from a planar continuum $X$ onto $Y$ such that $f^{-1}(y)$ is decomposable for each $y \in Y$. Must $Y$ be (hereditarily) locally connected?

One can prove that $Y$ is hereditarily decomposable. It would be interesting to know what is a characterization of the continua $Y$ in terms of intrinsic properties.

Added in proof. The answer to the problem is affirmative: E. Dyer, Continuous collection of decomposable continua, Proc. Amer. Math. Soc. 6 (1955), pp. 351-360. Moreover, one can prove that $Y$ must be regular.

References


F$_c$-sections of Borel sets

by

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Abstract. It is shown that if $E$, $F$ are compact metric spaces and $A$ is a Borel subset of $E \times F$, then $(x \times F; A(x) = F_{x}$ in $F)$ is coanalytic in $E$.

Introduction. Throughout this paper, $E$ and $F$ are compact metric spaces. If $A$ is a set that contains $E \times F$ and $x \in E$, let $A(x) = \{ y \in F : (x,y) \in A \}$, which is called a section of $A$. It is already known that if $A$ is Borel in $E \times F$, then $(x \times E; A(x) = F_{x}$ in $F)$ is closed in $F$ and $(x \in E; A(x) = F_{x}$ in $F)$ is coanalytic. I refer for instance to [1] and [4]. It follows from a result in my recent paper [2] that the set $(x \in E; A(x) = F_{x}$ in $F)$ is a universally measurable subset of $E$. We will obtain here the following refinement:

THEOREM 1. If $A$ is Borel in $E \times F$, then $(x \in E; A(x) = F_{x}$ in $F)$ is coanalytic in $E$.

The main point in the proof of this result is a useful description of the fact that a set in $F$ is $F_{c}$.

If $L$ is a compact metric space, then $K(L)$ consists of all closed subsets of $L$ and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [4]).

LEMMA 2. Let $P$ be a Polish subspace of the compact metric space $L$. Then the subspace $F(P)$ of $K(L)$ consisting of those compact sets $K$ in $L$ such that $K = K \cap P$, is Polish.

We denote by $s$ the set of all finite complexes $c$ in $\bigcup N$, where $N = \{ 0 \}$.

PROPOSITION 3. Let $A$ be Borel in $E$ and $B = F \Delta A$. There is a compact metric space $G$ and a $G$ subset $H$ of $F$ such that $A = \pi(H)$, if $\pi : F \times G \to F$ is the projection. Let $B = \bigcup B_{i,0}$ be an analytical representation of $B$, where the $B_{i,0}$ are closed in $F$ and $B_{i,0+1} \subset B_{i,0}$. Take a countable base $U_{i,0}$ for the topology of $F \times G$.

Then $A$ is not $F_{c}$ in $F$ if and only if there exists $(p_{i}, K_{i}) \in \bigcap A_{i}$ satisfying:

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1. \( K_0 \neq \emptyset \).
2. \( \forall c \in \mathcal{A}, \forall n \in \mathbb{N}: \pi(K_{n_0}) \subset \pi(K_n \cap U_n) \).
3. \( \forall c \in \mathcal{A}, \forall n \in \mathbb{N}: U_n \cap K_c \neq \emptyset \Rightarrow B_{p_{n_0}, \ldots, p_{n_k}} \cap \pi(K_n) \neq \emptyset \).

The proof of this proposition is rather technical. Let us first show how to derive Theorem 1.

Proof of Theorem 1. It is clear that \( E \) can be assumed \( \sigma \)-dimensional. Let \( A \) be Borel in \( E \times F \) and \( B = (E \times F) \setminus A \). There is a compact metric space \( G \) and a \( G_2 \) subset \( H = E \times F \times G \) such that \( A = \pi(H) \), if \( \pi: E \times F \times G \rightarrow E \times F \) is the projection. Let \( B = \bigcup \bigcup B_{i,k} \) be an analytic representation of \( B \), where the \( B_{i,k} \) are closed in \( E \times F \times G \) and \( B_{i,k+1} \subset B_{i,k} \). Since \( E \) is \( \sigma \)-dimensional, there is a countable base \( \{U_n\}_n \) for the topology of \( E \times F \times G \) such that \( U_n(x) = U_n(x) \) whenever \( x \in E \). If \( x \in E \), then it follows from Proposition 3 that \( A(x) \) is not \( F_{\alpha} \) in \( F \) if and only if there exists \( (\rho, K_{n_0}) \in \prod \prod (N \times F(H)) \) verifying the following conditions:

1. \( \forall c \in \mathcal{A}, K_c \subset \{x \} \times F \times G \).
2. \( K_0 \neq \emptyset \).
3. \( \forall c \in \mathcal{A}, \forall n \in \mathbb{N}: \pi(K_{n_0}) \subset \pi(K_n \cap U_n) \).
4. \( \forall c \in \mathcal{A}, \forall n \in \mathbb{N}: U_n \cap K_c \neq \emptyset \Rightarrow B_{\rho_n, \ldots, \rho_{n_k}} \cap \pi(K_n) \neq \emptyset \).

Remark that \( \Omega = \prod (N \times F(H)) \) is Polish.

To obtain that \( (x \in E: A(x) \neq \text{not } F_{\alpha}) \) is analytic in \( E \), it is enough to prove that the subset of \( E \times F \) consisting of those elements \( (x, (\rho, K_n)) \) satisfying conditions (1), (2), (3), (4) above is analytic in \( E \times F \). The reader will easily verify that this set is in fact closed. So the proof is complete.

Thus it remains to prove Proposition 3. We introduce some notations. If \( c \in \mathcal{A} \), let \( B(c) = \bigcup \bigcup B_{i,k} \). Suppose \( c \in \mathcal{A} \) and \( X \) closed in \( H \), then \( [c, X] \) will denote the \( \sigma \)-algebra generated by \( \{c, X\} \) and \( \pi(X) \). We also need the following:

Lemma 4. If \( c \in \mathcal{A} \) and \( X \) closed in \( H \) satisfy \( [c, X] \), then there exists \( p \in \mathcal{N} \) such that \( [c, p] \subset X \).

We now prove the following:

Lemma 5. Let \( c \in \mathcal{A} \) and \( X \) closed in \( H \). Then there exists a nonempty closed subset \( Y \) of \( H \) with \( \pi(Y) = \pi(X) \), so that \( [c, Y \cap U] \) whenever \( U \) is open and \( U \cap Y \neq \emptyset \).

Proof. If the claim is true, then for every nonempty closed subset \( Y \) of \( H \cap \pi^{-1}(\pi(X)) \) there is an open set \( U \) of \( F \times G \) such that \( U \cap Y \neq \emptyset \) and \( [c, Y \cap U] \) does not hold. A standard construction yields us then a countable closed covering \( \bigcup_{\lambda} \{Y_{\lambda}\} \) of \( H \cap \pi^{-1}(\pi(X)) \) so that \( [c, Y] \) does not hold for each \( n \). Hence there is a sequence \( (p_\lambda) \subseteq F_{\alpha} \) of \( F_{\alpha} \)-sets of \( F \) such that \( \pi(Y_{\lambda}) \cap A \neq \emptyset \), and \( B(c) \cap A = \emptyset \). Clearly the set

\[ F = \bigcup \{\pi(Y_{\lambda}) \cap \{\pi(Y_{\lambda}) \cap \bigcup_{\lambda} B_{p_{\lambda}, \ldots, p_{\lambda_k}} \cap \pi(K_n) \neq \emptyset \} \}

is still \( F_{\alpha} \) in \( F \). Furthermore \( B(c) \cap F = \emptyset \) and \( \pi(X) \cap A \subset \bigcup \{\pi(Y_{\lambda}) \cap A \subset F \}

which contradicts \( [c, X] \).

Lemma 6. Assume \( A \neq F_{\alpha} \). Then for each \( c \in \mathcal{A} \) we can define \( p_n \in \mathcal{N} \) and \( K_n \in F(H) \) verifying:

1. \( K_0 \neq \emptyset \).
2. \( \pi(K_{n_0}) \subset \pi(K_n \cap U_n) \).
3. \( U_n \cap K_n \neq \emptyset \Rightarrow K_n \neq \emptyset \).
4. \( U_n \cap K_n \neq \emptyset \Rightarrow \{p_n, \ldots, p_k\} \subset K_n \cap U_n \).

Proof. The construction will be made by induction on the length of \( c \).

Since \( A \neq F_{\alpha} \), we have \( [0, \alpha] \). By successive application of Lemma 4 and Lemma 5 we find some \( p_n \in \mathcal{N} \) and some nonempty closed subset \( Y \) of \( H \), so that

\[ \{p_n, \ldots, p_k\} \subset \pi(K_n \cap U_n) \]

and \( \pi(Y) = \pi(X) \). If \( U \) is open and \( U \cap Y \neq \emptyset \), take \( K_n \in F(H) \). If \( U_n \cap K_n \neq \emptyset \), then \( [p_n, \ldots, p_k] \subset K_n \cap U_n \).

Again by successive application of Lemmas 4 and 5 we find some \( p_n \in \mathcal{N} \) and some nonempty closed subset \( Y \) of \( H \) with \( \pi(Y) = \pi(X) \). If \( U \) is open and \( U \cap Y \neq \emptyset \), take \( K_n \in F(H) \). If \( U_n \cap K_n \neq \emptyset \), then \( [p_n, \ldots, p_k] \subset K_n \cap U_n \).

It is easy to verify that this set is in fact closed. So the proof is complete.

We now prove the first part of Proposition 3. Assume \( A \neq F_{\alpha} \) and \( (p_n, K_n) \) be as in Lemma 6. We only have to verify condition 3. If \( U_n \cap K_n \neq \emptyset \), then \( K_n \neq \emptyset \) and accordingly there is some \( r \in \mathcal{N} \) with \( U_n \cap K_n = \emptyset \). Hence

\[ \{p_n, \ldots, p_k\} \cap U_n \cap K_n \subset \emptyset \].

In particular, we have that \( B_{p_n, \ldots, p_k} \cap \pi(K_n) \neq \emptyset \).

Finally, we pass to the proof of the second part of Proposition 3. Assume \( A \neq F_{\alpha} \) with each \( F \) closed in \( F \). We will show that the assumption of the existence of \( (p_n, K_n) \) in \( \prod (N \times F(H)) \) satisfying 1, 2, 3 leads to a contradiction.

By induction we define sequences \( (\alpha_n) \) and \( (\beta_n) \) verifying following properties:

1. \( K_0 \neq \emptyset \).
2. \( U_n \cap K_n \neq \emptyset \).
3. \( U_n \cap K_n \neq \emptyset \).
4. \( \pi(K_n \cap U_n) \subset F(H) \).
5. \( \pi(K_n \cap U_n) \subset K_n \cap U_n \).
Since \( A \subset \bigcup F_{ii} \), we have \( K_{0} \cap H = \bigcup (K_{0} \cap H \cap \pi^{-1}(F_{ii})) \). Because \( K_{0} \cap H \) is a nonempty \( G \) subset of \( F \times G \), there exist \( l_{i} \in N, n_{i} \in N \) and \( U \) open with \( U_{n_{i}} \subset U \), \( K_{0} \cap H \cap U_{n_{i+1}} \neq \emptyset \) and \( K_{0} \cap H \cap U \subset \pi^{-1}(F_{ii}) \).

Since \( K_{0} \cap U_{n_{i+1}} \neq \emptyset \), we have \( K_{0} \neq \emptyset \). Clearly \( K_{0} \cap U \subset \pi^{-1}(F_{ii}) \) and thus \( K_{0} \cap U_{n_{i}} \subset \pi^{-1}(F_{ii}) \).

Assume \( l_{i}, n_{1}, \ldots, l_{i}, n_{0} \) obtained. Since \( A \subset \bigcup F_{i+1,j} \), we have \( K_{n_{i}, \ldots, n_{0}} \cap H = \bigcup (K_{n_{i}, \ldots, n_{0}} \cap H \cap \pi^{-1}(F_{i+1,j})) \). Again \( K_{n_{i}, \ldots, n_{0}} \cap H \) is a nonempty \( G \) in \( F \times G \) and therefore there exist \( l_{i+1} \in N, n_{i+1} \in N \) and \( U \) open, such that

\[
U_{n_{i+1}} \subset U, K_{n_{i}, \ldots, n_{0}} \cap H \cap U_{n_{i+1}} \neq \emptyset
\]

and

\[
K_{n_{i}, \ldots, n_{0}} \cap H \cap U \subset \pi^{-1}(F_{i+1,j+1}).
\]

Because \( K_{n_{i}, \ldots, n_{0}} \cap U_{n_{i+1}} \neq \emptyset \), we have \( K_{n_{i}, \ldots, n_{0}} \cap U_{n_{i+1}} \neq \emptyset \). Furthermore

\[
K_{n_{i}, \ldots, n_{0}} \cap U_{n_{i}} \subset \pi^{-1}(F_{i+1,j+1})
\]

and hence

\[
K_{n_{i}, \ldots, n_{0}} \cap U_{n_{i}} \subset \pi^{-1}(F_{i+1,j+1})
\]

This completes the construction.

Take \( r = (n_{0}) \). Since \( (K_{i,j} \cap U_{n_{i}}) \subset (K_{i,j} \cap U_{n_{i}}) \), we have that \( (K_{i,j} \cap U_{n_{i}}) \subset (K_{i,j} \cap U_{n_{i}}) \). Because \( U_{n_{i}} \cap K_{i,j} \neq \emptyset \), it follows that \( B_{n_{i}, \ldots, n_{0}} \cap (K_{i,j} \cap U_{n_{i}}) \neq \emptyset \). Therefore we obtain for each \( k \) that

\[
F_{ii} \cap \cdots \cap F_{i+1,j+1} \cap B_{n_{i}, \ldots, n_{0}} \subset \pi(K_{i,j} \cap \cdots \cap F_{i+1,j+1} \cap B_{n_{i}, \ldots, n_{0}})
\]

By the compactness of \( F \), we conclude that \( \bigcap F_{ii} \cap \bigcap B_{n_{i}, \ldots, n_{0}} \neq \emptyset \). But

\[
\bigcap F_{ii} \subset A \text{ and } \bigcap B_{n_{i}, \ldots, n_{0}} \subset B,
\]

contradicting the fact that \( B \subset A \).

Thus Proposition 3 is established.

References

