from a planar continuum X onto Y such that $f^{-1}(y)$ is decomposable for each $y \in Y$. Must Y be (hereditarily) locally connected?

One can prove that Y is hereditarily decomposable. It would be interesting to know what is a characterization of the continua Y in terms of intrinsic properties.

Added in proof. The answer to the problem is affirmative: E. Dyer, *Continuous collections of decomposable continua*, Proc. Amer. Math. Soc. 6 (1955), pp. 351–360. Moreover, one can prove that Y must be regular.

References

[1] R. D. Anderson, *Open mappings of compact continua*, Proc. Nat. Acad. Sci. USA 42 (1956), pp. 347–349.

[2] E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. Math. 51 (1950), pp. 534–543.

[3] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653–663.

[4] — and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc. 90 (1959), pp. 171–192.

[5] E. L. Bethel, *A note on a reducible continuum*, Proc. Amer. Math. Soc. 16 (1965), pp. 1331–1333.

[6] J. H. Case and R. E. Chamberlin, *Chracterizations of tree-like continua*, Pacific J. Math. 10 (1960), pp. 73–84.

[7] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. 68 (1970), pp. 18–22.

[8] E. Dyer, *Irreducibility of the sum of the elements of a continuous collection of continua*, Duke Math. J. 20 (1953), pp. 589–592.

[9] — *Certain transformations which lower dimension*, Ann. Math. 63 (1956), pp. 15–19.

[10] M. E. Hamstrom, *Concerning continuous collections of continuous curves*, Bull. Amer. Math. Soc. 58 (1952), p. 204.

[11] J. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52 (1942), pp. 22–36.

[12] B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fund. Math. 25 (1935), pp. 568–577.

[13] J. Krasinkiewicz and P. Minc, *Continua and their open subsets with connected complements*, Fund. Math. 102 (1979), pp. 129–136.

[14] K. Kuratowski, *Sur le problème des curbes gauches en Topologie*, Fund. Math. 15 (1930), pp. 271–283.

[15] — *Topology*, Vol. 2, New York–London–Warszawa 1968.

[16] S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math. 25 (1935), pp. 327–328.

[17] E. Moise, *A theorem on monotone interior transformations*, Bull. Amer. Math. Soc. 55 (1949), pp. 810–811.

[18] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. 45 (1958), pp. 320–324.

[19] D. C. Wilson, *Open mappings of the universal curve onto continuous curves*, Trans. Amer. Math. Soc. 168 (1972), pp. 497–515.

Accepté par la Rédaction le 21. 9. 1977

$F_{\sigma\delta}$-sections of Borel sets

by

J. Bourgain (*) (Brussel)

Abstract. It is shown that if E, F are compact metric spaces and A is a Borel subset of $E \times F$, then $\{x \in E\colon A(x) \text{ is } F_{\sigma\delta} \text{ in } F\}$ is coanalytic in E.

Introduction. Throughout this paper, E and F are compact metric spaces. If A is a subset of $E \times F$ and $x \in E$, let $A(x) = \{y \in F;\ (x, y) \in A\}$, which is called a *section of* A. It is already known that if A is Borel in $E \times F$, then $\{x \in E;\ A(x)$ is closed in $F\}$ and $\{x \in E;\ A(x)$ is F_σ in $F\}$ are coanalytic. I refer for instance to [1] and [4]. It follows from a result in my recent paper [2] that the set $\{x \in E;\ A(x)$ is $F_{\sigma\delta}$ in $F\}$ is a universally measurable subset of E. We will obtain here the following refinement:

THEOREM 1. *If A is Borel in $E \times F$, then $\{x \in E;\ A(x)$ is $F_{\sigma\delta}$ in $F\}$ is coanalytic in E.*

The main point in the proof of this result is a useful description of the fact that a set in F is $F_{\sigma\delta}$.

If L is a compact metric space, then $\underline{\underline{K}}(L)$ consists of all closed subsets of L and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [4]).

LEMMA 2. *Let P be a Polish subspace of the compact metric space L. Then the subspace $\underline{\underline{F}}(P)$ of $\underline{\underline{K}}(L)$ consisting of those compact sets K in L such that $K = \overline{K \cap P}$, is Polish.*

We denote by $\mathscr{R}$ the set of all finite complexes c in $\bigcup_k N^k$, where $N^0 = \{\emptyset\}$.

PROPOSITION 3. *Let A be Borel in F and $B = F \setminus A$. There is a compact metric space G and a G_δ subset H of $F \times G$ so that $A = \pi(H)$, if $\pi\colon F \times G \to F$ is the projection. Let $B = \bigcup_\nu \bigcap_k B_{\nu|k}$ be an analytical representation of B, where the $B_{\nu|k}$ are closed in F and $B_{\nu|k+1} \subset B_{\nu|k}$. Take a countable base $(U_n)_n$ for the topology of $F \times G$.*

Then A is not $F_{\sigma\delta}$ in F if and only if there exists $(p_c, K_c)_{c \in \mathscr{R}}$ in $\prod_{c \in \mathscr{R}} (N \times \underline{\underline{F}}(H))$ satisfying:

(*) Aspirant, N. F. W. O., Belgium.

1. $K_\varphi \neq \emptyset$,

2. $\forall c \in \mathcal{R}$, $\forall n \in N$: $\pi(K_{cn}) \subset \pi(K_c \cap \overline{U}_n)$,

3. $\forall c \in \mathcal{R}$, $\forall n \in N$: $U_n \cap K_c \neq \emptyset \Rightarrow B_{p_\varphi, \dots, p_c, p_{cn}} \cap \pi(K_{cn}) \neq \emptyset$.

The proof of this proposition is rather technical. Let us first show how to derive Theorem 1.

Proof of Theorem 1. It is clear that E can be assumed o-dimensional. Let A be Borel in $E\times F$ and $B = (E\times F)\setminus A$. There is a compact metric space G and a G_δ subset H of $E\times F\times G$ so that $A = \pi(H)$, if π: $E\times F\times G \to E\times F$ is the projection. Let $B = \bigcup_\nu \bigcap_k B_{\nu|k}$ be an analytic representation of B, where the $B_{\nu|k}$ are closed in $E\times F$ and $B_{\nu|k+1} \subset B_{\nu|k}$. Since E is o-dimensional, there is a countable base $(U_n)_n$ for the topology of $E\times F\times G$ such that $\overline{U}_n(x) = \overline{U_n(x)}$ whenever $x \in E$. If $x \in E$, then it follows from Proposition 3 that $A(x)$ is not $F_{\sigma\delta}$ in F if and only if there exists $(p_c, K_c)_{c\in\mathcal{R}}$ in $\prod_{c\in\mathcal{R}} (N\times \underline{\underline{F}}(H))$ verifying the following conditions:

1. $\forall c \in \mathcal{R}$: $K_c \subset \{x\}\times F\times G$,

2. $K_\varphi \neq \emptyset$,

3. $\forall c \in \mathcal{R}$, $\forall n \in N$: $\pi(K_{cn}) \subset \pi(K_c \cap \overline{U}_n)$,

4. $\forall c \in \mathcal{R}$, $\forall n \in N$: $U_n \cap K_c \neq \emptyset \Rightarrow B_{p_\varphi, \dots, p_c, p_{cn}}(x) \cap \pi(K_{cn})(x) \neq \emptyset$.

Remark that $\Omega = \prod_c (N\times \underline{\underline{F}}(H))$ is Polish.

To obtain that $\{x \in E;\ A(x) \text{ is not } F_{\sigma\delta}\}$ is analytic in E, it is enough to prove that the subset of $E\times\Omega$ consisting of those elements $(x, (p_c, K_c)_c)$ satisfying conditions (1), (2), (3), (4) above is analytic in $E\times\Omega$. The reader will easily verify that this set is in fact closed. So the proof is complete.

Thus it remains to prove Proposition 3. We introduce some notations. If $c \in N^k$, let $B(c) = \bigcup_{\nu|k=c} \bigcap_k B_{\nu|k}$. Suppose $c \in \mathcal{R}$ and X closed in H, then $[c, X]$ will mean that there is no $F_{\sigma\delta}$-set P satisfying $\pi(\overline{X}) \cap A \subset P$ and $B(c) \cap P = \emptyset$. The following lemma is straightforward.

Lemma 4. *If $c \in \mathcal{R}$ and X closed in H satisfy $[c, X]$, then there exists $p \in N$ such that $[(c, p), X]$.*

We also need the following:

Lemma 5. *Let $c \in \mathcal{R}$ and X closed in H with $[c, X]$. Then there exists a nonempty closed subset Y of H with $\pi(Y) \subset \pi(\overline{X})$, so that $[c, Y\cap \overline{U}]$ whenever U is open and $U \cap Y \neq \emptyset$.*

Proof. If the claim is untrue, then for every nonempty closed subset Y of $H\cap\pi^{-1}(\pi(\overline{X}))$ there is an open set U of $F\times G$ such that $U\cap Y \neq \emptyset$ and $[c, Y\cap\overline{U}]$ does not hold. A standard construction yields us then a countable closed covering $(Y_n)_n$ of $H\cap\pi^{-1}(\pi(\overline{X}))$ so that $[c, Y_n]$ does not hold for each n. Hence there is

a sequence $(P_n)_n$ of $F_{\sigma\delta}$-subsets of F such that $\pi(\overline{Y}_n) \cap A \subset P_n$ and $B(c) \cap P_n = \emptyset$. Clearly the set

$$P = \bigcup_n \pi(\overline{Y}_n) \cap \bigcap_n (\pi(\overline{Y}_n)^c \cup P_n)$$

is still $F_{\sigma\delta}$ in F. Furthermore $B(c) \cap P = \emptyset$ and $\pi(\overline{X}) \cap A \subset \bigcup_n \pi(\overline{Y}_n) \cap A \subset P$, which contradicts $[c, X]$.

Lemma 6. *Assume A not $F_{\sigma\delta}$. Then for each $c \in \mathcal{R}$ we can define $P_c \in N$ and $K_c \in \underline{\underline{F}}(H)$, verifying:*

1. $K_\varphi \neq \emptyset$,

2. $\pi(K_{cn}) \subset \pi(K_c \cap \overline{U}_n)$,

3. $U_n \cap K_c \neq \emptyset \Rightarrow K_{cn} \neq \emptyset$,

4. $U_n \cap K_c \neq \emptyset \Rightarrow [(p_\varphi, \dots, p_c), K_c \cap \overline{U}_n \cap H]$.

Proof. The construction will be made by induction on the length of c.

Since A is not $F_{\sigma\delta}$, we have $[\emptyset, H]$. By successive application of Lemma 4 and Lemma 5 we find some $p_\varphi \in N$ and a nonempty closed subset Y of H, so that $[p_\varphi, Y\cap\overline{U}]$ if U is open and $U\cap Y \neq \emptyset$. Take $K_\varphi = \overline{Y} \in \underline{\underline{F}}(H)$. If $U_n \cap K_\varphi \neq \emptyset$, then also $U_n \cap Y \neq \emptyset$ and thus $[P_\varphi, K_\varphi \cap H \cap \overline{U}_n]$.

Assume now $p_c \in N$ and $K_c \in \underline{\underline{F}}(H)$ obtained for all $c \in \mathcal{R}$ with length at most k. Let $c \in N^k$ and $n \in N$ fixed. If $U_n \cap K_c = \emptyset$, take $p_{cn} \in N$ arbitrarily and $K_{cn} = \emptyset$. If $U_n \cap K_c \neq \emptyset$, then $[(p_\varphi, \dots, p_c), K_c \cap \overline{U}_n \cap H]$ holds.

Again by successive application of Lemmas 4 and 5 we find some $p_{cn} \in N$ and a nonempty closed subset Y of H with $\pi(Y) \subset \pi(K_c \cap \overline{U}_n)$, so that $[(p_\varphi, \dots, p_c, p_{cn}), Y\cap\overline{U}]$ whenever U is open and $U\cap Y \neq \emptyset$. Take $K_{cn} = \overline{Y} \in \underline{\underline{F}}(H)$. If $U_r \cap K_{cn} \neq \emptyset$, then $U_r \cap Y \neq \emptyset$ and thus $[(p_\varphi, \dots, p_c, p_{cn}), K_{cn} \cap H \cap \overline{U}_r]$. This completes the construction.

We are now able to prove the first part of Proposition 3. Assume A is not $F_{\sigma\delta}$ and let $(p_c, K_c)_c$ be as in Lemma 6. We only have to verify condition 3. If $U_n \cap K_c \neq \emptyset$, then $K_{cn} \neq \emptyset$ and therefore there is some $r \in N$ with $U_r \cap K_{cn} \neq \emptyset$. Hence $[(p_\varphi, \dots, p_c, p_{cn}), K_{cn} \cap \overline{U}_r \cap H]$. In particular, we have that $B_{p_\varphi, \dots, p_c, p_{cn}} \cap \pi(K_{cn}) \neq \emptyset$.

Finally, we pass to the proof of the second part of Proposition 3. Assume $A = \bigcap_k \bigcup_l F_{kl}$ with each F_{kl} closed in F. We will show that the assumption of the existence of $(p_c, K_c)_c$ in $\prod_c (N\times\underline{\underline{F}}(H))$ satisfying 1, 2, 3 leads to a contradiction. By induction we define sequences $(l_k)_k$ and $(n_k)_k$, verifying following properties:

1. $K_{n_1, \dots, n_k} \neq \emptyset$,

2. $U_{n_1} \cap K_\varphi \neq \emptyset$,

3. $U_{n_{k+1}} \cap K_{n_1, \dots, n_k} \neq \emptyset$,

4. $\pi(K_\varphi \cap \overline{U}_{n_1}) \subset F_{1 l_1}$,

5. $\pi(K_{n_1, \dots, n_k} \cap \overline{U}_{n_{k+1}}) \subset F_{k+1, l_{k+1}}$.

Since $A\subset\bigcup_l F_{1l}$, we have $K_\varphi\cap H=\bigcup_l (K_\varphi\cap H\cap\pi^{-1}(F_{1l}))$. Because $K_\varphi\cap H$ is a nonempty G_δ subset of $F\times G$, there exist $l_1\in N$, $n_1\in N$ and U open with $\overline{U}_{n_1}\subset U$, $K_\varphi\cap H\cap U_{n_1}\neq\emptyset$ and $K_\varphi\cap H\cap U\subset\pi^{-1}(F_{1l_1})$.

Since $K_\varphi\cap U_{n_1}\neq\emptyset$, we have $K_{n_1}\neq\emptyset$. Clearly $K_\varphi\cap U\subset\pi^{-1}(F_{1l_1})$ and thus $K_\varphi\cap\overline{U}_{n_1}\subset\pi^{-1}(F_{1l_1})$.

Assume $l_1, n_1, \ldots, l_k, n_k$ obtained. Since $A\subset\bigcup_l F_{k+1,l}$, we have $K_{n_1,\ldots,n_k}\cap H=\bigcup_l (K_{n_1,\ldots,n_k}\cap H\cap\pi^{-1}(F_{k+1,l}))$. Again $K_{n_1,\ldots,n_k}\cap H$ is a nonempty G_δ in $F\times G$ and therefore there exist $l_{k+1}\in N$, $n_{k+1}\in N$ and U open, such that

$$\overline{U}_{n_{k+1}}\subset U,\ K_{n_1,\ldots,n_k}\cap H\cap U_{n_{k+1}}\neq\emptyset$$

and

$$K_{n_1,\ldots,n_k}\cap H\cap U\subset\pi^{-1}(F_{k+1,l_{k+1}}).$$

Because $K_{n_1,\ldots,n_k}\cap U_{n_{k+1}}\neq\emptyset$, we have $K_{n_1,\ldots,n_k,n_{k+1}}\neq\emptyset$. Furthermore

$$K_{n_1,\ldots,n_k}\cap U\subset\pi^{-1}(F_{k+1,l_{k+1}})$$

and hence

$$K_{n_1,\ldots,n_k}\cap\overline{U}_{n_{k+1}}\subset\pi^{-1}(F_{k+1,l_{k+1}}).$$

This completes the construction.

Take $v=(n_k)_k$. Since $\pi(K_{v|k+1})\subset\pi(K_{v|k}\cap\overline{U}_{n_{k+1}})$, we have that $\pi(K_{v|k+1})\subset F_{k+1,l_{k+1}}$. Because $U_{n_{k+1}}\cap K_{v|k}\neq\emptyset$, it follows that $B_{p_\varphi,\ldots,p_{v|k},p_{v|k+1}}\cap\pi(K_{v|k+1})\neq\emptyset$. Therefore we obtain for each k that

$$\begin{aligned}F_{1l_1}\cap\ldots\cap F_{k+1,l_{k+1}}\cap B_{p_\varphi,\ldots,p_{v|k},p_{v|k+1}} &\supset\pi(K_{v|1})\cap\ldots\cap\pi(K_{v|k+1})\cap B_{p_\varphi,\ldots,p_{v|k},p_{v|k+1}}\\ &=\pi(K_{v|k+1})\cap B_{p,\ldots,p_{v|k},p_{v|k+1}}\neq\emptyset.\end{aligned}$$

By the compactness of F, we conclude that $\bigcap_k F_{kl_k}\cap\bigcap_k B_{p_{v|0},\ldots,p_{v|k}}\neq\emptyset$. But $\bigcap_k F_{kl_k}\subset A$ and $\bigcap_l B_{p_{v|0},\ldots,p_{v|k}}\subset B$, contradicting the fact that $B=F\setminus A$.

Thus Proposition 3 is established.

References

[1] V. Ja. Arsenin, A. A. Liapunov and E. A. Cegolkov, *Arbeiten zur deskriptiven Mengenlehre*, Mathematische Forschungsberichte, Berlin 1955.

[2] J. Bourgain, *Decompositions in the product of a measure space and a Polish space*, Fund. Math. 105 (1979), pp. 61–71.

[3] J. P. R. Christensen, *Borel structures*, Notes in Math. 10 (1974).

[4] C. Dellacherte. *Ensembles analytiques: Théorèmes de séparation et applications*, Séminaire de Probabilités IX, Berlin–Heidelberg–New York 1975.

[5] F. Hausdorff, *Set Theory*, New York 1962.

[6] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.

[7] W. Sierpiński, *Les ensembles projectifs et analytiques*, Mémorial des Sciences Mathématiques, fasc. 112, Paris 1950.

VRIJE UNIVERSITEIT BRUSSEL

Accepté par la Rédaction le 22. 9. 1977