It is easy to write formulas \( \psi(x, y, z, X, Y, Z_1, Z_2, Z_3) \) in the topological language stating (in \( T \)) that \( x, y \in X \) and \( X \in Z_1 \), \( Y \in Z_2 \), \( x \sim z \pmod{Z_3} \) and there exists \( y' \in Y \) such that \( y \sim y' \pmod{Z_1} \) and \( y' \sim z \pmod{Z_3} \).

In order to interpret DL in \( T \) it is enough to find \( X, Y, Z_1, Z_2, Z_3 \subseteq \mathbb{R}^2 \) such that \( \{X\} = 2^x \) and \( \psi \) defines a one-one correspondence between \( \{(x, y) : x, y \in X\} \) and \( \{(x, z) : z \in \mathbb{R}^2\} \). Let \( Q \) be the set of rational numbers. Choose \( X = R \times \{0\}, Y = \{0\} \times R, Z_1 = R \times Q, Z_2 = Q \times R \) and \( Z_3 = \{(a, b) : a - b \in Q\} \). Then \( \psi(x, y, z, X, Y, Z_1, Z_2, Z_3) \) holds iff there exist \( a, b \in R \) such that \( x = (a, 0), y = (b, 0) \) and \( z = (a, b) \).

**References**


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**Borel sets with \( F_{\delta\delta} \)-sections**

by

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Abstract. Let \( E, F \) be compact metric spaces. We characterize Borel sets \( A \) in \( E \times F \) with \( F_{\delta\delta} \)-sections.

Introduction. We consider two fixed compact metric spaces \( E \) and \( F \). The class \( \mathcal{A} \) will consist of the Borel subsets \( A \) of \( E \times F \) such that for each \( x \in E \) the section \( A(x) = \{y \in F : (x, y) \in A\} \) is closed in \( F \). We will prove the following:

**Theorem 1.** If \( A \) is a Borel subset of \( E \times F \) such that each section \( A(x) \) is \( F_{\delta\delta} \) in \( F \), then \( A \) belongs to the class \( \mathcal{A} \).

This is an extension of the work of J. Saint-Raymond (see [13]), who established:

**Theorem 2.** If \( A \) is a Borel subset of \( E \times F \) such that each section \( A(x) \) is \( F_{\delta} \) in \( F \), then \( A \) belongs to the class \( \mathcal{Q} \).

Theorem 1 is also related to my earlier paper [2].

Preliminaries. \( N \) will denote the set of all positive integers. Let \( \mathcal{A} = \bigcup N^\infty \), taking \( N^0 = \{0\} \). Thus \( \mathcal{A} \) consists of the finite complexes of integers. If \( c \in \mathcal{A} \), let \( |c| \) be the length of \( c \). If \( c, d \in \mathcal{A} \), we write \( c < d \) if \( c \) is an initial section of \( d \). Let \( \rho_n \) be an enumeration of all prime numbers. If we associate \( 0 \) to \( \emptyset \) and the integer \( n \) to the complex \( c = (n_1, ..., n_k) \), a one-one map of \( \mathcal{A} \) into \( N \) is established. The induced ordering of \( \mathcal{A} \) will be called the standard ordering. Let \( \mathcal{N} = N^\infty \). If \( v \in \mathcal{A} \) and \( c \in \mathcal{N} \), we write \( c < v \) if \( c \) is an initial section of \( v \).

If \( L \) is a compact metric space, then \( K(L) \) consists of all closed subsets of \( L \) and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [7]).

**Lemma 1.** Let \( P \) be a Polish subspace of the compact metric space \( L \). Then the subspace \( K(P) \) of \( K(L) \) consisting of those compact sets \( K \) in \( L \) such that \( K = K \cap P \) is Polish.

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A Banach space $\mathcal{V}$ on a set $\Omega$ will be a family of subsets of $\Omega$ containing the empty set. A subset of $\Omega$ is said to be $\mathcal{V}$-analytic if it is the result of Souslin operation performed on members of $\mathcal{V}$. For more details, I refer to [7].

I also remember the separation theorem of Novikov, which will be often used in this paper:

**Lemma 2.** Let $(A_n)$ be a sequence of analytic subsets of the Polish space $\mathcal{P}$ satisfying $\bigcap_n A_n = \emptyset$. Then there exists a sequence $(B_n)$ of Borel subsets of $\mathcal{P}$ such that $A_n \subseteq B_n$ for each $n$ and $\bigcap_n B_n = \emptyset$.

The reader can find a proof of this result in [7].

If $\mathcal{A}$ is a subset of $\mathcal{E} \times \mathcal{F}$, let $\mathcal{A}^x = \{y \in \mathcal{A} : x < \pi(y)\}$ be the subset of $\mathcal{E} \times \mathcal{F}$ defined by $\mathcal{A}^x = \{y \in \mathcal{A} : x < \pi(y)\}$ for $x \in \mathcal{E}$. Consider for each $x \in \mathcal{E}$ a finite covering $(U_i)_{i \in I}$ of $\mathcal{F}$ by open sets with diameter less than $2^{-x}$. It is easily verified that $\mathcal{A}^x = \bigcap \bigcup \{U_i \in \mathcal{U} : x < \pi(y)\}$.

Therefore we obtain:

**Lemma 3.** If $\mathcal{A}$ is analytic in $\mathcal{E} \times \mathcal{F}$, then also $\mathcal{A}^x$ is analytic.

**Lemma 4.** If $\mathcal{A}$ and $\mathcal{B}$ are analytic in $\mathcal{E} \times \mathcal{F}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$, then there exists a member $C$ of $\mathcal{F}$ such that $A \cap C \subseteq C \cap B = \emptyset$.

**Proof.** Take the open set $U_i$ as before. For each $x \in \mathcal{E}$ the set $\mathcal{K}_x = \mathcal{B} \cap \mathcal{E}$ is analytic in $\mathcal{E} \times \mathcal{F}$. Since $\mathcal{K}_x \cap \mathcal{B}_x = \emptyset$, a sequence $(T_i)$ of Borel subsets of $\mathcal{E} \times \mathcal{F}$ is obtained such that $\mathcal{K}_x \subseteq T_i$ for each $x$ and $\bigcap T_i = \emptyset$. Now, for each $x$ and each $i$, we have $\mathcal{K}_x \subseteq T_i$, and $\mathcal{K}_x \subseteq \bigcap T_i = \emptyset$. Hence there exist Borel sets $B_i$ in $\mathcal{B}$ with $\mathcal{E} \times \mathcal{F} \cap (B_i \cap T_i) = \emptyset$. For each $x \in \mathcal{E}$ take $\mathcal{K}_x = \bigcup_{i \in I} (B_i \times U_i)$, which belongs to $\mathcal{F}$. Then $A \cap C_i \cap (C \cap B) = \emptyset$.

The set $C = \bigcup_i C_i$ satisfies the required properties.

**A result about transfinite systems.** The proof of various results in the remainder of the text is considerably shortened by the use of the following lemma:

**Lemma 5.** Let $\mathcal{K}$ be a Polish space $\mathcal{P}$ be given. Assume for all $x \in \mathcal{E}$ and $x < \pi_0$, a subset $\mathcal{K}_x$ of $\mathcal{P}$ can be defined, such that the following conditions are satisfied:

1. If $x \in \mathcal{E}$, then $\mathcal{K}_x$ is analytic in $\mathcal{P}$.
2. If $x \in \mathcal{E}$ and $x < \pi_0$, then $\mathcal{K}_x = \mathcal{K}_{x < \pi_0}$.
3. If $x \in \mathcal{E}$, $x < \pi_0$, and $x \in \mathcal{P}$, then there exists $\mathcal{K}_x \in \mathcal{K}_{x < \pi_0}$ with $x = \pi_0(k)$.

**Proof.** Suppose $\mathcal{K}_x \neq \emptyset$ for each $x < \pi_0$. Then there is some $\xi \in \prod \mathcal{P}$ such that $\xi_0(k) \in \mathcal{K}_x$ for each $k \in \mathcal{N}$.

**Proof.** If $x \in \mathcal{E}$, let $\mathcal{K}_x = \mathcal{B} \times \mathcal{P}$ be a continuous map with image $\mathcal{K}_x$.

By induction we will define for every $x < \pi_0$ elements $\pi_{x,1}, \ldots, \pi_{x,y}$, such that the following conditions are satisfied:

1. $\pi_{x,1} = \pi_0(k)$ if $k \geq \mathcal{L}$.
2. For each $x < \pi_0$, there exists $\xi \in \prod \mathcal{P}$ so that $\xi_0(k) \in \mathcal{K}_x$ for each $k = 1, \ldots, y$.

Since $\mathcal{K}_x \neq \emptyset$ for every $x < \pi_0$ and $\mathcal{K}_x = \mathcal{B} \times \mathcal{P}$, there must be some $\pi_{x,1} < \mathcal{N}$ such that $\pi_0(\mathcal{K}_x) \neq \emptyset$ for each $x < \pi_0$. Assume now $\pi_{x,1}, \ldots, \pi_{x,y}$ obtained. Let $x < \pi_0$, then there exists $\xi \in \prod \mathcal{P}$ such that $\xi_0(k) \in \mathcal{K}_x$ for each $k = 1, \ldots, y-1$. Therefore there is $\xi \in \mathcal{K}_x$ with $x < \pi_{x,y}$. Then $\mathcal{K}_x \neq \emptyset$ and $\mathcal{K}_x = \mathcal{B} \times \mathcal{P}$.

Again there must exist $\mathcal{K}_x \in \mathcal{K}_{x < \pi_0}$ such that for each $x < \pi_0$, there is $\xi \in \prod \mathcal{P}$ with $\xi_0(k) \in \mathcal{K}_x$ and $\mathcal{K}_x \neq \emptyset$ for each $k = 1, \ldots, y$. Hence there is $\xi \in \prod \mathcal{P}$ satisfying $\xi_0(k) \in \mathcal{K}_x$ for each $k < \pi_0$ and the lemma is established.

**Results about closed coverings.** Let $\mathcal{V}$ be the Banach space on $\prod \mathcal{P}$ consisting of the open subsets of $\prod \mathcal{P}$ which are of the form $\{x \in \prod \mathcal{P} : L \cap \mathcal{V} \neq \emptyset\}$, where $\mathcal{V}$ ranges over the open sets in $\prod \mathcal{P}$. Let $\mathcal{P} \neq \emptyset$ be the set of $\mathcal{V}$-analytic subsets of $\prod \mathcal{P}$.

The following two lemmas are obvious.

**Lemma 6.** If $\mathcal{A}$ is an analytic subset of $\mathcal{E} \times \mathcal{F}$, then the set $\{x \in \mathcal{E} \times \mathcal{F} : L \cap \mathcal{A} \neq \emptyset\}$ is a member of $\mathcal{P} \neq \emptyset$.

**Proof.** If $x \in \mathcal{E} \times \mathcal{F}$, then $L \cap \mathcal{A} \neq \emptyset$.

**Lemma 7.** If $\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is an analytic subset of $\mathcal{E} \times \mathcal{F}$.

**Proof.** Let $\mathcal{A}$ be an analytic subset of $\mathcal{E} \times \mathcal{F}$ such that $\mathcal{A}(x)$ is closed for each $x \in \mathcal{E}$. Then $\mathcal{A} \subseteq \mathcal{A}(x) \times \mathcal{A}(x) = \mathcal{A}$ is analytic.

**Proof.** There is a system $(\mathcal{A}, \mathcal{A}_x, \mathcal{A}_y)$ of open subsets of $\mathcal{E} \times \mathcal{F}$ such that $\mathcal{A}(x)$ is closed for each $x \in \mathcal{E}$.
of $\mathcal{X} \times E \times F$ defined by $\Sigma(x, x) = A(x) \cap Q_{\mu}(x)$. It is clear that $\Sigma$ is analytic in $\mathcal{X} \times E \times F$. Hence $\pi_2(\Gamma) = (x) \times A(x) \notin \Gamma$.

If $\Gamma \notin \mathcal{G}$, let $\Psi(\Gamma)$ consist of the members of $\mathcal{G}$ such that $\langle x \rangle \times A(x) \notin \Gamma$ for each $x \in E$.

**Lemma 9.** Let $\Gamma \notin \mathcal{G}$ and let $S$ be analytic in $E \times F \times G$ such that $S(x)$ is closed and $\langle x \rangle \times S(x) \notin \Gamma$ for each $x \in E$. Then there exists $T \in \Psi(\Gamma)$ containing $S$.

**Proof.** There is a system $\{\Omega_\alpha\}_{\alpha \in \mathfrak{X}}$ of open subsets of $E \times F \times G$ such that $\Gamma = \bigcup_{\alpha \in \mathfrak{X}} \Gamma_\alpha$, where $\Gamma_\alpha = \{L \in \mathcal{K}(E \times F \times G); L \cap \Omega_\alpha \neq \emptyset\}$. We will consider the space $\mathcal{B} = \bigcup_{\Gamma_\alpha} F_{\alpha}$, where each $F_{\alpha} = F$. If $r \in E$ and $\pi \in \mathcal{N}$, take

$$\Omega^{*_\pi} = \{(x, y') \in E \times F^*; (x, y') \in \Omega_\pi\}$$

and define $\Omega^* = \bigcup_{\pi \in \mathcal{N}} \Omega^{*_\pi}$, which is an analytic subset of $E \times F^*$. Let further for each $x \in E$ the set $S^*_x = \{(x, y'') \in E \times F^*; (x, y'') \in S\}$ and let $S^* = \bigcap_{x \in E} S^*_x$. It is easy to deduce from the hypothesis that $S^* \cap \Omega^* = \emptyset$.

It follows that there is a sequence $\{B_\alpha\}$ of Borel sets in $E \times F^*$ with $S^*_x \cap \Omega^* \subseteq B^{*_\pi}$ for each $x \in E$ and $\Omega^* \subseteq \bigcap_{\mathcal{G}} B_\alpha$. Let $r \in E$ be fixed. Since $S^*_r \cap (\Omega^* \times \mathcal{G}) = \emptyset$, we obtain that $S$ and $\pi \times r \times (\Omega^* \times \mathcal{G})$ are disjoint analytic sets. We now use the fact that each section $S(x)$ is closed to obtain a set $T_x$ in $\mathcal{G}$ such that $S(x) \cap T_x = \emptyset$. Then $T_x \cap (\Omega^* \times \mathcal{G}) = \emptyset$. We claim that the set $T = \bigcap_{x \in E} T_x$ satisfies, we only have to verify that $\langle x \rangle \times T(x) \notin \Gamma$ for each $x \in E$. Assume not, then there is $x \in \mathcal{X}$ such that $T(x) \cap \Omega_\mu(x) \neq \emptyset$ for each $x \in E$. Therefore $\bigcap_{x \in E} T(x) \cap \Omega_\mu(x) \neq \emptyset$, a contradiction.

We will use the following stability property of $\mathcal{G}$:

**Lemma 10.** Let $\Gamma \notin \mathcal{G}$ and let $A$ be an analytic subset of $E \times F$. If $A \subseteq \mathcal{K}(E \times F)$ then $A \cap L$ cannot be covered by countably many closed sets not belonging to $\Gamma$, then $A \notin \mathcal{G}$.

**Proof.** Take a compact metric space $G$ and a $G^\sharp$-subset $H$ of $E \times F \times G$ satisfying $\pi(A, H) = \mathfrak{K}(E \times F \times G)$. Let $\pi \in \mathfrak{K}(E \times F \times G)$ be fixed. Remark that $A \cap L = \pi(H \cap \pi^{-1}(L))$, where $H \cap \pi^{-1}(L)$ is a $G^\sharp$. It follows that $L \in A$ if and only if there exists a nonempty closed subset $T$ of $H \cap \pi^{-1}(L)$ such that if $U \in \pi \cap L \neq \emptyset$, then $\pi(T \cap U) = \pi(T \cap U) \notin \Gamma$. Hence $L \subseteq A$ if and only if there exists a nonempty set $M$ in $\mathcal{K}(E \times F \times G)$ satisfying:

1. $M \in \mathcal{K}(E \times F \times G)$.
2. $\pi(M \cap U) \notin \Gamma$ whenever $U \in \mathcal{G}$ and $M \cap U \neq \emptyset$.
3. $\pi(M) \subseteq L$.

Thus it remains to show that $\mathcal{K}(E \times F \times G)$ consists of the nonempty compact subsets of $E \times F$ belonging to $\mathcal{G}$. We will prove that the set $\mathcal{A} = \{(L, M) \in \mathcal{K}(E \times F \times G); M \neq \emptyset \}$ and $L, M$ satisfy $(1), (2), (3)$ is $\mathcal{G}$-$\times \mathcal{G}$-analytic, where $\times \mathcal{G}$ is the product topology on $\mathcal{K}(E \times F \times G)$ consisting of the closed sets. Because $\mathcal{A} = \pi_{1}(E \times F \times G)$, we will then obtain that $A$ is $\mathcal{G}$-analytic (see [7]).

1. Since $\mathcal{K}(E \times F \times G)$ is a $G^\sharp$-subset of $\mathcal{K}(E \times F \times G)$, $\pi_2(\Gamma)$ is $\mathcal{G}$-$\times \mathcal{G}$-analytic.

2. Clearly $\pi(A) \subseteq \mathcal{K}(E \times F \times G)$ satisfies $\pi(A \cap U) \subseteq \emptyset$. Because the map $\mathcal{K}(E \times F \times G) \to \mathcal{K}(E \times F \times G); M \mapsto \pi(M \cap U)$ is $\mathcal{G}$-$\times \mathcal{G}$ measurable, from the $

3. Let $(Y_t)$ be a countable base for the topology of $E \times F$. For each $t \in E$, we have that $G_t = \{L \in \mathcal{K}(E \times F); L \cap Y_t \neq \emptyset\}$ and $\mathcal{A}_t = \{M \in \mathcal{K}(E \times F \times G); M \cap \pi^{-1}(Y_t) \neq \emptyset\}$. But the set $\{(L, M) \in \mathcal{K}(E \times F \times G); \pi(M) \subseteq L\}$ is precisely $\bigcap_{t \in E} \{G_t \times \mathcal{A}_t \} \cup \mathcal{K}(E \times F \times G)$ and hence $\mathcal{G}$-$\times \mathcal{G}$ analytic.

So the proof is complete.

**Corollary 11.** Let $\Gamma \notin \mathcal{G}$ and let $A$ be an analytic subset of $E \times F$. Then the set $\{x \in E; \langle x \rangle \times A(x) \cap \Omega_\mu(x) \neq \emptyset \} \subseteq \mathcal{G}$ is $\mathcal{G}$-$\times \mathcal{G}$-analytic.

**Proof.** The set $A$ considered in Lemma 10 is an analytic subset of $\mathcal{K}(E \times F)$.

We only have to remark that the map $E \to \mathcal{K}(E \times F); x \mapsto \langle x \rangle \times A(x)$ is continuous to complete the proof.

Combining Lemma 6 and Corollary 11 we obtain immediately:

**Corollary 12.** Let $A$ and $B$ be analytic subsets of $E \times F$. Then the set $\{x \in E; \langle x \rangle \times A(x) \subseteq \mathcal{G} \} \subseteq \mathcal{G}$ is analytic.

Before we pass to the proof of the theorem, let us mention the following easy corollary:

**Corollary 13.** Let $A$ and $B$ be analytic subsets of $E \times F$ such that $A(x)$ is contained in an $F^\sharp$-set which is disjoint from $B(x)$, for each $x \in E$. Then $A$ can be separated from $B$ by a member of $\mathcal{G}$.

This result is due to J. Saint-Raymond (see [13]).

The remainder of this section is devoted to the proof of Theorem 3. Let $\mathfrak{H}$ be a compact metric space and set $H$ be an $\mathfrak{H}$-subset of $E \times F \times G$ such that $A = \pi(H)$, where $x \in \mathfrak{H} \times E \times F \times G$ is the projection. If $\mathfrak{H}$ is a subset of $H$, take $D(\mathfrak{H}) = \{(x, y, z) \in \mathfrak{H}; \text{for each neighborhood } U \text{ of } \langle x, y, z \rangle \text{ the set } \{x \in \pi(\mathfrak{H} \cap U)\} \in \Gamma\}$. 

**Lemma 14.** If $\mathfrak{H}$ is analytic, then $D(\mathfrak{H})$ is analytic. If moreover $B$ is a Borel subset of $H$ with $D(\mathfrak{H}) \subseteq B$, then $\pi(\mathfrak{H} \cap U) \subseteq B$ is contained in a member of $\mathcal{G}$.
Proof. Let $(U_i)$ be a countable base for the topology of $H$. For each $i \in \mathbb{N}$, the set $\pi(X \cap U_i)$ is analytic by Lemma 3 and thus

$$E_i = \{x \in E : \pi(x \cap U_i) \not\in \Gamma\}$$

is coanalytic by Lemma 8. Hence $D(X) = X \setminus \bigcup_i \{E_i \in X \cap U_i\}$ is analytic.

If $D(X) = B$, then $X \setminus B = \bigcup \{E_i \in X \cap U_i\}$, where $X \setminus B$ is analytic and each set $E_i \in X \cap U_i$ is analytic in $E \times F \times G$. Therefore there are analytic sets $(D_i)$ so that $D_i \in (E \times F) \cap U_i$ and $X \setminus B = \bigcup D_i$. We obtain that $\pi(X \cap U_i) \not\in \Gamma \cup \{D_i\}$ for all $i \in \mathbb{N}$.

Let $\lambda, \gamma$ be the transfinite system obtained as follows:

$\lambda_0 = H$,

$\lambda_{\alpha+1} = D(\lambda_\alpha)$.

If $\gamma$ is a limit ordinal, take $\lambda_\gamma = \bigcup_{\alpha < \gamma} \lambda_\alpha$.

It is easily verified that the sets $\lambda_\alpha$ are closed in $\lambda(x)$ for all $x \in E$ and the system $(\lambda_\alpha)_{\alpha < \gamma}$ is decreasing. Using Lemma 14, we obtain

**Lemma 15.** For each $\alpha < \omega_1$, the set $\lambda_\alpha$ is analytic.

**Lemma 16.** If $\alpha < \omega_1$, and $B$ is a Borel subset of $H$ containing $\lambda_\alpha$, then $\pi(H \setminus B)$ is contained in a member of $\hat{\Theta}(\Gamma)$. Proof. By induction on $\alpha < \omega_1$. If $\alpha = 0$, the statement is obvious. Let the statement be true for $\alpha < \omega_1$ and let $B$ be a Borel subset of $H$ containing $\lambda_\alpha = D(\lambda_\alpha)$. By Lemma 14, we obtain $D^{+} \in \hat{\Theta}(\Gamma)$, with $\pi(H \setminus B) \subseteq D^{+}$. Hence $H$ is contained in $B \cup \pi^{-1}(D') \cap H_i$, which is still Borel. By induction hypothesis, there is $D'' \in \hat{\Theta}(\Gamma)$, with $D'' \subseteq \pi(H \setminus B \cup \pi^{-1}(\Gamma))$.

Clearly $\pi(H \setminus B) \subseteq D'' \cup D'$. Finally let $\gamma$ be a limit ordinal and $(\lambda_\alpha)$ an increasing sequence of ordinals converging to $\gamma$ and satisfying the lemma. If $B$ is a Borel set containing $\lambda_\alpha$, then $\lambda \cap (H \setminus B) = \emptyset$. Thus there is a sequence $(B_\alpha)$ of Borel sets such that $\lambda \cap B = B_\alpha \cap B = \emptyset$. Let $n \in \mathbb{N}$ be fixed. Since $\lambda \cap B = \emptyset$, we obtain $D_\alpha \subseteq \hat{\Theta}(\Gamma)$, so that $\pi(H \setminus (B \cup B_\alpha)) \subseteq D \alpha$.

If we take $D = \bigcup_{\alpha < \omega_1} D_\alpha$, we get $\pi(H \setminus B) \subseteq D$, completing the proof.

**Lemma 17.** There exists $\alpha < \omega_1$ such that $\lambda_\alpha = \emptyset$.

Proof. Assume $\lambda_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. Then $A$ is contained in a member of $\hat{\Theta}(\Gamma)$. Thus it remains to prove:

**Lemma 18.** Let $A$ be an analytic subset of $E \times F$. Then for each $\alpha < \omega_1$ and the set $A \in \hat{\Theta}(\Gamma)$, $E \times F \cap \lambda_\alpha = \emptyset$.

**Proof.** Assume $\lambda_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. We take $Q = (L \in \mathbb{F}(H) : \pi(L) = \emptyset)$, which is a Polish subspace of $\mathbb{F}(E \times F \times G)$. For $k \in \mathbb{N}$ and $\alpha < \omega_1$, define $\mathcal{Q}_\alpha = \{(L_1, \ldots, L_k) \in Q^k : L_1 \cap H \cap D(L_{k+1} \cap H) \text{ if } 1 \leq k \leq k \}$ and $L_i \in H \cap H_i$. We verify that the conditions of Lemma 5 are satisfied.

1. We show that $Z = \{(L, M) \in Q^2 : L \cap H \cap D(M \cap H) \text{ is analytic in } Q^2\}$.

Let $(U_i)$ be a countable base for the topology of $E \times F \times G$. For each $i \in \mathbb{N}$, consider $Z_i = \{(L, M) : L \cap U_i = \emptyset \} \cup \{(M \in Q : \pi(M \cap H_i) \cap E)\}$, which is easily seen to be analytic. Therefore $Z_i$ is analytic, since $Z_i = \bigcup Z_i$.

2. This follows immediately from the fact that $(H_{\lambda_\alpha})_{\alpha < \gamma}$ is decreasing.

3. Suppose $k \in \mathbb{N}$, $\alpha < \omega_1$, and $(L_1, \ldots, L_k) \in \mathcal{Q}_\alpha$. Then $L_\alpha \in H \cap H \cap D(L_{\alpha+1} \cap H)$, and $L \cap H \cap D(L_{\alpha+1} \cap H)$ belongs to $Q$ and $(L_1, \ldots, L_k, L_{\alpha+1}) \in \mathcal{Q}_{\alpha+1}$, since $L_{\alpha+1} \cap H \cap D(L_{\alpha+1} \cap H)$. Thus the lemma applies. For each $\alpha < \omega_1$, the set $\mathcal{Q}_\alpha = \emptyset$, because $\pi(x \cap H \cap D(L_{\alpha+1} \cap H)$ whenever $H \cap D(L_{\alpha+1} \cap H)$. Hence there exists a sequence $(\lambda_\alpha)$ in $Q$ so that $L_\alpha, L_{\alpha+1} \subseteq \bigcup \mathcal{Z}_i$ for each $\alpha < \omega_1$. In particular there is $x \in E$ with $\pi(L_\alpha) = \emptyset$ for all $\alpha < \gamma$. If $(F_\alpha)$ is a sequence of closed sets not belonging to $\Gamma$, and covering $A \cap H \cap D(L_{\alpha+1} \cap H) \subseteq D$, we obtain that $\pi(x \cap H \cap D(L_{\alpha+1} \cap H)$. Let $L = \bigcup L_\alpha$, which belongs to $\mathcal{Q}(H)$. By the Baire category theorem, there is a set $\alpha \in N$ such that $\lambda \cap U_\alpha = \emptyset$.

Thus there is $\alpha < \omega_1$ with $L_\alpha \cap U_\alpha = \emptyset$ and therefore $\pi(L_\alpha, L_{\alpha+1} \cap H) = \bigcup \mathcal{Z}_i$, for which is the required (contradiction).

**F_{\alpha+1}-sections of Borel sets.** Let again $E$, $F$ be compact metric spaces. In [3], we obtained the following result:

**Theorem 4.** If $A$ and $B$ are analytic subsets of $E \times F$, then $\{x \in E : A \cap B \in \mathcal{D}(x)\}$ is contained in an $F_{\alpha+1}$-set which is disjoint from $B(x)$ is coanalytic in $E$.

In this section, we will prove:

**Theorem 5.** Let $A$ and $B$ be analytic subsets of $E \times F$ such that $A \cap B \in \mathcal{D}(x)$ is disjoint from $B(x)$, for all $x \in E$. Then $A$ can be separated from $B$ by a member of $\hat{\Theta}(\Gamma)$.

It clearly implies Theorem 2.

Assume $A$ and $B$ are analytic subsets of $E \times F$. Let $L = \bigcup L_{\alpha+1}$ be an analytic representation of $B$, where the $L_{\alpha+1}$ are closed in $E \times F$ and $B_{\alpha+1} \subseteq L_{\alpha+1}$. By induction on $\alpha < \omega_1$, we define for each $\alpha \in \mathbb{N}$ class $\mathcal{D}(\alpha)$ of subsets of $E \times F$, by taking:

$$\mathcal{D}(\alpha) = \{D \subseteq E \times F : D \cap B_{\alpha} = \emptyset\}$$

$$\mathcal{D}(\alpha) = \{D \subseteq E \times F : \text{for each } \alpha \in \mathbb{N} \text{ there is a countable closed covering } C_{\alpha}, \text{ of } D \text{ such that } D \cap C_{\alpha} \in \mathcal{D}(\alpha) \text{ for each } \alpha \in \mathbb{N}\}$$

**Lemma 18.** Let $A$ be an analytic subset of $E \times F$. Then for each $\alpha < \omega_1$ and the set $\Gamma(x) = \{L \subseteq \mathbb{F}(E) : A \cap L \in \mathcal{D}(\alpha)\}$ is a member of $\hat{\Theta}(\Gamma)$.
Proof. We proceed by induction on $\alpha < \omega_1$. Let $(O_\alpha)_\alpha$ be a decreasing sequence of open sets containing $B_\alpha$ such that $B_\alpha = \bigcap_\alpha O_\alpha$. Then
\[\Gamma^*(\alpha)(A) = \{ L \in \mathcal{E}(E \times F); A \cap L \cap B_\alpha \neq \emptyset \} = \bigcap_{\alpha < \beta} \{ L \in \mathcal{E}(E \times F); L \cap A \cap O_\beta \neq \emptyset \}\]
and hence $\mathcal{P}$-analytic by Lemma 6. Let now the property be established for all $\alpha < \beta$. Using the definition of $\mathcal{P}$-analytic, we obtain that $\Gamma^*(\alpha)(A) = \bigcup_{\alpha < \beta} \{ L \in \mathcal{E}(E \times F); A \cap L \cap O_\beta \neq \emptyset \}$ cannot be covered by countably many closed sets not belonging to $\bigcap_{\alpha < \beta} \Gamma^*(\alpha)(A)$. Since by induction hypothesis $\bigcap_{\alpha < \beta} \Gamma^*(\alpha)(A) \in \mathcal{P}$, it follows from Lemma 10 that also $\Gamma^*(\alpha)(A) \in \mathcal{P}$. This completes the proof.

Corollary 19. If $A$ is analytic in $E \times F$, then for each $\alpha < \omega_1$ and $\alpha \in \mathcal{P}$ the set \[\{ x \in E; \{ x \times A(x) \in \mathcal{P}(E) \} \} \text{ is coanalytic.}\]
Proof. It is the same as that of Corollary 11.

For each $\alpha \in \mathcal{P}$, take $B(\alpha) = \bigcup_{\alpha < \beta} \bigcap_{\alpha < \beta} B_{\beta \beta}$. We pass to the first step in the proof of Theorem 5.

Lemma 20. Let $A$ be an analytic subset of $E \times F$ and assume that there exist $\alpha < \omega_1$ and $\alpha \in \mathcal{P}$ such that $\{ x \times A(x) \in \mathcal{P}(E) \}$ for each $x \in E$. Then $A$ can be separated from $B(\alpha)$ by a member of $C_{\alpha\beta}$.

Proof. If $\alpha = 0$, then $\{ x \times A(x) \in \mathcal{P}(E) \}$ for all $x \in E$. Hence, by Lemma 4, there exists a set $D \in \mathcal{P}$ so that $A \in D$ and $D \cap B_\beta = \emptyset$. Let the property be true for all $x < \beta$ and assume \(\{ x \times A(x) \in \mathcal{P}(E) \}\) for each $x \in E$. Take $\alpha \in N$ fixed. Then the set \(\{ x \times A(x) \in \mathcal{P}(E) \}\) can be covered by countably many closed sets (of $F_\beta$) with \(A \cap F_\beta \in \mathcal{P}(E)\). Since by Lemma 18, we obtain by Theorem 3 a sequence $(\delta_\alpha)_\alpha$ in $\mathcal{P}(F_\alpha)$ with $A = \bigcup \delta_\alpha$. Let $r \in N$ be also fixed. Since for each $x \in E$, the set \(\{ x \times A(x) \in \mathcal{P}(E) \}\), there exists $\alpha < \beta$ such that \(\{ x \times A(x) \in \mathcal{P}(E) \}\). Hence, using Corollary 19, the sets $C(r, p, a) = \{ x \in E; \{ x \times (A \cap \{ A(x) \in \mathcal{P}(E) \}) \} \in \mathcal{P}(\pi, p) \}$ are coanalytic and they cover $E$. Therefore there is a sequence $(F_\alpha(p, a))_{\alpha \in \alpha}$ of disjoint Borel sets satisfying $(F_\alpha(p, a)) \in C(r, p, a)$ and $E = \bigcup (F_\alpha(p, a))$. For each $\alpha < \beta$, we introduce the set $A_{\alpha\beta} = A \cap \bigcup (F_\alpha(p, a)) \times \mathcal{E}(E \times F)$, which is still analytic. Because $\{ x \times A_{\alpha\beta}(x) \in \mathcal{P}(\pi, p) \}$ for each $x \in E$, the induction hypothesis applies. Thus we obtain a member $D_{\alpha\beta}$ of $C_{\alpha\beta}$ separating $A_{\alpha\beta}$ from $B(\alpha)$.

If we define $D_\alpha = \bigcup_{\alpha \in \alpha} B_{\alpha\beta}$ and $D_\alpha = B(\alpha)$. Then the set $D_\alpha = \bigcup_{\alpha \in \alpha} \bigcap_{\alpha \in \alpha} \bigcup (E \times F) \cap (D_{\alpha\beta}) \cap B(\alpha)$ belongs to $\mathcal{P}_{\alpha\beta}$, $D_\alpha \supseteq A$ and $D_\alpha \cap B(\pi, p) = \emptyset$. We only have to take $D = \bigcup_{\alpha \in \alpha} D_\alpha$. The proof of Theorem 5 will be complete if the following property holds:

Lemma 21. If $A$ is an analytic subset of $E \times F$, then one of the following 2 alternatives must occur:

1. There exists $\alpha < \omega_1$ such that \(\{ x \times A(x) \in \mathcal{P}(E) \}\) for all $x \in E$.
2. There exists $x \in E$ such that no $F_\alpha$-subset of $F$ separates $A(x)$ from $B(x)$.

Proof. There is a compact metric space $G$ and a $G$-subset $H$ of $E \times F \times G$ so that $A = \pi(H)$, where $\pi: E \times F \times G \to E \times F$ is the projection. Take a countable base $(U_n)$ for the topology of $E \times F \times G$. Let $(\alpha_n)$ be the standard ordering of $\mathcal{P}$. We will again make use of Lemma 5. We introduce for each $k \in N$ and $\alpha < \omega_1$, a subset $\mathcal{P}_{\alpha\beta}$ of $\mathcal{P}(E \times F)\times\pi$:

\[\mathcal{P}_{\alpha\beta}\]

consists of the elements $(x, p_n) \in \mathcal{P}(E \times F)\times\pi$ such that:

1. $K_\alpha \neq \emptyset$.
2. There is some $x \in E$ with $K_\alpha \subseteq \{ x \times \pi \times \pi \}$.
3. If $U$ is open in $E \times F \times G$ and $U \cap K_\alpha \neq \emptyset$, then

\[
\{ x \in E \times F \times G; \exists (x, p_n) \in \mathcal{P}(E \times F)\times\pi \}
\]

We claim that the conditions 1, 2, 3 of Lemma 5 are satisfied:

1. In fact $\mathcal{P}_{\alpha\beta}$ is closed in $\mathcal{P}(E \times F)\times\pi$.
2. It is obviously satisfied.

Assume $\alpha = (\alpha_n, K_\alpha)$ such that an element of $\mathcal{P}(E \times F)\times\pi$ exists for each $\varphi \in \mathcal{P}_{\alpha\beta}$ and $\alpha_n \neq \alpha$. Let $\alpha_n \neq \alpha$. Then $\mathcal{P}_{\alpha\beta}$ is a countable covering of $\mathcal{P}(E \times F)\times\pi$. If $U_\alpha \cap K_\alpha = \emptyset$, then $\mathcal{P}(E \times F)\times\pi$ is given by $\varphi \in \mathcal{P}(E \times F)\times\pi$, where $\varphi \neq \varphi_n$. Clearly $\varphi \in \mathcal{P}_{\alpha\beta}$. Assume now $U_\alpha \cap K_\alpha \neq \emptyset$. Because $\alpha \varphi \in \mathcal{P}_{\alpha\beta}$ and we get $\mathcal{P}(U_\alpha \cap K_\alpha) \subseteq \mathcal{P}(E \times F)\times\pi$. Therefore there must be some $\pi \in N$ such that there is no countable closed covering $(F_\alpha(p_n))$ of $\check{\mathcal{P}}(U_\alpha \cap K_\alpha) \subseteq \mathcal{P}(E \times F)\times\pi$. For all $r \in N$. Remark that $\pi(U_\alpha \cap K_\alpha)$ is the image of $\pi^{-1}(\mathcal{P}(U_\alpha \cap K_\alpha)) \subseteq \mathcal{P}(E \times F)\times\pi$. A standard argument yields a nonempty closed subset $Y$ of $\pi(U_\alpha \cap K_\alpha) \subseteq \mathcal{P}(E \times F)\times\pi$ such that $\pi(U \cap Y) \subseteq \pi(U_\alpha \cap K_\alpha) \subseteq \mathcal{P}(E \times F)\times\pi$. 

\[\pi(U \cap Y) \subseteq \pi(U_\alpha \cap K_\alpha) \subseteq \mathcal{P}(E \times F)\times\pi\]

whenever $U$ is open in $E \times F \times G$ and $U \cap Y \neq \emptyset$. It is clear that $Y \notin F(H)$. Let

$$r \in (N \times F(H))^{m+1}$$

be given by $i^k = \sigma$ and $i_{k+1} = (x, Y)$, which is in $S_{m+1}$.

Thus Lemma 5 applies and we have to distinguish 2 cases:

Case I. There is $\eta < \omega_1$ with $S_{\eta+1} = \emptyset$.

Take $x = n + 1$ and assume the existence of $x \in E$ with $[x] \times A(x) \notin F(p)$. Since

$$[x] \times A(x) = \pi((x \times F \times G) \cap H),$$

an integer $p_n$ and a nonempty closed subset $Y$ of $((x \times F \times G) \cap H)$ can be found with $(x \times A(x)) \cap \pi(Y \cap U) \notin F(p_n)$ whenever $U$ is open in $E \times F \times G$ and $U \cap Y \neq \emptyset$. It follows that $(p_n, Y) \in S_{\eta+1}$, a contradiction.

Case II. There is $\xi \in (N \times F(H))^{\omega}$ such that $\xi(k) \notin S_k$ for each $k \in N$.

Let $\xi = (p_n, K_n)_{n \in N}$ and take $x \in E$ such that $K_n = [x] \times F \times G$. It is clear that $\xi$ satisfies the following properties:

1. $\forall e \in \mathcal{E} : K_e \subset [x] \times F \times G$.
2. $K_n \neq \emptyset$.
3. $\forall e \in \mathcal{E}, \forall n \in N : \pi(K_n) = \pi(K_e \cap U)$.
4. $\forall e \in \mathcal{E}, \forall n \in N : U_e \cap K_e \neq \emptyset \Rightarrow B_{p_1, \ldots, p_n} \cap \pi(K_n) \neq \emptyset$.

It is shown in [3] that under this hypothesis, no $F_{\omega}$-subset of $F$ separates $A(x)$ from $B(x)$.

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References