It is easy to write formula $\psi(x, y, z, X, Y, Z_1, Z_2, Z_3)$ in the topological language stating (in T) that $x, y \in X$ and $X \in Z_1^*$, $Y \in Z_2^*$, $x \sim z (\mathrm{mod} Z_2)$ and there exists $y^1 \in Y$ such that $y \sim y^1 (\mathrm{mod} Z_3)$ and $y^1 \sim z (\mathrm{mod} Z_1)$.

In order to interpret DL in T it is enough to find $X, Y, Z_1, Z_2, Z_3 \subset R^2$ such that $|X| = 2^{\aleph_0}$ and ψ defines a one-one correspondence between $\{\langle x, y\rangle: x, y \in X\}$ and $\{z: z \in R^2\}$. Let Q be the set of rational numbers. Choose $X = R \times \{0\}$, $Y = \{0\} \times R$, $Z_1 = R \times Q$, $Z_2 = Q \times R$ and $Z_3 = \{\langle a, b\rangle: a - b \in Q\}$. Then $\psi(x, y, z, X, Y, Z_1, Z_2, Z_3)$ holds iff there exist $a, b \in R$ such that $x = \langle a, 0\rangle$, $y = \langle b, 0\rangle$ and $z = \langle a, b\rangle$.

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Borel sets with $F_{\sigma\delta}$-sections

by

J. Bourgain (Brussel)

Abstract. Let E, F be compact metric spaces. We characterize Borel sets A in $E \times F$ with $F_{\sigma\delta}$-sections.

Introduction. We consider two fixed compact metric spaces E and F. The class $\mathscr{C}$ will consist of the Borel subsets A of $E \times F$ such that for each $x \in E$ the section $A(x) = \{y \in F;\ (x, y) \in A\}$ is closed in F. We will prove the following:

THEOREM 1. *If A is a Borel subset of $E \times F$ such that each section $A(x)$ is $F_{\sigma\delta}$ in F, then A belongs to the class $\mathscr{C}_{\sigma\delta}$.*

This is an extension of the work of J. Saint-Raymond (see [13]), who established:

THEOREM 2. *If a is a Borel subset of $E \times F$ such that each section $A(x)$ is F_σ in F, then A belongs to the class $\mathscr{C}_\sigma$.*

Theorem 1 is also related to my earlier paper [2].

Preliminaries. N will denote the set of all positive integers. Let $\mathscr{R} = \bigcup_k N^k$, taking $N^0 = \{\emptyset\}$. Thus $\mathscr{R}$ consists of the finite complexes of integers. If $c \in \mathscr{R}$, let $|c|$ be the length of c. If $c, d \in \mathscr{R}$, we write $c < d$ if c is an initial section of d. Let $(p_k)_k$ be an enumeration of all prime numbers. If we associate 0 to $\emptyset$ and the integer $p_1^{n_1} \dots p_k^{n_k}$ to the complex $c = (n_1, \dots, n_k)$, a one-one map of $\mathscr{R}$ into N is established. The induced ordering of $\mathscr{R}$ will be called the *standard ordering*. Let $\mathscr{N} = N^N$. if $\nu \in \mathscr{N}$ and $c \in \mathscr{R}$, we write $c < \nu$ if c is an initial section of ν.

If L is a compact metric space, then $\underline{K}(L)$ consists of all closed subsets of L and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [7]).

LEMMA 1. *Let P be a Polish subspace of the compact metric space L. Then the subspace $\underline{F}(P)$ of $\underline{K}(L)$ consisting of those compact sets K in L such that $K = \overline{K \cap P}$, is Polish.*

A pavage $\mathscr{P}$ on a set Ω will be a family of subsets of Ω containing the empty set. A subset of Ω is said to be *$\mathscr{P}$-analytic* if it is the result of Souslin operation performed on members of $\mathscr{P}$. For more details, I refer to [7].

I also remember the separation theorem of Novikov, which will be often used in this paper:

LEMMA 2. *Let $(A_n)_n$ be a sequence of analytic subsets of the Polish space P satisfying $\bigcap_n A_n = \emptyset$. Then there exists a sequence $(B_n)_n$ of Borel subsets of P such that $A_n \subset B_n$ for each n and $\bigcap_n B_n = \emptyset$.*

The reader can find a prove of this result in [7].

If A is a subset of $E\times F$, let $\bar{A}^s$ be the subset of $E\times F$ defined by $\bar{A}^s(x) = \overline{A(x)}$ for $x\in E$. Consider for each $r\in N$ a finite covering $(U_{ri})_i$ of F by open sets with diameter less than 2^{-r}. It is easily verified that $\bar{A}^s = \bigcap_r \bigcup_i [\pi_E(A\cap(E\times U_{ri}))\times \bar{U}_{ri}]$. Therefore we obtain:

LEMMA 3. *If A is analytic in $E\times F$, then also $\bar{A}^s$ is analytic.*

LEMMA 4. *If A and B are analytic in $E\times F$ and $\bar{A}^s\cap B = \emptyset$, then there exists a member C of $\mathscr{C}$ such that $A\subset C$ and $B\cap C = \emptyset$.*

Proof. Take the open set U_{ri} as before. For each $r\in N$ the set

$$S_r = B\cap \bigcup_i [\pi_E(A\cap(E\times U_{ri}))\times \bar{U}_{ri}]$$

is analytic in $E\times F$. Since $\bigcap_r S_r = \emptyset$, a sequence $(T_r)_r$ of Borel subsets of $E\times F$ is obtained such that $S_r\subset T_r$ for each r and $\bigcap_r T_r = \emptyset$. Now, for each r and each i, we have that $[\pi_E(A\cap(E\times U_{ri}))\times \bar{U}_{ri}]\cap(B\setminus T_r) = \emptyset$ and thus $\pi_E(A\cap(E\times U_{ri}))$ and $\pi_E((B\setminus T_r)\cap(E\times \bar{U}_{ri}))$ are disjoint analytic sets. Therefore there exist Borel sets B_{ri} in E with $A\cap(E\times U_{ri})\subset B_{ri}\times \bar{U}_{ri}\subset((E\times F)\setminus B)\cup T_r$. For each $r\in N$, take $C_r = \bigcup_i (B_{ri}\times \bar{U}_{ri})$, which belongs to $\mathscr{C}$. Then $A\subset C_r$ and $C_r\cap(B\setminus T_r) = \emptyset$. The set $C = \bigcap_r C_r$ satisfies the required properties.

A result about transfinite systems. The proof of various results in the remainder of the text is considerably shortened by the use of the following lemma:

LEMMA 5. *Let for each $k\in N$ a Polish space P_k be given. Assume for all $k\in N$ and $\alpha<\omega_1$ a subset $\mathscr{S}_k^\alpha$ of $\prod_{l=1}^k P_l$ be defined, such that following conditions are satisfied:*

1. *If $k\in N$, then $\mathscr{S}_k^0$ is analytic in $\prod_{l=1}^k P_l$.*
2. *If $k\in N$ and $\alpha<\beta$, then $\mathscr{S}_k^\beta\subset\mathscr{S}_k^\alpha$.*
3. *If $k\in N$, $\alpha<\omega_1$ and $\sigma\in\prod_{l=1}^k P_l$ satisfies $\sigma|l\in\mathscr{S}_l^{\alpha+1}$ for each $l = 1,\dots,k$, then there exists $\iota\in\mathscr{S}_{k+1}^\alpha$ with $\sigma = \iota|k$.*

Suppose $\mathscr{S}_1^\alpha \neq \emptyset$ for every $\alpha<\omega_1$. Then there is some $\xi\in\prod_k P_k$ such that $\xi|k\in\mathscr{S}_k^0$ for each $k\in N$.

Proof. If $\pi\in\mathscr{R}$, let $\mathscr{N}(\pi) = \{\nu\in\mathscr{N};\ \pi<\nu\}$. For each $k\in N$, let $\varphi_k\colon \mathscr{N}\to\prod_{l=1}^k P_l$ be a continuous map with image $\mathscr{S}_k^0$.

By induction we will define for every $k\in N$ elements $\pi_{1k},\dots,\pi_{kk}$ of N^k, such that the following conditions are satisfied:

1. $\pi_{lk} = \pi_{l,k+1}|k$ if $k\geqslant l$.
2. For each $\alpha<\omega_1$, there exists $\sigma\in\prod_{l=1}^k P_l$ so that $\sigma|l\in\varphi_l(\mathscr{N}(\pi_{lk}))\cap\mathscr{S}_l^\alpha$ for each $l = 1,\dots,k$.

Since $\mathscr{S}_1^\alpha\neq\emptyset$ for every $\alpha<\omega_1$ and $\mathscr{S}_1^0 = \bigcup_p \varphi_1(\mathscr{N}(p))$, there must be some $\pi_{11}\in N$ such that $\varphi_1(\mathscr{N}(\pi_{11}))\cap\mathscr{S}_1^\alpha\neq\emptyset$ for each $\alpha<\omega_1$. Assume now $\pi_{1k},\dots,\pi_{kk}$ obtained. If $\alpha<\omega_1$, then there exists $\sigma\in\prod_{l=1}^k P_l$ so that $\sigma|l\in\varphi_l(\mathscr{N}(\pi_{lk}))\cap\mathscr{S}_l^{\alpha+1}$ for each $l = 1,\dots,k$. Therefore there is $\iota\in\mathscr{S}_{k+1}^\alpha$ with $\iota|k = \sigma$ and thus $\iota|l\in\varphi_l(\mathscr{N}(\pi_{lk}))\cap\mathscr{S}_l^\alpha$ for each $l = 1,\dots,k$. For $l = 1,\dots,k$ we have $\varphi_l(\mathscr{N}(\pi_{lk})) = \bigcup_p \varphi_l(\mathscr{N}(\pi_{lk},p))$ and also $\mathscr{S}_{k+1}^0 = \bigcup_{\pi\in N^{k+1}}\varphi_{k+1}(\mathscr{N}(\pi))$. Again there must exist $p_1,\dots,p_k\in N$ and $\pi_{k+1,k+1}\in N^{k+1}$ such that for each $\alpha<\omega_1$ there is $\sigma\in\prod_{l=1}^{k+1} P_l$ with $\sigma|l\in\varphi_l(\mathscr{N}(\pi_{lk},p_l))\cap\mathscr{S}_l^\alpha$ for each $l = 1,\dots,k$ and $\sigma\in\varphi_{k+1}(\mathscr{N}(\pi_{k+1,k+1}))\cap\mathscr{S}_{k+1}^\alpha$. For every $l = 1,\dots,k$ let $\pi_{l,k+1} = (\pi_{lk},p_l)$. Then the construction is complete. For each $l\in N$, let $\pi_l\in\mathscr{N}$ be defined by $\pi_l|k = \pi_{lk}$ if $k\geqslant l$. Obviously $\varphi_l(\pi_l)\in\mathscr{S}_l^0$. If $k\geqslant l+1$, then $[\varphi_l(\mathscr{N}(\pi_{lk}))\times P_{l+1}]\cap\varphi_{l+1}(\mathscr{N}(\pi_{l+1,k}))\neq\emptyset$, implying $\varphi_{l+1}(\pi_{l+1})|l = \varphi_l(\pi_l)$. Hence there is $\xi\in\prod_k P_k$ satisfying $\xi|k = \varphi_k(\pi_k)$ for each $k\in N$ and the lemma is established.

Results about closed coverings. Let $\mathscr{P}$ be the pavage on $\underline{\underline{K}}(E\times F)$ consisting of the open subsets of $K(E\times F)$ which are of the form $\{L\in\underline{\underline{K}}(E\times F);\ L\cap\Omega\neq\emptyset\}$, where Ω ranges over the open sets in $E\times F$. Let P^* be the set of $\mathscr{P}$-analytic subsets of $\underline{\underline{K}}(E\times F)$.

The following two lemma's are obvious.

LEMMA 6. *If A is an analytic subset of $E\times F$, then the set $\{L\in\underline{\underline{K}}(E\times F);\ L\cap A\neq\emptyset\}$ is a member of $\mathscr{P}^*$.*

LEMMA 7. *If $\Gamma\in\mathscr{P}^*$, $L\in\Gamma$ and $M\in\underline{\underline{K}}(E\times F)$ contains L, then $M\in\Gamma$.*

LEMMA 8. *Let $\Gamma\in\mathscr{P}^*$ and let A be an analytic subset of $E\times F$ such that $A(x)$ is closed for each $x\in E$. Then $\{x\in E;\ \{x\}\times A(x)\in\Gamma\}$ is analytic.*

Proof. There is a system $(\Omega_\pi)_{\pi\in\mathscr{R}}$ of open subsets of $E\times F$ such that $\Gamma = \bigcup_\nu\bigcap_r\Gamma_{\nu|r}$, where $\Gamma_{\nu|r} = \underline{\underline{K}}\{L\in(E\times F);\ L\cap\Omega_{\nu|r}\neq\emptyset\}$. For each $r\in N$, let Σ_r be the subset of $\mathcal{N}\times E\times F$ defined by $\Sigma_r(\nu, x) = A(x)\cap\Omega_{\nu|r}(x)$. It is clear that Σ_r is analytic in $\mathcal{N}\times E\times F$. Hence $\pi_E(\bigcap_r \pi_{\mathcal{N}\times E}(\Sigma_r))$ is analytic in E. But this set is precisely

$$\{x\in E;\ \{x\}\times A(x)\in\Gamma\}.$$

If $\Gamma\in\mathcal{P}^*$, let $\mathcal{C}(\Gamma)$ consist of the members A of $\mathcal{C}$ such that $\{x\}\times A(x)\notin\Gamma$ for each $x\in E$.

LEMMA 9. *Let $\Gamma\in\mathcal{P}^*$ and let S be analytic in $E\times F$ such that $S(x)$ is closed and $\{x\}\times S(x)\notin\Gamma$ for each $x\in E$. Then there exists $T\in\mathcal{C}(\Gamma)$ containing S.*

Proof. There is a system $(\Omega_\pi)_{\pi\in\mathcal{R}}$ of open subsets of $E\times F$ such that $\Gamma = \bigcup_\nu\bigcap_r\Gamma_{\nu|r}$, where $\Gamma_{\nu|r} = \{L\in\underline{\underline{K}}(E\times F);\ L\cap\Omega_{\nu|r}\neq\emptyset\}$. We will consider the space $F^* = \prod_r F_r$, where each $F_r = F$. If $r\in N$ and $\pi\in N^r$, take

$$\Omega_\pi^* = \{(x, y^*)\in E\times F^*;\ (x, y_r^*)\in\Omega_\pi\}$$

and define $\Omega^* = \bigcup_\nu\bigcap_r\Omega^*_{\nu|r}$ which is an analytic subset of $E\times F^*$. Let further for each $r\in N$ the set $S_r^* = \{(x, y^*)\in E\times F^*;\ (x, y_r^*)\in S\}$ and let $S^* = \bigcap_r S_r^*$. It is easy to deduce from the hypothesis that $S^*\cap\Omega^* = \emptyset$.

It follows that there is a sequence $(B_r^*)_r$ of Borel sets in $E\times F^*$ with $S_r^*\cap\Omega^*\subset B_r^*$ for each r and $\bigcap_r B_r^* = \emptyset$. Let $r\in N$ be fixed. Since $S_r^*\cap(\Omega^*\setminus B_r^*) = \emptyset$, we obtain that S and $\pi_{E\times F_r}(\Omega^*\setminus B_r^*)$ are disjoint analytic sets. We now use the fact that each section $S(x)$ is closed to obtain a set T_r in $\mathcal{C}$ such that $S\subset T_r$ and $T_r\cap\pi_{E\times F_r}(\Omega^*\setminus B_r^*) = \emptyset$. If $T_r^* = \{(x, y^*)\in E\times F^*;\ (x, y_r^*)\in T_r\}$ then $T_r^*\cap(\Omega^*\setminus B_r^*) = \emptyset$. We claim that the set $T = \bigcap_r T_r$ satisfies. We only have to verify that $\{x\}\times T(x)\notin\Gamma$ for each $x\in E$. Assume not, then there is $\nu\in\mathcal{N}$ such that $T(x)\cap\Omega_{\nu|r}(x)\neq\emptyset$ for each $r\in N$. Therefore $\bigcap_r T_r^*\cap\Omega^*\neq\emptyset$, implying $\bigcap_r B_r^*\neq\emptyset$, a contradiction.

We will use the following stability property of $\mathcal{P}^*$:

LEMMA 10. *Let $\Gamma\in\mathcal{P}^*$ and let A be an analytic subset of $E\times F$. If $\Lambda = \{L\in\underline{\underline{K}}(E\times F);$ $A\cap L$ can not be covered by countably many closed sets not belonging to $\Gamma\}$, then $\Lambda\in\mathcal{P}^*$.*

Proof. Take a compact metric space G and a G_δ-subset H of $E\times F\times G$ satisfying $A = \pi(H)$, where π: $E\times F\times G\to E\times F$ is the projection. Let $\mathcal{U}$ be a countable base for the topology of $E\times F\times G$. Let $L\in\underline{\underline{K}}(E\times F)$ be fixed. Remark that $A\cap L = \pi(H\cap\pi^{-1}(L))$, where $H\cap\pi^{-1}(L)$ is a G_δ. It follows that $L\in\Lambda$ if and only if there exists a nonempty closed subset Y of $H\cap\pi^{-1}(L)$ such that if $U\in\mathcal{U}$ and $U\cap Y\neq\emptyset$, then $\overline{\pi(Y\cap U)} = \overline{\pi(\overline{Y}\cap U)}\in\Gamma$. Hence $L\in\Lambda$ if and only if there exists a nonempty set M in $\underline{\underline{K}}(E\times F\times G)$ satisfying:

1. $M\in\underline{\underline{F}}(H)$,
2. $\overline{\pi(M\cap U)}\in\Gamma$ whenever $U\in\mathcal{U}$ and $M\cap U\neq\emptyset$,
3. $\pi(M)\subset L$.

The set $\underline{\underline{K}}(E\times F)$ consisting of the nonempty compact subsets of $E\times F$ belongs to $\mathcal{P}$. We will prove that the set $\mathcal{A} = \{(L, M)\in\underline{\underline{K}}(E\times F)\times\underline{\underline{K}}(E\times F\times G);\ M\neq\emptyset$ and L, M satisfy (1), (2), (3)$\}$ is $\mathcal{P}\times\mathcal{K}$-analytic, where $\mathcal{K}$ is the pavage on $\underline{\underline{K}}(E\times F\times G)$ consisting of the closed sets. Because $\Lambda = \pi_{\underline{\underline{K}}(E\times F)}(\mathcal{A})$, we will then obtain that Λ is $\mathcal{P}$-analytic (see [7]).

1. Since $\underline{\underline{F}}(H)$ is a G_δ-subset of $\underline{\underline{K}}(E\times F\times G)$, $\underline{\underline{F}}(H)$ is $\mathcal{K}$-analytic.

2. Clearly $\{M\in\underline{\underline{K}}(E\times F\times G);\ M\cap U = \emptyset\}$ belongs to $\mathcal{K}$. Because the map $\underline{\underline{K}}(E\times F\times G)\to\underline{\underline{K}}(E\times F)$: $M\mapsto\overline{\pi(M\cap U)}$ is $\mathcal{P}$-measurable, we obtain that $\{M\in\underline{\underline{K}}(E\times F\times G);\ \overline{\pi(M\cap U)}\in\Gamma\}$ is $\mathcal{K}$-analytic.

3. Let $(V_i)_i$ be a countable base for the topology of $E\times F$. For each $i\in N$, we have that $\Gamma_i = \{L\in\underline{\underline{K}}(E\times F);\ L\cap V_i\neq\emptyset\}\in\mathcal{P}$ and $\Delta_i = \{M\in\underline{\underline{K}}(E\times F\times G);$ $M\cap\pi^{-1}(V_i) = \emptyset\}\in\mathcal{K}$. But the set $\{(L, M)\in\underline{\underline{K}}(E\times F)\times\underline{\underline{K}}(E\times F\times G);\ \pi(M)\subset L\}$ is precisely $\bigcap_i[(\Gamma_i\times\underline{\underline{K}}(E\times F\times G))\cup(\underline{\underline{K}}(E\times F)\times\Delta_i)]$ and hence $\mathcal{P}\times\mathcal{K}$-analytic.

So the proof is complete.

COROLLARY 11. *Let $\Gamma\in\mathcal{P}^*$ and let A be an analytic subset of $E\times F$. Then the set $\{x\in E;\ \{x\}\times A(x)$ can be covered by countably many closed sets not belonging to $\Gamma\}$ is coanalytic in E.*

Proof. The set Λ considered in Lemma 10 is an analytic subset of $\underline{\underline{K}}(E\times F)$. We must consider $\{x\in E;\ \{x\}\times F\notin\Lambda\}$. We only have to remark that the map $E\to\underline{\underline{K}}(E\times F)$: $x\mapsto\{x\}\times F$ is continuous to complete the proof.

Combining Lemma 6 and Corollary 11 we obtain immediately

COROLLARY 12. *Let A and B be analytic subsets of $E\times F$. Then the set $\{x\in E;\ A(x)$ is contained in an F_σ-set which is disjoint from $B(x)\}$ is coanalytic in E.*

THEOREM 3. *Let $\Gamma\in\mathcal{P}^*$ and let A be an analytic subset of $E\times F$. Assume that for each $x\in E$ the set $\{x\}\times A(x)$ can be covered by countably many closed sets not belonging to Γ. Then A is contained in a member of $\mathcal{C}(\Gamma)_\sigma$.*

Before we pass to the proof of the theorem, let us mention the following easy corollary:

COROLLARY 13. *Let A and B be analytic subsets of $E\times F$. Assume that $A(x)$ is contained in an F_σ-set which is disjoint from $B(x)$, for each $x\in E$. Then A can be separated from B by a member of $\mathcal{C}_\sigma$.*

This result is due to J. Saint-Raymond (see [13]).

The remainder of this section is devoted to the proof of Theorem 3. Let G be a compact metric space and let H be a G_δ-subset of $E\times F\times G$ such that $A = \pi(H)$, where π: $E\times F\times G\to F$ is the projection. If $\mathcal{H}$ is a subset of H, take $D(\mathcal{H}) = \{(x, y, z)\in\mathcal{H};$ for each neighborhood U of (x, y, z) the set $\{x\}\times\overline{\pi(\mathcal{H}\cap U)(x)}\in\Gamma\}$.

LEMMA 14. *If $\mathcal{H}$ is analytic, then $D(\mathcal{H})$ is analytic. If moreover B is a Borel subset of H with $D(\mathcal{H})\subset B$, then $\pi(\mathcal{H}\setminus B)$ is contained in a member of $\mathcal{C}(\Gamma)_\sigma$.*

Proof. Let $(U_i)_i$ be a countable base for the topology of H. For each $i\in N$, the set $\overline{\pi(\mathscr{H}\cap U_i)}^s$ is analytic by Lemma 3 and thus

$$E_i = \{x\in E;\ \{x\}\times\overline{\pi(\mathscr{H}\cap U_i)(x)}\notin\Gamma\}$$

is coanalytic by Lemma 8. Hence $D(\mathscr{H}) = \mathscr{H}\setminus\bigcup_i [(E_i\times F\times G)\cap U_i]$ is analytic. If $D(\mathscr{H})\subset B$, then $\mathscr{H}\setminus B\subset\bigcup_i[(E_i\times F\times G)\cap U_i]$, where $\mathscr{H}\setminus B$ is analytic and each set $(E_i\times F\times G)\cap U_i$ coanalytic in $E\times F\times G$. Therefore there are analytic sets $(D_i)_i$ so that $D_i\subset (E_i\times F\times G)\cap U_i$ and $\mathscr{H}\setminus B = \bigcup_i D_i$. We obtain that $\pi(\mathscr{H}\setminus B)\subset\bigcup_i\pi(D_i)$. If $i\in N$ is fixed, then $\overline{\pi(D_i)}^s$ is an analytic set contained in $\overline{\pi(\mathscr{H}\cap U_i)}^s\cap(E_i\times F)$. It follows from the definition of E_i that $\{x\}\times\overline{\pi(D_i)(x)}\notin\Gamma$ for all $x\in E$. Applying Lemma 9 there exists $A_i\in\mathscr{C}(\Gamma)$ satisfying $\overline{\pi(D_i)}^s\subset A_i$. The set $\bigcup_i A_i$ satisfies.

Let $(H_\alpha)_{\alpha<\omega_1}$ be the transfinite system obtained as following:

$H_0 = H$,

$H_{\alpha+1} = D(H_\alpha)$.

If γ is a limit ordinal, take $H_\gamma = \bigcap_{\alpha<\gamma} H_\alpha$.

It is easily verified that the sets $H_\alpha(x)$ are closed in $H(x)$ for all $x\in E$ and the system $(H_\alpha)_{\alpha<\omega_1}$ is decreasing. Using Lemma 14, we obtain

LEMMA 15. *For each $\alpha<\omega_1$ the set H_α is analytic.*

LEMMA 16. *If $\alpha<\omega_1$ and B is a Borel subset of H containing H_α, then $\pi(H\setminus B)$ is contained in a member of $\mathscr{C}(\Gamma)_\sigma$.*

Proof. By induction on $\alpha<\omega_1$. If $\alpha=0$, then the statement is obvious. Let the statement be true for $\alpha<\omega_1$ and let B be a Borel subset of H containing $H_{\alpha+1} = D(H_\alpha)$. By Lemma 14, we obtain D' in $\mathscr{C}(\Gamma)_\sigma$ with $\pi(H_\alpha\setminus B)\subset D'$. Hence H_α is contained in $B\cup(\pi^{-1}(D')\cap H)$, which is still Borel. By induction hypothesis, there is D'' in $\mathscr{C}(\Gamma)_\sigma$ with $D''\supset\pi(H\setminus(B\cup\pi^{-1}(D')))$.

Clearly $\pi(H\setminus B)\subset D'\cup D''$. Finally let γ be a limit ordinal and $(\alpha_n)_n$ an increasing sequence of ordinals converging to γ and satisfying the lemma. If B is a Borel set containing H_γ, then $\bigcap_n(H_{\alpha_n}\setminus B)=\emptyset$. Thus there is a sequence $(B_n)_n$ of Borel sets such that $H_{\alpha_n}\setminus B\subset B_n$ and $\bigcap_n B_n=\emptyset$. Let $n\in N$ be fixed. Since $H_{\alpha_n}\subset B\cup B_n$, we obtain D_n in $\mathscr{C}(\Gamma)_\sigma$ so that $\pi(H\setminus(B\cup B_n))\subset D_n$. If we take $D=\bigcup_n D_n$, we get $\pi(H\setminus B)\subset D$, completing the proof.

It follows from the preceding lemma that if $H_\alpha=\emptyset$ for some $\alpha<\omega_1$, then A is contained in a member of $\mathscr{C}(\Gamma)_\sigma$. Thus it remains to prove:

LEMMA 17. *There exists $\alpha<\omega_1$ such that $H_\alpha=\emptyset$.*

Proof. Assume $H_\alpha\neq\emptyset$ for each $\alpha<\omega_1$. We take $Q=\{L\in\underline{\underline{F}}(H);\ \pi_E(L)$ is a unique point of $E\}$, which is a Polish subspace of $\underline{\underline{K}}(E\times F\times G)$. If $k\in N$ and

$\alpha<\omega_1$, define $\mathscr{S}^\alpha_k = \{(L_1,\dots,L_k)\in Q^k;\ L_l\cap H\subset D(L_{l+1}\cap H)$ if $1\leq l<k$ and $L_k\cap H\subset H_\alpha\}$. We verify that the conditions of Lemma 5 are satisfied.

1. We show that $Z=\{(L,M)\in Q^2;\ L\cap H\subset D(M\cap H)\}$ is analytic in Q^2. Let $(U_i)_i$ be a countable base for the topology of $E\times F\times G$. For each $i\in N$, consider $Z_i = [\{L\in Q;\ L\cap U_i=\emptyset\}\times Q]\cup[Q\times\{M\in Q;\ \overline{\pi(M\cap U_i)}\in\Gamma\}]$ which is easily seen to be analytic. Therefore Z is analytic, since $Z=\bigcap_i Z_i$.

2. This follows immediately from the fact that $(H_\alpha)_{\alpha<\omega_1}$ is decreasing.

3. Suppose $k\in N$, $\alpha<\omega_1$ and $(L_1,\dots,L_k)\in\mathscr{S}^{\alpha+1}_k$. If $\pi_E(L_k)=\{x\}$, then $L_k\cap H\subset\{x\}\times H_{\alpha+1}(x)\in D(\{x\}\times H_\alpha(x))$. The set $L_{k+1}=\overline{\{x\}\times H_\alpha(x)}$ belongs to Q and $(L_1,\dots,L_k,L_{k+1})\in\mathscr{S}^\alpha_{k+1}$ since $L_{k+1}\cap H=\{x\}\times H_\alpha(x)$. Thus the lemma applies. For each $\alpha<\omega_1$ the set $\mathscr{S}^\alpha_1\neq\emptyset$, because $\overline{\{x\}\times H_\alpha(x)}\in\mathscr{S}^\alpha_1$ whenever $H_\alpha(x)\neq\emptyset$. Hence there exists a sequence $(L_k)_k$ in Q so that $(L_k,L_{k+1})\in Z$ for each $k\in N$. In particular there is $x\in E$ with $\pi_E(L_k)=\{x\}$ for all $k\in N$. If $(F_r)_r$ is a sequence of closed sets not belonging to Γ and covering $\{x\}\times A(x)$, we obtain that $\{x\}\times H(x)\subset\bigcup_r\pi^{-1}(F_r)$. Let $L=\overline{\bigcup_k L_k}$, which belongs to $\underline{\underline{F}}(H)$. By the Baire category theorem, there is $r\in N$ and $i\in N$ such that $L\cap U_i\neq\emptyset$ and $L\cap H\cap U_i\subset\pi^{-1}(F_r)$.

Thus there is $k\in N$ with $L_k\cap U_i\neq\emptyset$ and therefore $\overline{\pi(L_{k+1}\cap U_i)}\in\Gamma$, since c$L_k, L_{k+1})\in Z_i$. But $\overline{\pi(L_{k+1}\cap U_i)}\subset\overline{\pi(L\cap H\cap U_i)}\subset F_r$, which is the required (ontradiction.

$F_{\sigma\delta}$-sections of Borel sets. Let again E, F be compact metric spaces. In [3], we obtained the following result:

THEOREM 4. *If A and B are analytic subsets of $E\times F$, then $\{x\in E;\ A(x)$ is contained in an $F_{\sigma\delta}$-set which is disjoint from $B(x)\}$ is coanalytic in E.*

In this section, we will prove:

THEOREM 5. *Let A and B be analytic subsets of $E\times F$ such that $A(x)$ is contained in an $F_{\sigma\delta}$-set which is disjoint from $B(x)$, for all $x\in E$. Then A can be separated from B by a member of $\mathscr{C}_{\sigma\delta}$.*

It clearly implies Theorem 2.

Assume B a fixed analytic subset of $E\times F$. Let $B=\bigcup_\nu\bigcap_k B_{\nu|k}$ be an analytic representation of B, where the $B_{\nu|k}$ are closed in $E\times F$ and $B_{\nu|k+1}\subset B_{\nu|k}$. By induction on $\alpha<\omega_1$, we define for each $\pi\in\mathscr{R}$ a class $\mathscr{D}^\alpha(\pi)$ of subsets of $E\times F$, by taking:

$\mathscr{D}^0(\pi)=\{D\subset E\times F;\ \bar D\cap B_\pi=\emptyset\}$.

$\mathscr{D}^\beta(\pi)=\{D\subset E\times F;$ for each $p\in N$ there is a countable closed covering $(F_r)_r$ of D such that $D\cap F_r\in\bigcup_{\alpha<\beta}\mathscr{D}^\alpha(\pi,p)$ for each $r\in N\}$.

LEMMA 18. *Let A be an analytic subset of $E\times F$. Then for each $\alpha<\omega_1$ and $\pi\in\mathscr{R}$ the set $\Gamma^\alpha(\pi)(A)=\{L\in\underline{\underline{K}}(E\times F);\ A\cap L\notin\mathscr{D}^\alpha(\pi)\}$ is a member of $\mathscr{P}^*$.*

Proof. We proceed by induction on $\alpha<\omega_1$. Let $(O_n)_n$ be a decreasing sequence of open sets containing B_π such that $B_\pi = \bigcap_n \bar{O}_n$. Then

$$\Gamma^0(\pi)(A) = \{L \in \underline{\underline{K}}(E\times F);\ \overline{A \cap L} \cap B_\pi \neq \emptyset\}$$
$$= \bigcap_n \{L \in \underline{\underline{K}}(E\times F);\ L \cap A \cap O_n \neq \emptyset\}$$

and hence $\mathscr{P}$-analytic by Lemma 6. Let now the property be established for all $\alpha<\beta$. Using the definition of $\mathscr{D}^\beta(\pi)$, we obtain that $\Gamma^\beta(\pi)(A) = \bigcup_p \{L \in \underline{\underline{K}}(E\times F);\ A \cap L$ can not be covered by countably many closed sets not belonging to $\bigcap_{\alpha<\beta} \Gamma^\alpha(\pi, p)(A)\}$. Since by induction hypothesis $\bigcap_{\alpha<\beta} \Gamma^\alpha(\pi, p)(A) \in \mathscr{P}^*$, it follows from Lemma 10 that also $\Gamma^\beta(\pi)(A) \in \mathscr{P}^*$. This completes the proof.

COROLLARY 19. *If A is analytic in $E\times F$, then for each $\alpha<\omega_1$ and $\pi \in \mathscr{R}$ the set $\{x \in E;\ \{x\}\times A(x) \in \mathscr{D}^\alpha(\pi)\}$ is coanalytic.*

Proof. It is the same as that of Corollary 11.

For each $\pi \in \mathscr{R}$, take $B(\pi) = \bigcup_{\pi<\nu} \bigcap_k B_{\nu|k}$. We pass to the first step in the proof of Theorem 5.

LEMMA 20. *Let A be an analytic subset of $E\times F$ and assume that there exist $\alpha<\omega_1$ and $\pi \in \mathscr{R}$ such that $\{x\}\times A(x) \in \mathscr{D}^\alpha(\pi)$ for each $x \in E$. Then A can be separated from $B(\pi)$ by a member of $C_{\sigma\delta}$.*

Proof. If $\alpha = 0$, then $\overline{A(x)} \cap B_\pi(x) = \emptyset$ for all $x \in E$. Hence, by Lemma 4, there exists a set D in $\mathscr{C}$ so that $A \subset D$ and $D \cap B_\pi = \emptyset$. Let the property be true for all $\alpha<\beta$ and assume $\{x\}\times A(x) \in \mathscr{D}^\beta(\pi)$ for each $x \in E$. Take $p \in N$ fixed. The set $\{x\}\times A(x)$ can be covered by countably many closed sets $(F_r)_r$ with $A \cap F_r \in \bigcup_{\alpha<\beta} \mathscr{D}^\alpha(\pi, p)$ or $F_r \notin \Gamma_p = \bigcap_{\alpha<\beta} \Gamma^\alpha(\pi, p)(A)$ for each $x \in E$. Because $\Gamma_p \in \mathscr{P}^*$ by Lemma 18, we obtain by Theorem 3 a sequence $(A_r^p)_r$ in $\mathscr{C}(\Gamma_p)$ with $A \subset \bigcup_r A_r^p$. Let $r \in N$ be also fixed. Since for each $x \in E$, the set $\{x\}\times A_r^p(x) \notin \Gamma_p$, there exists $\alpha<\beta$ such that $\{x\}\times(A_r^p \cap A)(x) \in \mathscr{D}^\alpha(\pi, p)$. Hence, using Corollary 19, the sets $C(r, p, \alpha) = \{x \in E;\ \{x\}\times(A_r^p \cap A)(x) \in \mathscr{D}^\alpha(\pi, p)\}$ are coanalytic and they cover E. Therefore there is a sequence $(B(r, p, \alpha))_{\alpha<\omega_1}$ of disjoint Borel sets satisfying $B(r, p, \alpha) \subset C(r, p, \alpha)$ and $E = \bigcup_{\alpha<\beta} B(r, p, \alpha)$. For each $\alpha<\beta$, we introduce the set $A_{rp\alpha} = A_r^p \cap A \cap (B(r, p, \alpha)\times F)$, which is still analytic. Because $\{x\}\times A_{rp\alpha}(x) \in \mathscr{D}^\alpha(\pi, p)$ for each $x \in E$, the induction hypothesis applies. Thus we obtain a member $D_{rp\alpha}$ of $\mathscr{C}_{\sigma\delta}$ separating $A_{rp\alpha}$ from $B(\pi, p)$.

If we define $D_{rp} = \bigcup_{\alpha<\beta} [D_{rp\alpha} \cap (B(r, p, \alpha)\times F)]$, it is easily seen that D_{rp} is also $\mathscr{C}_{\sigma\delta}$, $D_{rp} \supset A_r^p \cap A$ and $D_{rp} \cap B(\pi, p) = \emptyset$. The set

$$D_p = \bigcup_r A_r^p \cap \bigcap_r [((E\times F)\setminus A_r^p) \cup D_{rp}]$$

belongs to $\mathscr{C}_{\sigma\delta}$, $D_p \supset A$ and $D_p \cap B(\pi, p) = \emptyset$. We only have to take $D = \bigcap_p D_p$.

The proof of Theorem 5 will be complete if the following property holds:

LEMMA 21. *If A is an analytic subset of $E\times F$, then one of the following 2 alternatives must occur:*

1. *There exists $\alpha<\omega_1$ such that $\{x\}\times A(x) \in \mathscr{D}^\alpha(\varphi)$ for all $x \in E$.*
2. *There exists $x \in E$ such that no $F_{\sigma\delta}$-subset of F separates $A(x)$ from $B(x)$.*

Proof. There is a compact metric space G and a G_δ-subset H of $E\times F\times G$ so that $A = \pi(H)$, where $\pi\colon E\times F\times G \to E\times F$ is the projection. Take a countable base $(U_n)_n$ for the topology of $E\times F\times G$. Let $(c_k)_k$ be the standard ordering of $\mathscr{R}$. We will again make use of Lemma 5. We introduce for each $k \in N$ and $\alpha<\omega_1$ a subset $\mathscr{S}_k^\alpha$ of $[N\times \underline{\underline{F}}(H)]^k$:

$\mathscr{S}_1^\alpha$ consists of the elements (p_φ, K_φ) of $N\times \underline{\underline{F}}(H)$ such that:

1. $K_\varphi \neq \emptyset$.
2. There is some $x \in E$ with $K_\varphi \subset \{x\}\times F\times G$.
3. If U is open in $E\times F\times G$ and $U \cap K_\varphi \neq \emptyset$ then

$$\overline{\pi(U \cap K_\varphi)} \cap A \notin \mathscr{D}^\alpha(p_\varphi).$$

If $c_k = (d, n)$ with $d \in \mathscr{R}$ and $n \in N$, then $\mathscr{S}_k^\alpha$ consists of the elements $(p_{c_l}, K_{c_l})_{1\leqslant l\leqslant k}$ of $[N\times \underline{\underline{F}}(H)]^k$ such that:

1. $U_n \cap K_d \neq \emptyset \Rightarrow K_{dn} \neq \emptyset$.
2. $\pi(K_{dn}) \subset \pi(K_d \cap \bar{U}_n)$.
3. If U is open in $E\times F\times G$ and $U \cap K_{dn} \neq \emptyset$, then

$$\overline{\pi(U \cap K_{dn})} \cap A \notin \mathscr{D}^\alpha(p_\varphi, \ldots, p_d, p_{dn}).$$

We claim that the conditions 1, 2, 3 of Lemma 5 are satisfied.

1. In fact $\mathscr{S}_k$ is closed in $[N\times \underline{\underline{F}}(H)]^k$.
2. Is obviously satisfied.
3. Assume $\sigma = (p_{c_l}, K_{c_l})_{1\leqslant l\leqslant k}$ an element of $[N\times \underline{\underline{F}}(H)]^k$ with $\sigma|l \in \mathscr{S}_l^{+1}$ for each $l = 1, \ldots, k$. Now $c_{k+1} = (c_l, n)$ for some $l = 1, \ldots, k$ and $n \in N$. If $U_n \cap K_{c_l} = \emptyset$, let $\iota \in [N\times \underline{\underline{F}}(H)]^{k+1}$ be given by $\iota|k = \sigma$ and $\iota_{k+1} = (p, \varphi)$, where $p \in N$ is chosen arbitrarily. Clearly $\iota \in \mathscr{S}_{k+1}^\alpha$. Assume now $U_n \cap K_{c_l} \neq \emptyset$. Because $\sigma|l \in \mathscr{S}_l^{\alpha+1}$, we get $\overline{\pi(U_n \cap K_{c_l})} \cap A \notin \mathscr{D}^{\alpha+1}(p_\varphi, \ldots, p_{c_l})$. Therefore there must be some $p \in N$ such that there is no countable closed covering $(F_r)_r$ of $\overline{\pi(U_n \cap K_{c_l})} \cap A$ with $\overline{\pi(U_n \cap K_{c_l})} \cap A \cap F_r \in \mathscr{D}^\alpha(p_\varphi, \ldots, p_{c_l}, p)$ for all $r \in N$. Remark that $\overline{\pi(U_n \cap K_{c_l})} \cap A$ is the image of $\pi^{-1}(\overline{\pi(U_n \cap K_{c_l})}) \cap H$ by π. A standard argument yields us a nonempty closed subset Y of $\pi^{-1}(\overline{\pi(U_n \cap K_{c_l})}) \cap H$ so that

$$\overline{\pi(U \cap Y)} \cap \overline{\pi(U_n \cap K_{c_l})} \cap A \notin \mathscr{D}^\alpha(p_\varphi, \ldots, p_{c_l}, p),$$

whenever U is open in $E\times F\times G$ and $U\cap Y\neq\emptyset$. It is clear that $\overline{Y}\in \underline{\underline{F}}(H)$. Let $\imath\in[N\times\underline{\underline{F}}(H)]^{k+1}$ be given by $\imath|k=\sigma$ and $\imath_{k+1}=(p,\overline{Y})$, which is in $\mathscr{S}^{\alpha}_{k+1}$.

Thus Lemma 5 applies and we have to distinguish 2 cases:

Case I. There is $\eta<\omega_1$ with $\mathscr{S}^{\eta}_1=\emptyset$.

Take $\alpha=\eta+1$ and assume the existence of $x\in E$ with $\{x\}\times A(x)\notin\mathscr{D}^{\alpha}(\varphi)$. Since $\{x\}\times A(x)=\pi((\{x\}\times F\times G)\cap H)$, an integer p_{φ} and a nonempty closed subset Y of $(\{x\}\times F\times G)\cap H$ can be found with $(\{x\}\times A(x))\cap\overline{\pi(Y\cap U)}\notin\mathscr{D}^{\eta}(p_{\varphi})$ whenever U is open in $E\times F\times G$ and $U\cap Y\neq\emptyset$. It follows that $(p_{\varphi},\overline{Y})\in\mathscr{S}^{\eta}_1$, a contradiction.

Case II. There is $\xi\in[N\times\underline{\underline{F}}(H)]^{N}$ such that $\xi|k\in\mathscr{S}^{0}_{k}$ for each $k\in N$.

Let $\xi=(p_c,K_c)_{c\in\mathscr{R}}$ and take $x\in E$ such that $K_{\varphi}\subset\{x\}\times F\times G$. It is clear that ξ satisfies the following properties:

1. $\forall c\in\mathscr{R}$: $K_c\subset\{x\}\times F\times G$.
2. $K_{\varphi}\neq\emptyset$.
3. $\forall c\in\mathscr{R}$, $\forall n\in N$: $\pi(K_{cn})\subset\pi(K_c\cap\overline{U}_n)$.
4. $\forall c\in\mathscr{R}$, $\forall n\in N$: $U_n\cap K_c\neq\emptyset\Rightarrow B_{p_{\varphi},\ldots,p_c,p_{cn}}\cap\pi(K_{cn})\neq\emptyset$.

It is shown in [3] that under this hypothesis, no $F_{\sigma\delta}$-subset of F separates $A(x)$ from $B(x)$.

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