construction of § 6 implies that the Hanf number is at least $(2^{\aleph_0})^+$, but the proof of Theorem 5.1 does not carry over as written. The application of the Erdös–Rado theorem uses the fact that an arbitrary model of a Scott sentence has at most $\aleph_0$ $L_{\infty\omega}$ 2-types. For models of an arbitrary $L_{\omega_1\omega}$ sentence this cannot be assumed.

In § 6 we invoked Theorem 4.1 to produce $\mathfrak{B}$ of power $2^{\aleph_0}$ satisfying $\varphi_{\mathfrak{A}}$. It turns out that (in the notation of § 4) $N(d, S_d) = \{d\}$ in $\mathfrak{A} = \mathfrak{A}^*$ and, in the construction of $\mathfrak{B}$, $\mathfrak{C}$ is all of $B^+ - B$. To find a case where, by contrast, the simple expedient $Q_0^{\mathfrak{B}} = B^+ - B^*$ does not suffice one may examine $\mathfrak{A} - P_1^{\mathfrak{A}}$. The gaps created by removing $P_1^{\mathfrak{A}}$ reappear in B^+, but may not be added to $Q_0^{\mathfrak{B}}$, as a back and forth argument confirms. In general, $\mathfrak{C}$ is the largest subset of B^+ which may be added to $Q_0^{\mathfrak{A}^*}$ without modifying the $Q_i^{\mathfrak{A}^*}$, $i>0$.

It is not difficult to show that the model $\mathfrak{B}$ of $\varphi_{\mathfrak{A}}$ in the case $N(d, S_d) = \{d\}$ of § 4 has at least $2^{\aleph_0}$ nonisomorphic elementary submodels. It follows that for any denumerable $\mathfrak{A}$, if $\varphi_{\mathfrak{A}}$ has more than one (nonisomorphic) model, then it has at least $2^{\aleph_0}$. In the case of the $\mathfrak{A}$ of § 6 it can be shown that $\mathfrak{B}$ has a "universal" property: if $\mathfrak{C} \models \varphi_{\mathfrak{A}}$, then $\mathfrak{C} < \mathfrak{B}$. We do not know whether all $\varphi_{\mathfrak{A}}$ with spectrum $\mathrm{Card}^{\leqslant 2^{\aleph_0}}$ possess this property. Another question about $\varphi_{\mathfrak{A}}$ with spectrum $\mathrm{Card}^{\leqslant 2^{\aleph_0}}$ stems from the fact that in Theorem 2.1 we are able to give structural characterizations of orderings whose spectra are Card and $\{\aleph_0\}$, but none for these. It is clear from our analysis that orderings in which the union of the scattered orbits is dense, yet which have a nonscattered orbit, possess this spectrum. However, there are also dense orderings with the same spectrum but no scattered orbit.

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Continuous monotone decompositions of planar curves

by

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Abstract. In this paper we prove that if X is a planar curve and $f\colon X \to Y$ is a monotone open surjection onto a nondegenerate continuum Y such that either (1) $f^{-1}(y)$ is a λ-dendroid for each $y \in Y$, or (2) $f^{-1}(y)$ is locally connected for each $y \in Y$, then f is a homeomorphism. We give also some examples showing that the theorem is in a sense the best possible.

1. Introduction. In this paper we are going to discuss continuous monotone decomposition of certain metric continua. It is well-known that the investigation of continuous monotone decompositions of continua is equivalent to investigation of continuous monotone and open transformations of continua. All the results of this paper are expressed in the language of mappings. In [12] B. Knaster showed that there is a continuous monotone and open map f from an irreducible continuum onto the unit interval $I = \{0, 1\}$ such that each fiber $f^{-1}(t)$ is nondegenerate. Since then there has been a remarkable interest in investigations of structure of the fibers for such mappings defined on irreducible continua (see e.g. [10] and [17]). These investigations were in some sense closed by E. Dyer [8] who proved that for every continuous monotone and open surjection with nondegenerate fibres from an irreducible continnum onto a nondegenerate continuum there is an indecomposable fibre.

From this result it follows in particular that in the above Knaster's example there must be some $t \in I$ such that $f^{-1}(t)$ is indecomposable. There are examples of irreducible and nonirreducible continua admitting monotone open surjections onto nondegenerate continua such that all fibers of the surjections are indecomposable (even pseudoarcs) (see [1], [4] and [11]). It should be noted that the set of indecomposable fibres in such situations is of a particular Borel type. In fact, we have the following theorem easily resulting from a theorem of Mazurkiewicz.

1.1. Theorem. *Let $f\colon X \to Y$ be a continuous monotone open surjection from a continuum X. Then the set of all $y \in Y$ such that $f^{-1}(y)$ is indecomposable (hereditarily indecomposable) is a G_δ-subset of Y.*

To prove the above theorem let us first observe that by the theorem of Mazurkiewicz the set of all indecomposable (hereditarily indecomposable) subcontinua of X constitute a G_δ-set in the hyperspace $C(X)$ (see [15], p. 207, Remark 5). From our assumptions it follows that the map $f^{-1}\colon Y \to C(X)$ is an embedding (see [15], p. 68). Combining these facts one easily obtains the conclusion of the theorem.

1.2. Remark. Let f, X and Y be as in 1.1. It is easily seen that the set $\{y \in Y\colon f^{-1}(y)$ is snake-like$\}$ is a G_δ-subset of Y. This fact together with 1.1 give a simple proof of Theorem 1 in [5].

Theorem 1.1 indicates that some sorts of constructions are not possible. In particular it is not possible to have a continuous decomposition of a circle-like continuum into arcs and a countable collection of indecomposable continua with the quotient space being a circle, as is claimed in [4], p. 191, 1.14–18.

Let us remark that the G_δ-set occurring in Theorem 1.1 can be empty even in the case where X is a locally connected curve (i.e. 1-dimensional continuum). Moreover, there is a continuous monotone open map f from the Menger curve onto the Hilbert cube such that each fibre of f is locally connected (homeomorphic to the Menger curve) (see [1] and [19]). The essential feature of this example is the cyclicity of locally connected fibres, because, as has been proved by E. Dyer [9], there is no continuous monotone open map f from a curve onto a nondegenerate continuum such that each fibre of f is a nondegenerate dendrite (locally connected continuum containing no simple closed curve). The fact that the Menger curve in the example of Anderson–Wilson is not planar is also essential. We prove that if X is a planar curve, then each continuous monotone open map on X with locally connected fibres is a homeomorphism (see 2.8).

Nevertheless one can construct an example of a planar locally connected hereditarily decomposable continuum X admitting a continuous monotone and open map f onto the unit interval I such that each fibre of f is a nondegenerate arcwise connected continuum (see 3.6) (observe that X must be 1-dimensional (a curve) according to a result of Mazurkiewicz [16]). Again the fibres in this example are not acyclic. This is also essential, because we prove that each continuous monotone and open map f defined on a planar curve such that the fibres of f are λ-dendroids must be a homeomorphism (see 2.7). (By a λ-dendroid we mean a hereditarily decomposable and hereditarily unicoherent continuum. It is known that λ-dendroids are acyclic curves (see [6] and [7])). We show also by an example that one cannot replace the stipulation of hereditary decomposability of such fibres by decomposability of the fibres. Indeed we shall construct an example of an acyclic planar curve (tree-like curve) admitting a continuous monotone and open map f onto I such that each fibre of f is a decomposable snake-like continuum (see 3.2).

Finally we shall show that for curves in E^3 the situation is substantially different. To this effect we construct an example of a dendroid (= arcwise connected λ-dendroid) in E^3 admitting a continuous monotone and open map f onto I such that each fibre of f is nondegenerate (see 3.3).

1.3. Remark. In connection with Example 3.2 let us note that the existence of such an example follows from a theorem of R. D. Anderson [1]. Unfortunately no proof of that theorem is given in [1].

Standard notation: $I = [0, 1]$, $\mathring{I} = (0, 1)$, $\dot{D}$ = the boundary of a disc D and $\mathring{D}$ = the interior of D.

2. On triviality of certain decompositions of planar curves. In this section we shall prove two theorems (2.6 and 2.7) asserting that some kinds of decompositions of planar curves must be trivial. The proofs of these theorems havily depend on the following result.

2.1. Theorem (E. Dyer). *Let X and Y be nondegenerate continua and let $f\colon X \to Y$ be a monotone open surjection. Then there is a dense G_δ-subset A of Y having the following property: for each $y \in A$, for each continuum $B \subset f^{-1}(y)$ with nonvoid interior with respect to $f^{-1}(y)$, and for each open set $U \subset X$ such that $\overline{U} \cap B = \emptyset$, there is a continuum $Z \subset X$ containing B and a neighborhood V of y in Y such that $(f|Z)^{-1}(V) \cap U = \emptyset$ and $f|Z\colon Z \to Y$ is a monotone surjection.*

This theorem immediately follows from an argument of Dyer [8], but in this form it is not stated in [8]. Therefore we briefly recall the argument. For each $y \in Y$ and $\varepsilon > 0$ we define, following Dyer, a nonnegative number $C(y, \varepsilon)$ as follows: if $f^{-1}(y) \setminus K(f^{-1}(y) \setminus B, \varepsilon) = \emptyset$, where $K(S, \varepsilon)$ is the ε-ball around S, for each proper subcontinuum B of $f^{-1}(y)$ then let $C(y, \varepsilon) = 0$. Otherwise let $C(y, \varepsilon)$ be the lower upper bound of reals $\eta > 0$ for which there exist a proper subcontinuum B of $f^{-1}(y)$ and a point $p \in f^{-1}(y) \setminus K(f^{-1}(y) \setminus B, \varepsilon)$ having the following property: for each $\delta > 0$ there is $z \in Y$ such that two different components of $f^{-1}(z) \cap \overline{K(B, \eta)}$ meet $\overline{K(p, \delta)}$.

From Lemmas 3 and 4 of [8] we conclude that $M = \{y \in Y\colon$ there is an $\varepsilon > 0$ such that $C(y, \varepsilon) \neq 0\}$ is of the 1st category F_σ subset of Y. Let $A = Y \setminus M$. Take $y \in A$ and a (proper) subcontinuum B of $f^{-1}(y)$ with nonvoid interior with respect to $f^{-1}(y)$. There are a point $p \in B$ and $\varepsilon > 0$ such that $p \in f^{-1}(y) \setminus K(f^{-1}(y) \setminus B, \varepsilon)$. Let $0 < \eta < \inf\{\varrho(a, b)\colon a \in U, b \in B\}$.

Since f is open and $C(y, \varepsilon) = 0$, there is an open neighborhood V of y in Y and a closed neighborhood W of p in X such that for each $z \in V$ exactly one component D_z of $f^{-1}(z) \setminus U$ meets W. Let $Z = f^{-1}(Y \setminus V) \cup \bigcup_{z \in V} D_z$. Observe that Z is a compact subset of X such that $f|Z$ is a monotone map onto Y. It follows that Z is a continuum because Y is a continuum.

This completes the argument.

2.2. Remark. From the above theorem of Dyer it follows that every continuous monotone decomposition of an irreducible continuum into nondegenerate elements contains uncountably many indecomposable elements (comp. Theorem 2 in [5] and questions following it on p. 1333).

Before we state the promised theorems we first prove the following four lemmas needed in the proofs of these theorems.

2.3. Lemma. *Let X be a nondegenerate continuum which is not a triod and which can not be mapped onto an indecomposable continuum. Then there exist two disjoint subcontinua B_0 and B_1 of X with nonvoid interiors.*

Proof. By [13], 3.5 there exist two disjoint continua B'_0 and B'_1 in X such that every neighbourhood of B'_i contains a neighborhood of B'_i with connected complement. To complete the proof it suffices to show that every neighborhood U of B'_i contains a continuum with nonvoid interior.

Let $V \subset \overline{V} \subset U$ be a neighborhood of B'_i with connected complement. By our assumption it follows that there are at most two components of $\overline{V}$ meeting V. For otherwise there would exist three disjoint closed sets F_0, F_1, F_2 such that $\overline{V} = F_0 \cup F_1 \cup F_2$ and $F_j \cap V \neq \emptyset$. Then $F_j \cap V$ would be an open subset of X and X would be a triod with links $F_j \cup (X \setminus V)$ and the center $X \setminus V$. Hence the component of $\overline{V}$ containing B'_i has nonvoid interior and is a subset of U, which completes the proof.

2.4. Lemma. *Let X be a continuum in the sphere S^2 and let $f\colon X \to Y$ be an open surjection such that for each $y \in Y$ the fiber $f^{-1}(y)$ is a nondegenerate continuum which can not be mapped onto an indecomposable continuum. Then there exists a nondegenerate continuum $E \subset Y$ and two disjoint subcontinua D_0 and D_1 of X such that $f(D_0) = f(D_1) = E$ and the map $f|D_i\colon D_i \to E$ is monotone for $i = 0, 1$.*

Proof. Let $T \subset Y$ be the set of all points such that $f^{-1}(y)$ contains a triod for $y \in T$. By the Moore triodic theorem T is countable. Let $A \subset Y$ be the set as in Theorem 2.1. Pick a point $z \in A \setminus T$. By Lemma 2.3 there exist two disjoint continua $B_0, B_1 \subset f^{-1}(z)$ with nonvoid interiors relative to $f^{-1}(z)$. There exist two sets U_0 and U_1 open in X such that $f^{-1}(z) \subset U_0 \cup U_1$, $\overline{U}_0 \cap B_0 = \emptyset$ and $\overline{U}_1 \cap B_1 = \emptyset$. By Theorem 2.1 there exist two subcontinua Z_0 and Z_1 of X and a neighborhood V of z in Y such that $f|Z_i$ is a monotone surjection onto Y and

$$(f|Z_i)^{-1}(V) \cap U_i = \emptyset \quad \text{for} \quad i = 0, 1.$$

There is a neighborhood $W \subset V$ of z in Y such that $f^{-1}(W) \subset U_0 \cup U_1$. Let $E \subset W$ be an arbitrary nondegenerate continuum. One easily sees that E and the continua $D_i = (f|Z_i)^{-1}(E)$, $i = 0, 1$, satisfy the conclusion of the lemma. This completes the proof.

2.5. Lemma. *Let X be a continuum in the sphere S^2 and let $f\colon X \to Y$ be an open surjection such that for each $y \in Y$ the fiber $f^{-1}(y)$ is a nondegenerate continuum which can not be mapped onto an indecomposable continuum. Then Y is hereditarily locally connected.*

Proof. Suppose Y is not hereditarily locally connected. Then there exist a sequence of nondegenerate mutually disjoint continua $C_0, C_1, \ldots$ in Y such that $C_n \underset{n\to\infty}{\longrightarrow} C_0$ in the Hausdorff distance ([15], p. 47). By 2.4 there are three distinct points $y_0, y_1, y_2 \in C_0$ and two disjoint continua $D_0, D_1 \subset f^{-1}(C_0)$ such that $D_0 \cap f^{-1}(y_i) \neq \emptyset \neq D_1 \cap f^{-1}(y_i)$ for $i = 0, 1, 2$. There exist three points $p_i \in f^{-1}(y_i)$, $i = 0, 1, 2$, and a positive number ε such that no component of $S^2 \setminus (D_0 \cup D_1 \cup f^{-1}(y_0) \cup \cup f^{-1}(y_1) \cup f^{-1}(y_2))$ meets each ε-ball $K(p_i, \varepsilon)$, $i = 0, 1, 2$. Otherwise one can construct the Kuratowski skew-curve in S^2, which is impossible [14]. However by our supposition there exists an index n such that the continuum $f^{-1}(C_n)$ meets $K(p_i, \varepsilon)$ for each $i = 0, 1, 2$, and $f^{-1}(C_n)$ is disjoint from $D_0 \cup D_1 \cup f^{-1}(y_0) \cup \cup f^{-1}(y_1) \cup f^{-1}(y_2)$, a contradiction.

For a continuum X by $C(X)$ we denote the hyperspace of nonvoid subcontinua of X with the Hausdorff metric.

2.6. Lemma. *Let $\varphi\colon I \to C(I^2)$ be a mapping with disjoint values, i.e. $\varphi(r) \cap \varphi(s) = \emptyset$ for $r \neq s$, such that $\varphi(t) \cap I \times \{0\} \neq \emptyset \neq \varphi(t) \cap I \times \{1\}$ for each $t \in I$. Then $\varphi(t)$ is irreducible between $\varphi(t) \cap I \times \{0\}$ and $\varphi(t) \cap I \times \{1\}$ for each $t \in I$.*

Proof. For $z \in I^2$ let z' denote the first coordinate of z. For $r \in I$ let $p_r \in \varphi(r) \cap I \times \{0\}$ and $q_r \in \varphi(r) \cap I \times \{1\}$ be arbitrary points. Observe that

(0) if $p'_s \in [p'_{s_0}, p'_{s_1}]$, then $s \in [s_0, s_1]$.

If not, then the set $\{p_s, q_s\}$ separates $\dot{I}^2$ between p_{s_0} and p_{s_1}. But $\bigcup_{r \in [s_0, s_1]} \varphi(r)$ is a continuum containing p_{s_0} and p_{s_1}, and $\varphi(s)$ is a continuum containing p_s and q_s. It follows that for some $r \in [s_0, s_1]$ the continuum $\varphi(s)$ meets $\varphi(r)$. Hence $s = r$, a contradiction.

To prove the lemma it suffices to show that $\varphi(t)$ is irreducible between p_t and q_t. Suppose, to the contrary, that there are a continuum $C \subset \varphi(t)$ containing p_t and q_t, and a point $x \in \varphi(t) \setminus C$. Let $D \subset I^2$ be a disc missing C and containing x in its relative interior with respect to I^2. There is an interval $[t_0, t_1] \subset I$ containing t in its interior such that $\varphi(r) \cap D \neq \emptyset$ for each $r \in [t_0, t_1]$. It follows from (0) that p'_t lies in the interior of $[p'_{t_0}, p'_{t_1}]$. Hence $\{p_t, q_t\}$ separates $\dot{I}^2$ between p_{t_0} and p_{t_1}. Since $\varphi(t_0) \cup D \cup \varphi(t_1)$ is a continuum in I^2 containing p_{t_0} and p_{t_1}, and $C \subset I^2$ is a continuum containing p_t and q_t, then $C \cap (\varphi(t_0) \cup D \cup \varphi(t_1)) \neq \emptyset$. Hence $\varphi(t) \cap \varphi(t_i) \neq \emptyset$ for some $i = 0, 1$, a contradiction.

2.7. Theorem. *Let X be a curve in S^2 and let $f\colon X \to Y$ be an open surjection onto a nondegenerate continuum Y. If for each $y \in Y$ the fiber $f^{-1}(y)$ is a λ-dendroid, then f is a homeomorphism.*

Proof. Suppose the theorem fails. Using Lemma 2.5 we may assume (without loss of generality) that Y is an arc such that $f^{-1}(y)$ is nondegenerate for each $y \in Y$. Thus Lemma 2.4 implies that X contains a subcontinuum separating S^2. Hence X separates S^2 since $\dim X = 1$. On the other hand, since all fibers $f^{-1}(y)$ are acyclic, the continuum X is acyclic too (see [2]). This is a contradiction completing the proof.

2.8. Theorem. *Let X be a curve in S^2 and let $f\colon X \to Y$ be an open surjection onto a nondegenerate continuum Y. If for each $y \in Y$ the fiber $f^{-1}(y)$ is a locally connected continuum, then f is a homeomorphism.*

Proof. Suppose the theorem fails. By 2.5 we may assume that Y is an arc ab

such that $f^{-1}(y)$ is nondegenerate for each $y \in Y$. By Lemma 2.4 there is a nondegenerate continuum $E \subset Y$ and there are two continua D_0 and D_1 in X such that $E = f(D_0) = f(D_1)$ and $f|D_i\colon D_i \to E$ is a monotone map. Without loss of generality one can assume that $E = Y$. Let $\mathscr{D}$ be a decomposition of S^2 defined as follows:

$$\mathscr{D} = \{(f|\widehat{D_0})^{-1}(y), (f|\widehat{D_1})^{-1}(y)\}_{y \in Y} \cup \{\text{points of } S^2 \setminus (D_0 \cup D_1)\},$$

where

$$(f^{-1}|\widehat{D_i})^{-1}(y) = (f|D_i)^{-1}(y) \cup \{\text{the union of all components of } S^2 \setminus (f|D_i)^{-1}(y) \text{ each of which does not contain } D_{1-i}\}, \quad \text{for } i = 0, 1.$$

Observe that $\mathscr{D}$ is an upper semi-continuous decomposition of S^2 into continua nonseparating S^2. By the Moore theorem we have $S^2/\mathscr{D} = S^2$. Note that the image $\tilde{X}$ of X in $S^2/\mathscr{D}$ behaves similarly as X itself, i.e. $\tilde{X}$ is a curve and there exists an open monotone surjection $\tilde{f}$ of $\tilde{X}$ onto Y such $\tilde{f}^{-1}(y)$ is locally connected. Furthermore there exist two disjoint arcs $\tilde{D}_0$ and $\tilde{D}_1$ in $\tilde{X}$ such that $\tilde{f}$ restricted to each $\tilde{D}_i$ is a homeomorphism and each fiber of $\tilde{f}$ meets both $\tilde{D}_0$ and $\tilde{D}_1$. Let $L_a \subset \tilde{f}^{-1}(a)$ be an arc with endpoints $p_a \in \tilde{D}_0$ and $q_a \in \tilde{D}_1$, and let $L_b \subset \tilde{f}^{-1}(b)$ be an arc with endpoints $p_b \in \tilde{D}_0$ and $q_b \in \tilde{D}_1$. Note that $L_a \cup L_b \cup \tilde{D}_0 \cup \tilde{D}_1$ is a simple closed curve. Let Q_0 and Q_1 denote the complementary discs in $S^2/\mathscr{D}$ bounded by that curve. Let $M_i = \{y \in Y\colon f^{-1}(y) \cap Q_i \text{ is connected}\}$, $i = 0, 1$. Since $M_0 \cup M_1 = Y$, then either $\overline{M}_0$ or $\overline{M}_1$ has nonvoid interior in Y. Without loss of generality we may now assume that $M_0 = Y$. Let $Z = \tilde{X} \cap Q_0$ and let $g = \tilde{f}|Z$. Note that g is an open surjection onto Y such that $g^{-1}(y)$ is a locally connected continuum meeting both $\tilde{D}_0$ and $\tilde{D}_1$. By Lemma 2.6 it follows that each fiber $g^{-1}(y)$, for $y \in Y \setminus \{a, b\}$, is irreducible between $\tilde{D}_0 \cap g^{-1}(y)$ and $\tilde{D}_1 \cap g^{-1}(y)$; hence $g^{-1}(y)$ is an arc. But by a theorem of Dyer [9] (comp. 2.7) this is impossible. This contradiction completes the proof.

3. Some examples.

In this section we present the examples mentioned in the Introduction.

3.1. Lemma. *Let Y and Z be continua and let $\varphi\colon Y \to C(Z)$ be a mapping with disjoint values, i.e. $\varphi(x) \cap \varphi(y) = \emptyset$ for $x \neq y$. Then $X = \bigcup_{y \in Y} \varphi(y)$ is a continuum and there is a unique monotone open mapping $f\colon X \to Y$ onto Y such that $f^{-1}(y) = \varphi(y)$ for each $y \in Y$.*

Proof. The first statement is proved in [11], 1.2, p. 23. Define: $f(x) = y$ if $x \in \varphi(y)$. Let V be an open set in Y. First we shall show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$ and suppose there is a sequence $x_1, x_2, \ldots$ from $X \setminus f^{-1}(V)$ converging to x. Let $x_n \in \varphi(y_n) = f^{-1}(y_n)$. We can assume that $\{y_n\}$ converges to a point y_0. Since $y_n \notin V$, then $y_0 \notin V$. On the other hand $\varphi(y_0) \ni x$, which implies that $y_0 = f(x) \in V$, a contradiction. This proves that f is continuous.

Now we shall show that f is open. Let U be open in X. Then

$$f(U) = \{y\colon \varphi(y) \cap U \neq \emptyset\}$$

is open in Y because φ is continuous.

The monoteneity and uniqueness of f are evident, which completes the proof.

3.2. Example. There exists a planar tree-like continuum Y admitting a monotone open surjection f onto I such that $f^{-1}(t)$ is a decomposable snake-like continuum for each $t \in I$ (comp. [1]).

Proof. First we establish some notation. Let $I_0 = I \times \{0\}$ and $I_1 = I \times \{1\}$. Points of I_0 will be denoted by the letter p (with subscripts) and by the letter q (with subscripts) we denote points of I_1. By pq we mean an arc with endpoint p and q. By A we denote the collection of all arcs $p_s q_s$ in I^2 such that $p_s q_s \cap I_0 = \{p_s\}$ and $p_s q_s \cap I_1 = \{q_s\}$. If $z_1, z_2 \in I^2$, then $\overline{z_1 z_2}$ denotes the straight line segment between z_1 and z_2. If $p_1 q_1; p_2 q_2 \in A$ are disjoint then $[p_1 q_1, p_2 q_2]$ denotes the disc bounded by the simple closed curve $p_1 q_1 \cup p_2 q_2 \cup \overline{p_1 p_2} \cup \overline{q_1 q_2}$. If F is a family of sets, then F^* denotes the union of elements of F.

Let $U_1, U_2, \ldots$ be a base for open sets of I^2. One can construct two sequences:

(i) a sequence of discs $D_1, D_2, \ldots$ in I^2 and

(ii) a sequence of finite collections $A_1, A_2, \ldots$ contained in A such that setting

$$X_m = I^2 \setminus \bigcup_{j \leqslant m} \operatorname{Int}_{I^2} D_j, \quad m = 1, 2, \ldots,$$

the following conditions are satisfied for each $n \geqslant 1$:

$(1)_n$ $\{0\} \times I, \{1\} \times I \in A_n \subset A_{n+1}$ and two different elements of A_n are disjoint,

$(2)_n$ if $L_1, L_2 \in A_n$ are different and $[L_1, L_2] \cap A_n^* = L_1 \cup L_2$, then $[L_1, L_2] \cap X_n$ can be covered by $1/n$-chain $\mathscr{C}$ such that the first link of $\mathscr{C}$ contains $[L_1, L_2] \cap I_0$, and $[L_1, L_2] \cap X_n \subset \{x \in I^2\colon \varrho(x, L_i) < 1/n\}$ for $i = 1, 2$,

$(3)_n$ $A_n^* \cap \bigcup_{j \leqslant n} D_j = \emptyset$,

$(4)_n$ $D_{n+1} \cap (D_1 \cup \ldots \cup D_n) = \emptyset$,

$(5)_n$ $D_n \cap (I_0 \cup I_1)$ is an arc contained in I_1,

$(6)_n$ $U_n \setminus X_n \neq \emptyset$.

Having defined the sets $D_1, \ldots, D_n$ and $A_1, \ldots, A_n$ one can easily construct D_{n+1} and A_{n+1} using the following simple lemma, the proof of which is left to the reader.

Lemma. *Let P be a finite subcollection of mutually disjoint elements of A such that $\{0\} \times I, \{1\} \times I \in P$. Let $B \subset I_1$ be a closed set disjoint from P^*. Then for each $\varepsilon > 0$ there is a finite collection $Q \subset A$ of mutually disjoint arcs containing P such that $B \cap Q^* = \emptyset$ and if $L_1, L_2 \in Q$ are different and $[L_1, L_2] \cap Q^* = L_1 \cup L_2$, then $[L_1, L_2]$ can be covered by ε-chain whose the first link contains $[L_1, L_2] \cap I_0$, and $[L_1, L_2] \subset \{x \in I^2\colon \varrho(x, L_i) < \varepsilon\}$ for $i = 1, 2$.*

The collection A_{n+1} is obtained as the image of Q by a special homeomorphism h of I^2 onto X_n which sends B to $\bigcup_{j\leqslant n} D_j \cap X_n$ and P^* to A_n^*. The number $\varepsilon>0$ is chosen to fit condition $(2)_{n+1}$.

This completes the construction of D_n's and A_n's.

Let $X = \bigcap_{n\geqslant 1} X_n$. It follows from $(1)_n$, $(4)_n$ and $(5)_n$ that X is a continuum not separating the plane. By $(6)_n$ we infer that X is 1-dimensional. These conditions imply that X is tree-like [3]. Define a function $\varphi\colon I_0 \to C(X)$ in the following way. If $p \in \bigcup_{n\geqslant 1} A_n^*$, then there is exactly one element $L_p \in \bigcup_{n\geqslant 1} A_n$ containing p. Set $\varphi(p) = L_p$ in this case. Otherwise consider the sequence $E_1, E_2, \ldots$ of discs such that $p \in E_n = [L_1^n, L_2^n] \cap X_n$, where $L_1^n, L_2^n \in A_n$ are different and $[L_1^n, L_2^n] \cap A_n^* = L_1^n \cup L_2^n$. Note that $E_{n+1} \subset E_n$ for each $n\geqslant 1$. In this case set $\varphi(p) = \bigcap_{n\geqslant 1} E_n$. By $(2)_n$ the function φ is continuous and has disjoint snake-like values such that $\{p\} = \varphi(p) \cap I_0$ and for each $\varepsilon>0$ the continuum $\varphi(p)$ can be covered by an ε-chain whose only the first link contains p, i.e. p is an endpoint of $\varphi(p)$ (see [3], p. 660).

The continuum Y we define as the union of X and the reflection of X is the x-axis. Clearly, Y has the promised properties.

A. Lelek asked the question (credited by him to A. Stralka) whether there is a dendroid (different from an arc) admitting a continuous monotone and open map onto an arc. The answer to this question is given by the following.

3.3. Example. There exists (in E^3) a dendroid X and a continuous monotone and open retraction r from X onto an arc $I^* \subset X$ such that all fibres of r are non-degenerate.

Proof. First we establish some notation. Let M be the Cantor fan (i.e. the cone over the Cantor set) and let v be the vertex of M. Let $C(M)$ be the hyperspace of nonvoid subcontinua of M with the Hausdorff metric dist $(\cdot,\cdot)$. Let us denote

$$C(M, v) = \{A \in C(M)\colon\ v \in A\}.$$

Note that $C(M, v)$ is a (contractible) continuum. We first prove that there is a continuous map $f\colon I \to C(M, v)$ having the following properties

(i) $f(t) \neq \{v\}$ for each $t \in I$,

(ii) $\dim\{t \in I\colon\ p \in f(t)\} \leqslant 0$ for each $p \in M\setminus\{v\}$.

This will be done by applying the Baire category argument to the space $C(M, v)^I$.

For each $\varepsilon>0$ let us denote

$$\Phi_\varepsilon = \{f \in C(M, v)^I\colon \text{(i)} \,\&\, (\text{for each } p \in M\setminus K(v, \varepsilon) \text{ each component of } \{t\colon\ p \in f(t)\} \text{ has diameter less than } \varepsilon\},$$

where $K(v, \varepsilon)$ is the open ε-ball around v.

Observe that each map belonging to $\bigcap_n \Phi_{1/n}$ satisfies conditions (i) and (ii). We shall show that this set is nonvoid. By the Baire theorem it suffices to show that for each $\varepsilon>0$ we have

(a) Φ_ε is open in $C(M, v)^I$, and

(b) Φ_ε is dense in $C(M, v)^I$.

Proof of (a). Let $f \in \Phi_\varepsilon$ and suppose there is a sequence $f_1, f_2, \ldots$ from $C(M, v)^I$ converging to f such that for each $n\geqslant 1$ there is a point $p_n \in M\setminus K(v, \varepsilon)$ and there is a component T_n of the set $\{t\colon\ p_n \in f_n(t)\}$ such that $\operatorname{diam} T_n \geqslant \varepsilon$. One can assume that $p_n \to p$ and $T_n \to T$ (in $C(I)$). It is easily seen that $p \notin K(v, \varepsilon)$, $\operatorname{diam} T \geqslant \varepsilon$ and $T \subset \{t\colon\ p \in f(t)\}$, which is a contradiction.

Proof of (b). Let $g \in C(M, v)^I$ and let δ be a positive number. Take $t_0 = 0 < t_1 < \ldots < t_n = 1$ such that

$$\operatorname{diam}[t_i, t_{i+1}] < \tfrac{1}{2}\varepsilon, \tag{1}$$

$$\operatorname{diam} g([t_i, t_{i+1}]) < \tfrac{1}{6}\delta. \tag{2}$$

There exist $A_0, A_1, \ldots, A_n \in C(M, v)\setminus(\{v\})$ such that

$$A_i \cap A_j = \{v\} \quad \text{for} \quad i \neq j, \tag{3}$$

$$\operatorname{dist}(g(t_i), A_i) < \tfrac{1}{6}\delta. \tag{4}$$

To finish the proof we need the following remark. If $A, B \in C(M)\setminus(\{v\})$ and $A \cap B \neq \emptyset$, then there is an arc L in $C(M)\setminus(\{v\})$ joining A and B such that $\operatorname{diam} L \leqslant \operatorname{dist}(A, B)$. Such an arc can be easily chosen in the union of the segments $AA \cup B$ and $BA \cup B$ (see [15], p. 186, Th. 3).

By this remark there is a mapping $f\colon I \to C(M, v)$ such that $f(t_i) = A_i$, $f([t_i, t_{i+1}]) \subset C(M, v)\setminus(\{v\})$ and $\operatorname{diam} f([t_i, t_{i+1}]) < \frac{1}{2}\delta$. Observe that $d(f, g) < \delta$ and $f \in \Phi_\varepsilon$, which completes the proof of (b).

Thus there is a mapping $f\colon I \to C(M, v)$ satisfying conditions (i) and (ii). Let us define

$$X = \bigcup_{t\in I} f(t) \times \{t\} \quad (\subset M \times I),$$

$$I^* = \{v\} \times I \quad (\subset X)$$

and let $r\colon X \to I^*$ be given by $r(p, t) = (v, t)$. Observe that r is a monotone open retraction onto I^*. To complete the proof it remains to show that X is a dendroid.

Let $\Pi\colon X \to M$ be given by $\Pi(p, t) = p$. By (ii) it follows that $\dim \Pi^{-1}(p) \leqslant 0$ for each $p \neq v$ and $\dim \Pi^{-1}(v) = \dim I^* = 1$. Hence by [15], p. 114, Th. 1 the set X can be represented as a countable union of closed subsets each of dimension $\leqslant 1$. It follows that X is a curve. Since M is contractible one easily sees that X is contractible as well. This implies that X is a dendroid (see [6]).

In the next example we will need the following lemma.

3.4. Lemma. *Let D be a disc in S^2 and let $D_1, D_2, \ldots$ be a null-sequence of discs contained in D such that*

$$\mathring{D}_i \cap \mathring{D}_j = \emptyset \quad \textit{for} \quad i \neq j.$$

Then $X = D\setminus \bigcup_{i=1}^{\infty} \mathring{D}_i$ is a locally connected continuum.

Proof. It is clear that X is a continuum. Let $f_0\colon I\to D$ be any surjective map. One can easily construct a sequence of maps $f_1, f_2, \ldots$, where $f_n\colon I\to D$, satisfying the following conditions for each $n\geqslant 0$,

(1) $$f_{n+1}(t)=f_n(t) \quad \text{for} \quad t\notin f_n^{-1}(\mathring{D}_{n+1}),$$

(2) $$f_{n+1}(t)\in \dot{D}_{n+1} \quad \text{for} \quad t\in f_n^{-1}(\mathring{D}_{n+1}).$$

It follows from these conditions that $f_n(I)=D\setminus\bigcup_{i=1}^{n}\mathring{D}_i$ for $n\geqslant 1$ and that for each $n>m\geqslant 0$ we have

(3) $$d(f_m, f_n)\leqslant \max\{\operatorname{diam} D_{m+1}, \operatorname{diam} D_{m+2}, \ldots, \operatorname{diam} D_n\}.$$

Thus $f_n(I)\subset f_m(I)$ for $n>m$ and $X=\bigcap_{n\geqslant 1}^{\infty} f_n(I)$. Since $\operatorname{diam} D_n\to 0$, by (3) the sequence $f_1, f_2, \ldots$ converges to a map $f\colon I\to D$. Hence $X=f(I)$, which completes the proof.

In the statement of the next example we use some symbols which now we are going to define.

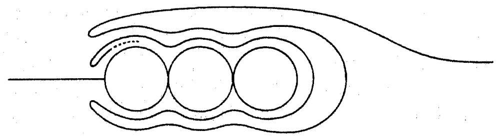
Let Q_ν denote a union of ν discs $D_1, D_2, \ldots, D_\nu$ forming a chain such that for $1\leqslant i\leqslant \nu-1$ the disc D_i meets D_{i+1} at a single point q_i lying on the boundaries of these discs. Let $[Q_\nu]$ denote the set $\{q_i\colon i=1,\ldots,\nu-1\}$ and let

$$\sigma(Q_\nu)=\max\{\operatorname{diam} D_i\colon i=1,\ldots,\nu\}.$$

Let Σ_ν denote a plane continuum defined as follows:

$$\Sigma_\nu=(\operatorname{Fr} Q_\nu)\cup R_1\cup R_2,$$

where R_1 is an arc disjoint from Q_ν except a point $q\in\dot{D}_1\setminus\{q_1\}$ being an endpoint of R_1, and R_2 is a (topological) closed half line missing $R_1\cup\operatorname{Fr} Q_\nu$ such that $\bar{R}_2\setminus R_2=\operatorname{Fr} Q_\nu$.



The set Σ_3

3.5. Example. There is an hereditarily decomposable locally connected plane continuum X admitting a monotone open map $f\colon X\to I$ onto I such that for each $t\in I$ the fiber $f^{-1}(t)$ is either an arc or $f^{-1}(t)$ is homeomorphic to Σ_ν for some $\nu\geqslant 1$ (ν depending on t). Moreover, there is a countable set $B\subset X$ such that each subcontinuum of X meeting two different fibers of f meets B.

Proof. For each integer $k\geqslant 1$ we now describe the continuum M_k which will be used in the construction of X.

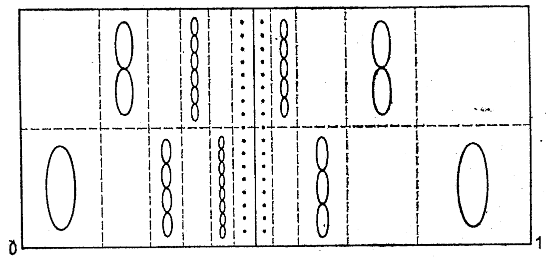
Let $\{a_n^l\}$, $l=0,1$; $n=0,1,\ldots$ be two sequences from I converging to $\frac{1}{2}$ such that $a_0^0=0<a_1^0<\ldots$ and $a_0^1=1>a_1^1>\ldots$ Let Q_{ij}^l be a copy of Q_ν for some $\nu=1,2,\ldots$ contained in the interior of the rectangle $[a_{kj+i}^l, a_{kj+i+1}^l]\times\left[\frac{i}{k},\frac{i+1}{k}\right]$ such that

(1) Q_{ij}^l is $\dfrac{1}{j+1}$-dense in that rectangle,

(2) $\sigma(Q_{ij}^l)\underset{j\to\infty}{\longrightarrow}0$.

The set M_k is now defined as follows

$$M_k=I^2\setminus\bigcup\{\operatorname{Int} Q_{ij}^l\colon l=0,1;\ i=0,\ldots,k-1;\ j=0,1,\ldots\}.$$



The continuum M_2

By L we will denote the segment $\{\frac{1}{2}\}\times I$. By (1) and (2) it is easy to see that

(3) each continuum in M_k meeting L and $M_k\setminus L$ meets the (countable) set $\{(\frac{1}{2}, i/k)\colon i=0,\ldots,k\}\cup\bigcup_{\substack{0\leqslant i\leqslant k-1\\ j>0}}[Q_{ij}^0]\cup[Q_{ij}^1]$.

Denote by A the set of centers of the intervals $[a_n^l, a_{n+1}^l]$, $l=0,1$; $n=0,1,\ldots$ Note that

(4) A is a countable set such that $\bar{A}=A\cup\{\frac{1}{2}\}\subset\dot{I}$.

Observe that there exist two mappings $F_k\colon I\to C(I^2)$ and $G_k\colon I^2\setminus\bar{A}\times I\to I^2$ having the following properties:

(5) G_k is an embedding such that $G_k(z)=z$ for $z\in\dot{I}^2\setminus\bar{A}\times I$ and $G_k\left((I\setminus\bar{A})\times\left[\frac{i}{k},\frac{i+1}{k}\right]\right)\subset I\times\left[\frac{i}{k},\frac{i+1}{k}\right]$ for $i=0,1,\ldots,k-1$,

(6) F_k has disjoint values such that $(t,0),(t,1)\in F_k(t)$ for each $t\in I$,

(7) $F_k(\frac{1}{2}) = L$, $F_k(t) \underset{\text{top}}{=} \Sigma_{v_t}$ for $t \in A$, and $F_k(t) = G_k(\{t\}\times I)$ for $t \notin \bar{A}$,

(8) $\bigcup_{t\in I} F_k(t) = M_k$.

Let $t_1, t_2, \ldots$ be a sequence of irrationals from I dense in I.

Now we show that it is possible to construct four sequences; (i) a sequence of sets $A_0, A_1, \ldots$, (ii) a sequence of sets $B_1, B_2, \ldots$, (iii) a sequence of mappings $\varphi_0, \varphi_1, \ldots$, where $\varphi_m\colon I \to C(I^2)$, and (iv) a sequence of mappings $\psi_0, \psi_1, \ldots$, where $\psi_m\colon I^2\setminus \bar{A}_m \times I \to I^2$, such that setting

(*) $$X_m = \bigcup_{t\in I} \varphi_m(t) \quad \text{for} \quad m = 0, 1, \ldots,$$

the following conditions are satisfied for each $n \geq 1$:

$(1)_n$ $A_n \subset \mathring{I}$ is a set of rationals such that $\bar{A}_n = A_n \cup \{t_1, \ldots, t_n\}$ and $A_{n-1}\subset A_n$,

$(2)_n$ $X_n \subset X_{n-1}$,

$(3)_n$ ψ_n is an embedding such that $\psi_n(z) = z$ for $z \in \mathring{I}^2\setminus \bar{A}_n \times I$,

$(4)_n$ $d(\psi_n, \psi_{n-1}|(I^2\setminus \bar{A}_n\times I)) < 1/2^n$,

$(5)_n$ φ_n has disjoint values such that $(t, 0), (t, 1) \in \varphi_n(t)$ for each $t \in I$ and $\varphi_n|\bar{A}_{n-1} = \varphi_{n-1}|\bar{A}_{n-1}$,

$(6)_n$ $d(\varphi_n, \varphi_{n-1}) < 1/2^n$,

$(7)_n$ $\varphi_n(t_n)$ is an arc with endpoints $(t_n, 0)$ and $(t_n, 1)$, $\varphi_n(t) \underset{\text{top}}{=} \Sigma_v$ for $t \in A_n$ and some $v = 1, 2, \ldots$ and $\varphi_n(t) = \psi_n(\{t\}\times I)$ for $t \notin \bar{A}_n$,

$(8)_n$ B_n is a countable subset of $\bigcup_{t\in \bar{A}} \varphi_n(t)$ such that each continuum in X_n meeting $\varphi_n(t_n)$ and $X_n \setminus \varphi_n(t_n)$ meets B_n,

$(9)_n$ each complementary domain of X_{n-1} is a complementary domain of X_n and if G is a complementary domain of X_n but not of X_{n-1}, then $\operatorname{diam} G < 1/2^n$.

Let $p\colon I^2 \to I$ be the projection, i.e. $p(x, y) = x$. Define $A_0 = \emptyset$, $\varphi_0 = p^{-1}$ and $\psi_0 = 1_{I^2}$ (thus $X_0 = I^2$) and assume the objects $A_0, \ldots, A_{n-1}$, $B_1, \ldots, B_{n-1}$, $\varphi_0, \ldots, \varphi_{n-1}$ and $\psi_0, \ldots, \psi_{n-1}$ have been constructed for $n \geq 1$. It remains to construct A_n, B_n, φ_n and ψ_n.

Since $t_n \notin \bar{A}_{n-1}$ there are two reals $r_0 < t_n < r_1$ such that $[r_0, r_1] \subset \mathring{I}\setminus \bar{A}_{n-1}$. By $(7)_{n-1}$ we can also assume that there is an integer $k \geq 1$ such that

$$\operatorname{diam}\psi_{n-1}\Big([r_0, r_1]\times\Big[\frac{i}{k}, \frac{i+1}{k}\Big]\Big) < \frac{1}{2^n} \quad \text{for} \quad i = 0, 1, \ldots, k-1.$$

Let $\beta\colon I \to [r_0, r_1]$ be a homeomorphism such that $\beta(A)$ is a subset of rationals, $\beta(\frac{1}{2}) = t_n$, $\beta(0) = r_0$ and $\beta(1) = r_1$. Then we define

$$A_n = A_{n-1} \cup \beta(A),$$

$$B_n = \psi_{n-1}\Big(\beta\times 1_I\Big(\bigcup_{\substack{0\leq i\leq k-1\\ j\geq 0}} [Q_{ij}^0] \cup [Q_{ij}^1] \cup \{(1/2, i/k)\colon i = 0, \ldots, k\}\Big)\Big),$$

$$\varphi_n(t) = \begin{cases} \varphi_{n-1}(t) & \text{for} \quad t \notin (r_0, r_1), \\ \psi_{n-1}\circ \beta\times 1_I(F_k\circ \beta^{-1}(t)) & \text{for} \quad t \in [r_0, r_1], \end{cases}$$

and

$$\psi_n(z) = \begin{cases} \psi_{n-1}(z) & \text{for} \quad z \in [I\setminus ((r_0, r_1) \cup \bar{A}_{n-1})]\times I, \\ \psi_{n-1}\circ \beta\times 1_I \circ G_k \circ (\beta\times 1_I)^{-1}(z) & \text{for} \quad z \in \big([r_0, r_1]\setminus \overline{\beta(A)}\big)\times I. \end{cases}$$

Using (1)–(8) and the properties stated in $(1)_{n-1}$–$(9)_{n-1}$ one can verify that conditions $(1)_n$–$(9)_n$ are satisfied, which completes the construction.

The maps $\varphi_0, \varphi_1, \ldots$ converge to a map $\varphi\colon I \to C(I^2)$ (see $(6)_n$). Observe that

(9) $$\varphi(t) = \varphi_n(t) \quad \text{for} \quad t \in \bar{A}_n \ (\text{see } (1)_n \text{ and } (5)_n),$$

(10) $$(t, 0), (t, 1) \in \varphi(t) \quad \text{for each } t \in I \ (\text{see } (5)_n).$$

Now we shall show that

(11) φ has disjoint values.

Let $r, s \in I$ be different. We may assume that $r < s$. Since $\{t_1, t_2, \ldots\}$ is dense in I there are two different numbers $t_i, t_j \in (r, s)$. For $n > \max\{i, j\}$ we have $\varphi(t_i) = \varphi_n(t_i)$ and $\varphi(t_j) = \varphi_n(t_j)$ by (9) and $(1)_n$. Hence by $(5)_n$ we have

$$\varepsilon = \inf\{\varrho(x, y)\colon x \in \varphi(t_i)\ \&\ y \in \varphi(t_j)\} > 0.$$

Suppose there is a point $z \in \varphi(r) \cap \varphi(s)$. Let U be a closed connected neighborhood of z in I^2 such that $\operatorname{diam} U < \varepsilon$. There is an index $n > \max\{i, j\}$ such that $\varphi_n(r) \cap \cap U \neq \emptyset \neq \varphi_n(s) \cap U$. The continuum $\varphi_n(r) \cup U \cup \varphi_n(s) \subset I^2$ contains $(r, 0)$ and $(s, 0)$ (see $(5)_n$) and the set $\{(r, 0), (s, 0)\}$ separates $\mathring{I}^2$ between $(t_i, 0)$ and $(t_i, 1)$ and also between $(t_j, 0)$ and $(t_j, 1)$. By $(5)_n$ it follows that $\varphi_n(t_i) \cap U \neq \emptyset \neq \varphi_n(t_j) \cap U$. But $\varphi_n(t_i) = \varphi(t_i)$ and $\varphi_n(t_j) = \varphi(t_j)$, which is impossible by the choice of U and ε. This completes the proof of (11).

Next we prove that

(12) $\varphi(t) \underset{\text{top}}{=} \Sigma_v$ for $t \in \bigcup_{n\geq 1} A_n$ and some $v \geq 1$, and $\varphi(t)$ is an arc for $t \notin \bigcup_{n\geq 1} A_n$.

If $t \in \bar{A}_n$, then $\varphi(t) = \varphi_n(t)$ by (9). This implies that $\varphi(t) \underset{\text{top}}{=} \Sigma_v$ for $t \in \bigcup_{n\geq 1} A_n$ and some $v \geq 1$ and $\varphi(t)$ is an arc for $t \in \bigcup_{n\geq 1} \bar{A}_n \setminus \bigcup_{n\geq 1} A_n$ (see $(1)_n$ and $(7)_n$). For $t = 0$ or 1 we have $\varphi(t) = \{0\}\times I$ or $\{1\}\times I$, respectively, by $(1)_n$, $(3)_n$ and $(7)_n$. If $t \in \mathring{I}\setminus \bigcup_{n\geq 1} \bar{A}_n$, then by $(7)_n$ we have

$$\varphi(t) = \lim_n \varphi_n(t) = \lim_n \psi_n(\{t\}\times I).$$

It follows from $(4)_n$ that the sequence $\psi_1|\{t\}\times I, \psi_2|\{t\}\times I, \ldots$ converges to a mapping $\psi\colon \{t\}\times I\to I^2$. Thus $\varphi(t) = \psi(\{t_n\}\times I)$ is a locally connected continuum. By (10), (11) and Lemma 2.6 we infer that $\varphi(t)$ is an arc with endpoints $(t, 0)$ and $(t, 1)$. This completes the proof of (12).

By [11], Lemma 1.2, p. 23 and (∗) the set X_m, $m\geqslant 0$, is a continuum. The promised continuum X is defined as follows

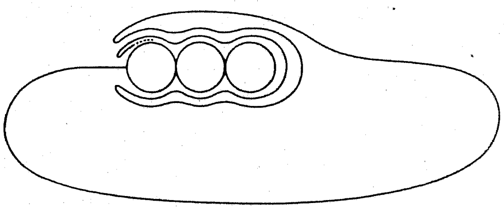
$$X = \bigcap_{m\geqslant 0} X_m \quad (\text{see } (2)_n).$$

By $(9)_n$ and Lemma 3.4 it follows that X is a locally connected continuum. Since $\varphi_n\to\varphi$, condition (∗) implies also that $X = \bigcup_{t\in I}\varphi(t)$. Combining (11) and Lemma 3.1 we infer that there is a monotone open map $f\colon X\to I$ onto I such that

(13) $$f^{-1}(t) = \varphi(t) \quad \text{for each } t\in I.$$

Thus by (12) it remains to show that X is hereditarily decomposable. Suppose X contains an indecomposable continuum Z. Then $f(Z) = [r, s]$ with $r<s$ by (12) and (13). Let $r<r'<s'<s$. Denote by C_τ, $\tau\in T$, the composants of Z such that $C_\tau\neq C_{\tau'}$ for $\tau\neq\tau'$. Since C_τ is dense in Z there is a continuum $C_\tau^*\subset C_\tau$ such that $r', s'\in f(C_\tau^*)$. There is an index n such that $r'<t_n<s'$. It follows that $C_\tau\cap f^{-1}(t_n)\neq\emptyset\neq C_\tau^*\setminus f^{-1}(t_n)$ for each $\tau\in T$. Since $f^{-1}(t_n) = \varphi(t_n) = \varphi_n(t_n)$, then by $(8)_n$ we get $C_\tau^*\cap B_n\neq\emptyset$ for each $\tau\in T$ (because $C_\tau^*\subset Z\subset X\subset X_n$). Since B_n is countable and T is uncountable there are two different $\tau, \tau'\in T$ such that $C_\tau^*\cap C_{\tau'}^*\neq\emptyset$. This is impossible because different composants of Z are disjoint. The final part of the argument shows also that the set $B = \bigcup_{n=1}^{n} B_n$ is a subset of X (see $(8)_n$, (9) and (13)) having the desired properties. This completes the proof.

Let Σ_ν^* denote the union of the continuum Σ_ν and an arc as indicated below.



Σ_3^*

Each continuum Σ_ν^* is arcwise connected.

3.6. Example. There is an hereditarily decomposable locally connected planar continuum X^* admitting an open monotone surjection $f^*\colon X^*\to I$ such that for each $t\in I$ the fiber $(f^*)^{-1}(t)$ is homeomorphic either to the circle or to Σ_ν^* for some $\nu\geqslant 1$ (ν depending on t). Moreover, there is a countable subset B^* of X^* such that each subcontinuum of X^* meeting two distinct fibres of f^* meets B^*.

Proof. We adopt the notation of the preceding example. Note that $\bigcup_{n\geqslant 1} A_n$ is a countable dense subset of I contained in $\mathring{I}$ (see $(1)_n$). Hence there is a homeomorphism $h\colon I\to I$ such that $h(0) = 0$, $h(1) = 1$ and $h(\bigcup_{n\geqslant 1} A_n)\cap\bigcup_{n\geqslant 1} A_n = \emptyset$. Let $X^* = X\times\{0, 1\}/\sim$ be the quotient space under the identification $(t, i, 0)\sim(h(t), i, 1)$ for $t\in I$ and $i = 0, 1$, and let $j\colon X\times\{0, 1\}\to X^*$ be the projection. Let $f^*\colon X^*\to I$ be such that

$$(f^*)^{-1}(t) = j(\varphi(t)\times\{0\}\cup\varphi(h(t))\times\{1\}) \quad \text{for} \quad t\in I.$$

Since $X = \bigcup_{t\in I}\varphi(t)\subset I^2$, it follows from (10), (12) and (13) that f^* has the required properties.

Let $B^* = j(B\times\{0, 1\}\cup(\bigcup_{n\geqslant 1} A_n)\times\{0, 1\}\times\{0\})$. To complete the proof it suffices to show that each subcontinuum E of X^* meeting two distinct fibres of f^* meets B^*.

Let $r_0<r_1$, $r_0, r_1\in I$, be such that $E\cap(f^*)^{-1}(r_i)\neq\emptyset$ for $i = 0, 1$. Since $\bigcup_{\geqslant 1} A_n$ is dense in I, there is $r\in\bigcup_{n\geqslant 1} A_n$ such that $r_0<r<r_1$. It follows that $E\cap(f^*)^{-1}(r)\neq\emptyset$. Let E^* be a component of $E\cap(f^*)^{-1}(r)$. If E^* meets the two-point subset $j(\{r, 0, 0), (r, 1, 0)\})$ of B^*, then we are done. Otherwise E^* is a subset of $X^*\setminus j(X\times\{i\})$ for $i = 0$ or $i = 1$, say $E^*\subset X^*\setminus j(X\times\{1\})$. Let V be a neigborhood of E^* in X^* such that $\overline{V}\cap j(X\times\{1\}) = \emptyset$ and let S be the component of $E\cap\overline{V}$ containing E^*. Since E is a continuum and meets two distinct fibres of f^*, the same holds for S. Hence S is a subcontinuum of $j(X\times\{0\})$ meeting two different sets of the form $j(\varphi(s)\times\{0\})$. Referring to the properties formulated in the preceding example we conclude that S meets $j(B\times\{0\})$. Hence E meets B^* because $S\subset E$. In the other possible case where $E^*\subset X^*\setminus j(X\times\{0\})$ we obtain the same conclusion using an analogous argument.

This completes the proof.

3.7. Remark. It is interesting to compare the locally connected hereditarily decomposable continua constructed in Examples 3.5 and 3.6 with the universal Sierpiński curve. Observe that they are obtained from the 2-sphere S^2 by removing from S^2 a specific null-sequence of disjoint open discs whose union is dense in S^2. Note also that the boundaries of certain pairs of the removed discs intersect. This property can not be avoided in any similar construction leading to such examples. For otherwise. by a result of Whyburn [18], we would obtain by such a procedure a curve homeomorphic to the Sierpiński curve (hence non-hereditarily decomposable).

In connection with Lemma 2.5 we have the following.

Problem. Suppose $f\colon X\to Y$ is a continuous monotone and open surjection from a planar continuum X onto Y such that $f^{-1}(y)$ is decomposable for each $y \in Y$. Must Y be (hereditarily) locally connected?

One can prove that Y is hereditarily decomposable. It would be interesting to know what is a characterization of the continua Y in terms of intrinsic properties.

Added in proof. The answer to the problem is affirmative: E. Dyer, *Continuous collections of decomposable continua*, Proc. Amer. Math. Soc. 6 (1955), pp. 351–360. Moreover, one can prove that Y must be regular.

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$F_{\sigma\delta}$-sections of Borel sets

by

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Abstract. It is shown that if E, F are compact metric spaces and A is a Borel subset of $E\times F$, then $\{x \in E\colon A(x) \text{ is } F_{\sigma\delta} \text{ in } F\}$ is coanalytic in E.

Introduction. Throughout this paper, E and F are compact metric spaces. If A is a subset of $E\times F$ and $x\in E$, let $A(x) = \{y\in F;\ (x,y)\in A\}$, which is called a *section of* A. It is already known that if A is Borel in $E\times F$, then $\{x\in E;\ A(x)$ is closed in $F\}$ and $\{x\in E;\ A(x)$ is F_σ in $F\}$ are coanalytic. I refer for instance to [1] and [4]. It follows from a result in my recent paper [2] that the set $\{x\in E;\ A(x)$ is $F_{\sigma\delta}$ in $F\}$ is a universally measurable subset of E. We will obtain here the following refinement:

THEOREM 1. *If A is Borel in $E\times F$, then $\{x\in E;\ A(x)$ is $F_{\sigma\delta}$ in $F\}$ is coanalytic in E.*

The main point in the proof of this result is a useful description of the fact that a set in F is $F_{\sigma\delta}$.

If L is a compact metric space, then $\underline{\underline{K}}(L)$ consists of all closed subsets of L and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [4]).

LEMMA 2. *Let P be a Polish subspace of the compact metric space L. Then the subspace $\underline{\underline{F}}(P)$ of $\underline{\underline{K}}(L)$ consisting of those compact sets K in L such that $K = \overline{K\cap P}$, is Polish.*

We denote by $\mathscr{R}$ the set of all finite complexes c in $\bigcup_k N^k$, where $N^0 = \{\emptyset\}$.

PROPOSITION 3. *Let A be Borel in F and $B = F\setminus A$. There is a compact metric space G and a G_δ subset H of $F\times G$ so that $A = \pi(H)$, if $\pi\colon F\times G\to F$ is the projection. Let $B = \bigcup_\nu\bigcap_k B_{\nu|k}$ be an analytical representation of B, where the $B_{\nu|k}$ are closed in F and $B_{\nu|k+1}\subset B_{\nu|k}$. Take a countable base $(U_n)_n$ for the topology of $F\times G$.*

Then A is not $F_{\sigma\delta}$ in F if and only if there exists $(p_c, K_c)_{c\in\mathscr{R}}$ in $\prod_{c\in\mathscr{R}}(N\times\underline{\underline{F}}(H))$ satisfying:

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