G)$ is trivial. Let $\mathcal{B} = \{b_m^k: k \leqslant n,\ m < \infty\}$. The uniformity $\mathcal{U}_X^*$ is a $\mathcal{B}$-uniformity. We shall verify that for each $\mathcal{B}$-pseudouniformity $\mathcal{U} \subset \mathcal{U}_X^*$ we have $\tilde{H}^k([x]_{\mathcal{U}}; G) = 0$, $k \leqslant n$, $x \in X$. Notice that for each $\mathcal{B}$-pseudouniformity $\mathcal{U}$ a family $\{\mathrm{st}(x, P): P \in \mathrm{ext}_{\beta Y}\mathcal{U}\}$ is a base of neighbourhoods of $[x]_{\mathcal{U}} = [x]_{\mathrm{ext}_{\beta Y}\mathcal{U}}$, $x \in X$. Hence a family $\{\mathrm{st}(x, P|Y): P \in \mathrm{ext}_{\beta Y}\mathcal{U}\}$ is also a neighbourhood base of $[x]_{\mathcal{U}}$. Now, from the definition of the relations b_m^k it follows that for each neighbourhood $u \in P \in \mathrm{ext}_{\beta Y}\mathcal{U}$ of $[x]_{\mathcal{U}}$ there exists a neighbourhood $u' \in P' \in \mathrm{ext}_{\beta Y}\mathcal{U}$, $(P', P) \in b_m^k$, such that $u' \cap Y \subset u \cap Y$ and the induced homomorphism $H^k(u \cap Y; G) \to H^k(u' \cap Y; G)$ is trivial. By Theorem 6.6.2 from [7] it follows that $\tilde{H}^k([x]_{\mathcal{U}}; G) = 0$.

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The category of abelian Hopf algebras

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Abstract. By abelian Hopf algebra we mean a commutative, cocommutative, connected, graded Hopf algebra over a field. In this paper we investigate the category $\mathfrak{H}$ of all abelian Hopf algebras and the full subcategory $\mathfrak{L}$ of $\mathfrak{H}$ consisting of all primitively generated Hopf algebras. In particular we give a complete description of injective objects in categories $\mathfrak{L}$ and $\mathfrak{H}$ and we prove that $\mathrm{gl.dim}\,\mathfrak{L} = 1$ and $\mathrm{gl.dim}\,\mathfrak{H} = 2$.

Introduction. Let K be an arbitrary field. A graded Hopf K-algebra which is commutative, cocommutative and connected will be called an *abelian Hopf algebra* (see [10], [18]). Denote by $\mathcal{H}$ the category of all abelian Hopf algebras. Recall that $\mathcal{H}$ is a locally noetherian Grothendieck category and an object H in $\mathcal{H}$ is noetherian if and only if H is finitely generated as a K-algebra (see [7], [10]). The tensor product $\otimes$ over K is the coproduct in $\mathcal{H}$. Let p be the characteristic of K. If $p = 0$ then $\mathrm{gl.dim}\,\mathcal{H} = 0$ (see [10]). Assume $p \geqslant 2$. In [10] Schoeller showed that $\mathcal{H} = \mathcal{H}^- \times \mathcal{H}^+$ where $\mathcal{H}^-$ is the full subcategory of $\mathcal{H}$ consisting of all Hopf algebras generated by elements of odd degrees and $\mathcal{H}^+$ consists all Hopf algebras which are zero in odd degrees. Furthermore, $\mathrm{gl.dim}\,\mathcal{H}^- = 0$ and $\mathcal{H}^+$ is a product of countably many $\prec$ categories each of which is equivalent to the full subcategory $\mathcal{H}_1$ of $\mathcal{H}^+$ consisting of all Hopf algebras generated by elements of degrees $2p^i$ where $i = 0, 1, 2, \ldots$

Let H be an object in $\mathcal{H}$ and Δ the comultiplication of H. An element x of H will be called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. From Theorem 6.3 in [7] it follows that each subobject of a primitively generated abelian Hopf algebra is also primitively generated. Denote by $\mathcal{L}$ (resp. $\mathcal{L}^-$, $\mathcal{L}^+$, $\mathcal{L}_1$) the full subcategory of $\mathcal{H}$ (resp. $\mathcal{H}^-$, $\mathcal{H}^+$, $\mathcal{H}_1$) consisting of all primitively generated Hopf algebras. Then $\mathcal{L}$ is a locally noetherian Grothendieck category, $\mathcal{L} = \mathcal{L}^- \times \mathcal{L}^+$ and $\mathcal{L}^+$ is a product of countably many categories each of which is equivalent to the category $\mathcal{L}_1$.

Let $\mathcal{K}$-GrMod denote the category of graded K-modules and let

$$P\colon \mathcal{H} \to K\text{-GrMod}$$

be the functor which assigns to each H from $\mathcal{H}_1$ the graded K-module $P(H)$ of all primitive elements of H. Moreover, let

$$Q\colon \mathcal{H}_1 \to K\text{-GrMod}$$