Fixed points and locally connected cyclic continua in E^3

by

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Abstract. In this paper is given an example of a locally connected continuum $Y \subset E^3$ such that Y separates E^3 and has the fixed point property.

It is well known that a planar locally connected acyclic continuum has the fixed point property. On the other hand, each locally connected continuum separating the plane admits a fixed point free mapping. The case differs in E^3. There is an example of a locally connected acyclic continuum contained in E^3 without the fixed point property (see [1]). K. Borsuk posed the problem whether there exists a locally connected continuum $Y \subset E^3$ which separates E^3 and has the fixed point property (see also [5] Problem 7. p. 68). The aim of this paper is to give such an example. The construction gives also an example of a locally connected and acyclic continuum lying in E^3 and containing a simple closed curve which is not contractible in it.

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1. Combinatorical preliminaries. In this section we introduce some notions which are needed later.

Let G be a free group with two generators, a and b. Let e denote the neutral element of G. Denote also $a_1 = bab^{-1}$ and $b_1 = aba^{-1}$. Let T be the set $\{a, a^{-1}, a_1, a_1^{-1}, b, b^{-1}, b_1, b_1^{-1}\}$. Define a function $i\colon T^2 \to \{-1, 0, 1\}$ by the formula

$$i(c, d) = \begin{cases} 1 & \text{if} \quad c = d, \\ -1 & \text{if} \quad c = d^{-1}, \\ 0 & \text{if} \quad c \neq d \neq c^{-1}. \end{cases}$$

Denote $i_a(c) = i(a, c) + i(a_1, c)$ and $i_b(c) = i(b, c) + i(b_1, c)$ for $c \in T$.

Let $\varkappa$ be a function from $\{1, 2, \ldots, n\}$ into T. Define an integer $\tau(\varkappa)$ as follows:

$$\tau(\varkappa) = \sum_{j=1}^{n-1} i_b(\varkappa(j)) \sum_{l=j+1}^{n} i_a(\varkappa(l)).$$

1.1. Lemma. *Let* $\varkappa$ *be a function from* $\{1,2,\ldots,n\}$ *into* T, *such that for some* r $(2\leqslant r\leqslant n-1)$ *one of the following conditions holds*:

(i) $\varkappa(r-1)=a$, $\varkappa(r)=b$ *and* $\varkappa(r+1)=a^{-1}$,

(ii) $\varkappa(r-1)=a$, $\varkappa(r)=b^{-1}$ *and* $\varkappa(r+1)=a^{-1}$,

(iii) $\varkappa(r-1)=b$, $\varkappa(r)=a$ *and* $\varkappa(r+1)=b^{-1}$,

(iv) $\varkappa(r-1)=b$, $\varkappa(r)=a^{-1}$ *and* $\varkappa(r+1)=b^{-1}$.

Let γ *be a function from* $\{1,2,\ldots,n-2\}$ *into* T *defined as follows*:

$$\gamma(j)=\begin{cases}\varkappa(j) & \text{for } j=1,2,\ldots,r-2,\\ \varkappa(j+2) & \text{for } j=r,r+1,\ldots,n-2\end{cases}$$

and

$$\gamma(r-1)=\begin{cases}b_1 & \text{in case (i)},\\ b_1^{-1} & \text{in case (ii)},\\ a_1 & \text{in case (iii)},\\ a_1^{-1} & \text{in case (iv)}.\end{cases}$$

Then

$$\tau(\gamma)-\tau(\varkappa)=\begin{cases}1 & \text{in case (i)},\\ -1 & \text{in case (ii)},\\ -1 & \text{in case (iii)},\\ 1 & \text{in case (iv)}.\end{cases}$$

Proof. Case (i).

$$\begin{aligned}\tau(\varkappa)&=\sum_{j=1}^{n-1}i_b(\varkappa(j))\sum_{l=j+1}^{n}i_a(\varkappa(l))\\ &=\sum_{j=1}^{r-2}i_b(\varkappa(j))\Big[\sum_{l=j+1}^{r-2}i_a(\varkappa(l))+0+\sum_{l=r+2}^{n}i_a(\varkappa(l))\Big]+\\ &\quad+\sum_{l=r+1}^{n}i_a(\varkappa(l))+\sum_{j=r+2}^{n-1}i_b(\varkappa(j))\sum_{l=j+1}^{n}i_a(\varkappa(l))\\ &=\sum_{j=1}^{r-2}i_b(\gamma(j))\Big[\sum_{l=j+1}^{r-1}i_a(\gamma(l))+\sum_{l=r}^{n-2}i_a(\gamma(l))\Big]-1+\\ &\quad+\sum_{l=r}^{n-2}i_a(\gamma(l))+\sum_{j=r}^{n-3}i_b(\gamma(j))\sum_{l=j+1}^{n-2}i_a(\gamma(l))\\ &=-1+\sum_{j=1}^{r-2}i_b(\gamma(j))\sum_{l=j+1}^{n-2}i_a(\gamma(l))+i_b(\gamma(r-1))\sum_{l=r}^{n-2}i_a(\gamma(l))+\\ &\quad+\sum_{j=r}^{n-3}i_b(\gamma(j))\sum_{l=j+1}^{n-2}i_a(\gamma(l))\\ &=\tau(\gamma)-1.\end{aligned}$$

The other cases are proved similarly.

1.2. Lemma. *Let* $\varkappa$ *be a function from* $\{1,2,\ldots,n\}$ *into* T. *Suppose that there are integers* r *and* s *such that* $1\leqslant r<s\leqslant n$ *and either*

(i) $\varkappa(j)\in\{a,a^{-1},a_1,a_1^{-1}\}$ *for* $r\leqslant j\leqslant s$ *and* $\sum_{j=r}^{s}i_a(\varkappa(j))=0$ *or*

(ii) $\varkappa(j)\in\{b,b^{-1},b_1,b_1^{-1}\}$ *for* $r\leqslant j\leqslant s$ *and* $\sum_{j=r}^{s}i_b(\varkappa(j))=0$.

Let γ *be a function from* $\{1,2,\ldots,n-s+r-1\}$ *into* T *defined by the formula*

$$\gamma(j)=\begin{cases}\varkappa(j) & \text{for } 1\leqslant j<r,\\ \varkappa(j+s-r+1) & \text{for } r\leqslant j\leqslant n-s+r-1.\end{cases}$$

Then $\tau(\varkappa)=\tau(\gamma)$.

Proof. Case (i).

$$\begin{aligned}\tau(\varkappa)&=\sum_{j=1}^{n-1}i_b(\varkappa(j))\sum_{l=j+1}^{n}i_a(\varkappa(l))\\ &=\sum_{j=1}^{r-1}i_b(\varkappa(j))\Big[\sum_{l=j+1}^{r-1}i_a(\varkappa(l))+0+\sum_{l=s+1}^{n}i_a(\varkappa(l))\Big]+0+\\ &\quad+\sum_{j=s+1}^{n-1}i_b(\varkappa(j))\sum_{l=j+1}^{n}i_a(\varkappa(l))\\ &=\tau(\gamma).\end{aligned}$$

Case (ii).

$$\begin{aligned}\tau(\varkappa)&=\sum_{j=1}^{r-1}i_b(\varkappa(j))\Big[\sum_{l=j+1}^{r-1}i_a(\varkappa(l))+0+\sum_{l=s+1}^{n}i_a(\varkappa(l))\Big]+\\ &\quad+\sum_{j=r}^{s}i_b(\varkappa(j))\Big[0+\sum_{l=s+1}^{n}i_a(\varkappa(l))\Big]+\sum_{j=s+1}^{n-1}i_b(\varkappa(j))\sum_{l=j+1}^{n}i_a(\varkappa(l))\\ &=\sum_{j=1}^{r-1}i_b(\gamma(j))\sum_{l=j+1}^{n-s+r-1}i_a(\gamma(l))+0+\sum_{j=r}^{n-s+r-2}i_b(\gamma(j))\sum_{l=j+1}^{n-s+r-1}i_a(\gamma(j))\\ &=\tau(\gamma).\end{aligned}$$

1.3. Lemma. *Let* $\varkappa$ *be a function from* $\{1,2,\ldots,n\}$ *into* T *such that*

$$e=\varkappa(1)\varkappa(2)\ldots\varkappa(n).$$

Then

$$\tau(\varkappa)=\sum_{j=1}^{n}i(b_1,\varkappa(j))-\sum_{j=1}^{n}i(a_1,\varkappa(j)).$$

Proof. In the case where $\varkappa(j)\in\{a,a^{-1},b,b^{-1}\}$ for all $j=1,\ldots,n$, one can prove using 1.2 that $\tau(\varkappa)=0$. Thus the lemma follows from 1.1.

1.4. Lemma. *Let* $\varkappa$ *be a function from* $\{1,2,\ldots,n\}$ *into* T *such that* $\tau(\varkappa)\neq 0$. *Suppose that* $N_1,N_2,\ldots,N_l$ *are mutually disjoint subsets of* $\{1,2,\ldots,n\}$ *such that* $\bigcup_{s=1}^{l}N_s=\{1,2,\ldots,n\}$ *and there is a number* l_0 $(1\leqslant l_0<l)$ *such that*

(i) $\varkappa(j)\in\{a,a^{-1},a_1,a_1^{-1}\}$ *for* $j\in N_s$ *and* $\sum_{j\in N_s}i_a(\varkappa(j))=0$ *for* $1\leqslant s\leqslant l_0$, *and*

(ii) $\varkappa(j)\in\{b,b^{-1},b_1,b_1^{-1}\}$ *for* $j\in N_s$ *and* $\sum_{j\in N_s}i_b(\varkappa(j))=0$ *for* $l_0<s\leqslant l$.

Then there are natural numbers s_1, s_2, j_1, j_2, j_3 and j_4 such that $1 \leq s_1 \leq l_0$, $l_0 < s_2 \leq l$, $1 \leq j_1 < j_2 < j_3 < j_4 \leq n$, and either $j_1, j_3 \in N_{s_2}$ and $j_2, j_4 \in N_{s_1}$ or $j_1, j_3 \in N_{s_1}$ and $j_2, j_4 \in N_{s_2}$.

Proof. Suppose that the lemma fails. Let n be the smallest natural number satisfying the assumption but not satisfying the conclusion of the lemma.

There are $l_1, l_2, \ldots, l_r$ such that, either $1 \leq l_s \leq l_0$ for all $s = 1, 2, \ldots, r$ or $l_0 < l_s \leq l$ for all $s = 1, 2, \ldots, r$, and there are j_1 and j_2 such that $1 \leq j_1 < j_2 \leq n$ and $\bigcup_{s=1}^{r} N_{l_s} = \{j_1, j_1+1, \ldots, j_2\}$.

Let γ be a function from $\{1, 2, \ldots, n-j_2+j_1-1\}$ into T defined by the formula

$$\gamma(j) = \begin{cases} \varkappa(j) & \text{for} \quad 1 \leq j < j_1, \\ \varkappa(j+j_2-j_1+1) & \text{for} \quad j_1 \leq j \leq n-j_2+j_1-1. \end{cases}$$

By 1.2 $\tau(\varkappa) = \tau(\gamma)$. Now from our supposition the lemma follows.

2. Perforated discs. Let C be a continuum lying in the plane E^2. By $\hat{C}$ we denote the union of C and of all bounded components of $E^2 - C$. By $\dot{C}$ we denote the boundary of C in E^2.

Let $C \subset E^2$ be a simple closed curve. By the orientation $+1$ of C we mean the clock-wise orientation. By the orientation -1 we mean the opposite one. Note that the orientation of C determines the orientation on any subarc of C.

2.1. Proposition. *Let C_1 and C_2 be two simple closed curves with the orientation $+1$. Let $I \subset C_1 \cap C_2$ be an arc. If $\hat{C}_1 \subset \hat{C}_2$, then the orientations of I determined by C_1 and C_2 agree.*

If $\hat{C}_1 \cap \hat{C}_2 = C_1 \cap C_2$, then the orientations of I determined by C_1 and C_2 are opposite.

By a perforated disc with n-holes ($n = 0, 1, \ldots$) we mean a two-dimensional continuum $F \subset E^2$ such that $\dot{F}$ is the union of $n+1$ mutually disjoint simple closed curves.

If $C \subset \dot{F}$ is a simple closed curve, then we say that F determines the orientation $+1$ on C if $F \subset \hat{C}$, and the orientation -1 in the opposite case.

2.2. Proposition. *Let F_1 and F_2 be two perforated discs. Let I be an arc contained in $\dot{F}_1 \cap \dot{F}_2$. If $F_1 \subset F_2$, then the orientations on I determined by F_1 and F_2 are identical. If $\dot{F}_1 \cap \dot{F}_2 = F_1 \cap F_2$ then the orientations on I determined by F_1 and F_2 are opposite.*

Let L be a one-dimensional polyhedron, and let $K \subset L$ be an oriented simple closed curve such that $K \cap \overline{L-K}$ is void or consists of a single point p. Denote by r the retraction of L onto K with $r(L-K) = \{p\}$. Let C be another oriented simple closed curve and let f be a mapping of C into L. By $w(f, C, K)$ we denote the integer j such that $(rf)_*(c) = k^j$, where c and k are generators of the fundamental groups of C and K, respectively, determining the chosen orientation of C and K.

Note that $w(f, C, K)$ is equal to the number of oriented components of $f^{-1}(K-\{p\})$ which f maps onto $K-\{p\}$ and preserves the orientations minus the number of the remaining components of $f^{-1}(K-\{p\})$ which are mapped onto $K-\{p\}$.

If F is a union of a finite number of mutually disjoint perforated discs and $f\colon \dot{F} \to L$, then by $w(f, F, K)$ we denote the sum of $w(f_{|C}, C, K)$ where C runs over all components C of $\dot{F}$ with orientations determined by the component of F which contains C ($f_{|C}$ denotes f restricted to C).

2.3. Proposition. *Let F be a perforated disc and let f_0 be a mapping of $\dot{F}$ into an oriented simple closed curve K. Then $w(f_0, F, K) = 0$ if and only if there exists a mapping $f\colon F \to K$ such that $f_{|\dot{F}} = f_0$.*

2.4. Proposition. *Let F_1 and F_2 be two plane sets such that*

(i) *each of the sets F_1, F_2 and $F_1 \cup F_2$ is a union of a finite number of mutually disjoint perforated discs, and*

(ii) $F_1 \cap F_2 = \dot{F}_1 \cap \dot{F}_2$.

Let f be a mapping of $\dot{F}_1 \cup \dot{F}_2$ into a one-dimensional polyhedron L containing an oriented simple closed curve K as in the preceding definition. Then $w(f, F_1, K) + w(f, F_2, K) = w(f, F_1 \cup F_2, K)$.

We finish the section by the following

2.5. Lemma. *Let F_0 be a disc and let F be a perforated disc. Suppose $f\colon F \to F_0$ is a mapping such that $f(\dot{F}) \subset \dot{F}_0$ and $w(f, F, \dot{F}_0) = 0$. Then there is a homotopy $H\colon F \times [0, 1] \to F_0$ such that*

(i) $H(u, 0) = f(u)$ *for* $u \in F$,

(ii) $H(v, t) = f(v)$ *for* $v \in \dot{F}$ *and* $t \in [0, 1]$ *and*

(iii) $H(u, 1) \in \dot{F}_0$ *for* $u \in F$.

Proof. By 2.3 there is a mapping $g\colon F \to \dot{F}_0$ such that $g_{|\dot{F}} = f_{|\dot{F}}$. Define $g_1\colon (F \times \{0\}) \cup (\dot{F} \times [0, 1]) \cup (F \times \{1\}) \to F_0$ by the formula

$$g_1(v, t) = \begin{cases} f(v) & \text{for} \quad (v, t) \in (F \times \{0\} \cup (\dot{F} \times [0, 1]), \\ g(v) & \text{for} \quad t = 1. \end{cases}$$

An extension of g_1 over the whole of $F \times [0, 1]$ has the desired properties.

3. Basic constructions. In this section we construct a locally connected acyclic continuum $X \subset E^3$. In Section 4 we shall show that X contains a simple closed curve A_0^1 which is not contractible in X. The continuum Y is the union of X and a disc D meeting X at A_0^1. In Section 5 we shall prove that Y has the fixed point property. In this section we shall only show that X has the fixed point property and Y satisfies the Lefschetz fixed point theorem.

The notation established in this section will be used freely throughout the rest of this paper.

Let S_1 and S_2 be two mutually disjoint circles. Fix two points, $z_1 \in S_1$ and $z_2 \in S_2$. Let Z_ν ($\nu = 1, 2$) denote $S_\nu \times [-2, -1]$ with points $(z_\nu, -2)$ and $(z_\nu, -1)$ identified. Attach Z_1 to Z_2 by homeomorphisms sending $\{z_1\} \times [-2, -1]$ onto

$S_2\times\{-2\}$ and $S_1\times\{-1\}$ onto $z_2\times[-2,-1]$. Denote the resulting space by U. One can assume that $Z_1\subset U$ and $Z_2\subset U$. There is an embedding of U into the Cartesian three-space E^3 such that

$$U\cap\{(x,y,z)\in E^3;\ z=0,\ x\geqslant 0,\ y\geqslant 0\}=S_1\times\{-1\},$$
$$U\cap\{(x,y,z)\in E^3;\ z=0,\ x\geqslant 0,\ y\leqslant 0\}=S_1\times\{-2\},$$
$$U\cap\{(x,y,z)\in E^3;\ z=0,\ x\leqslant 0,\ y\leqslant 0\}=S_2\times\{-2\},$$
$$U\cap\{(x,y,z)\in E^3;\ z=0,\ x\leqslant 0,\ y\geqslant 0\}=S_2\times\{-1\}$$

(comp. Fig. 1). Observe that U has the same homotopy type as one-point union of two circles, namely $S_1\times\{-1\}$ and $S_2\times\{-2\}$.

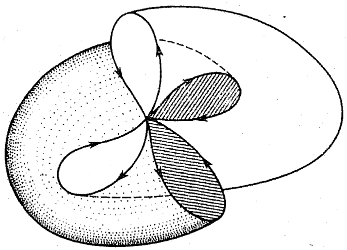

Fig. 1

Note that any orientation on S_ν $(\nu=1,2)$ determines, by the natural projection, orientations on $S_\nu\times\{-1\}$ and $S_\nu\times\{-2\}$. There are orientations on S_1 and S_2 such that if a, a_1, b and b_1 are elements of $\pi_1(U)$ corresponding, respectively, to the orientations on $S_1\times\{-1\}$, $S_1\times\{-2\}$, $S_2\times\{-2\}$ and $S_2\times\{-1\}$, then $a_1=bab^{-1}$ and $b_1=aba^{-1}$ (see Fig. 1). Fix these orientations on S_1 and S_2.

Let N denote the set natural numbers (including 0). In the set $(S_1\times[0,1]\times N)\cup$ $\cup(S_2\times[0,1]\times N)\cup(U\times N)$, for each $n\in N$ and $z\in S_1\cup S_2$ identify points $(z,-2,n)$ and $(z,1,n)$, $(z,-1,n)$ and $(z,0,n+1)$. Observe that the resulting space is a tangle of two canals, say C^1 and C^2. Denote this space by $C^1\cup C^2$. Points of $C^1\cup C^2$ we shall denote as points of $(S_1\times[0,1]\times N)\cup(S_2\times[0,1]\times N)\cup(U\times N)$.

For each $n\in N$ and $\nu=1,2$ let us adopt the following notation:

$$C_n^3=U\times\{n\},$$
$$Z_n^\nu=Z_\nu\times\{n\},$$
$$C_n^\nu=S_\nu\times[0,1]\times\{n\},$$
$$A_n^\nu=S_\nu\times\{0\}\times\{n\},$$
$$B_n^\nu=S_\nu\times\{1\}\times\{n\},$$

$$L_n^\nu=\{z_\nu\}\times[0,1]\times\{n\},$$
$$P_n=(z_1,n)=(z_2,n).$$

In this notation $C^\nu=\bigcup_{n=0}^{\infty}(C_n^\nu\cup Z_n^\nu)$.

Note that the fixed orientation on S_ν determines by the natural projection the orientations on A_n^ν and B_n^ν.

Let $r_1\colon [0,\infty)\to(-\infty,\infty)$ be defined by the formula

$$r_1(t)=\begin{cases}\dfrac{t+2}{t+1}(-1+\sin t) & \text{for}\quad 2n\pi\leqslant t\leqslant(2n+1)\pi,\ n\in N,\\[2mm] \dfrac{t+2}{t+1}(-1-2\sin t) & \text{for}\quad (2n+1)\pi\leqslant t\leqslant(2n+2)\pi,\ n\in N.\end{cases}$$

Define the function $\varphi_1\colon [0,\infty)\to E^3$ by the formula

$$\varphi_1(t)=\left(r_1(t),\ \frac{\sin t}{t+1},\ \frac{1}{t+1}\right).$$

Figure 2 shows the result of the projection of $\varphi_1([0,\infty))$ into the plane $\{(x,y,z)\in E^3;\ z=0\}$.

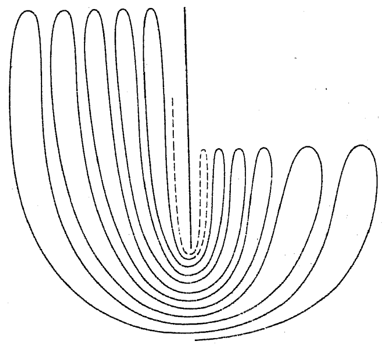

Fig. 2

Let $r_2\colon [0,\infty)\to(-\infty,\infty)$ be defined by the formula

$$r_2(t)=\tfrac{1}{2}(\sin(t+\pi)+1).$$

Define $\varphi_2\colon [0,\infty)\to E^3$ as follows:

$$\varphi_2(t)=\left(r_2(t),\ \frac{1}{t+1},\ \frac{1}{t+1}\right).$$

Figure 3 shows the image of the projection of $\varphi_1([0, \infty)) \cup \varphi_2([0, \infty))$ into the plane $\{(x, y, z) \in E^3;\ z = 0\}$.

Let $J_\nu(t)$ $(\nu = 1, 2)$ denote the straight segment between points $(r_\nu(t), 0, 0)$ and $\varphi_\nu(t)$. If $(\nu, t) \neq (\nu', t')$, then the intersection of $J_\nu(t)$ and $J_{\nu'}(t')$ is either void or consists of a single point $(r_\nu(t), 0, 0) = (r_{\nu'}(t'), 0, 0)$.

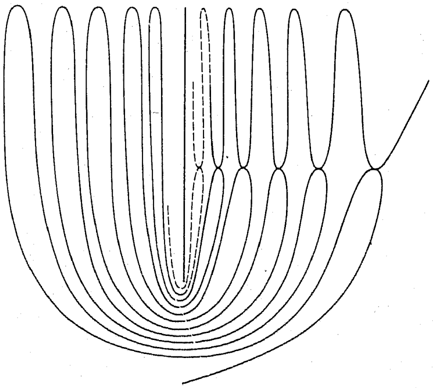

Fig. 3

Let us adopt the following notation: $P_\nu = \bigcup \{J_\nu(t);\ t \in [0, \infty)\}$ for $\nu = 1, 2$, and $J = \{(x, y, z) \in E^3;\ -1 \leqslant x \leqslant 1,\ y = z = 0\}$. There is an embedding of $C^1 \cup C^2$ into E^2 such that

1. $L_0^\nu = \varphi_\nu([0, \frac{1}{2}\pi])$ for $\nu = 1, 2$,
2. $L_n^\nu = \varphi_\nu([\frac{1}{2}\pi + 2(n-1)\pi, \frac{1}{2}\pi + 2n\pi])$ for $\nu = 1, 2$ and $n = 1, 2, \ldots$
3. $(P_1 \cup P_2) \cap (C^1 \cup C^2) = \varphi_1([0, \infty)) \cup \varphi_2([0, \infty))$,
4. $\operatorname{diam} S_\nu \times \{t\} \times \{n\} < 2^{-n}$ for $\nu = 1, 2$, $n = 1, 2, \ldots$ and $t \in [-2, -1] \cup [0, 1]$,
5. the sets $\{(x, y, z) \in C_n^1;\ x = -\frac{1}{2}\}$ and $\{(x, y, z) \in C_n^2;\ x = \frac{1}{2}\}$ are both unions of two disjoint circles, for $n \in N$.

Denote $P_1 \cup P_2 \cup C^1 \cup C^2$ by X. A schema of X is illustrated in Figure 4. Denote also by C_n^4 the component of $(C^1 \cup C^2) \cap \{(x, y, z) \in X;\ -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}\}$ containing C_n^3. Note that C_n^4 is of the homotopy type of C_n^3, so C_n^4 is of the homotopy type of a one-point union of two circles.

Let D be a disc in E^3 such that A_0^1 is its boundary and $D \cap X = A_0^1$. Denote $X \cup D$ by Y. Observe that X and Y are both locally connected, X is acyclic and Y separates E^3 into two components.

3.1. Lemma. *X and Y are* QANR-*spaces* (see [4], comp. [6]).

Proof. Since $\overline{\varphi_2([0, \infty))}$ is homeomorphic to the $(\sin(1/x))$-curve (the closure of $\{(x, y) \in E^2;\ y = \sin(1/x),\ 0 < x \leqslant 1\}$), $\overline{\varphi_2([0, \infty))} \in$ QANR (see [6]). If we identify points $(\frac{1}{2} - x, 0)$ and $(\frac{1}{2} + x, 0)$ for $0 \leqslant x \leqslant \frac{1}{2}$, in the $(\sin(1/x)$-curve we obtain a continuum homeomorphic to $\overline{\varphi_1([0, \infty))}$. Hence, by [4, 4.3], $\overline{\varphi_1([0, \infty))} \in$ QANR. Let F_ν $(\nu = 1, 2)$ be a continuum obtained from $\overline{\varphi_\nu[0, \infty)} \times S_1$ by the identification of each $\{x\} \times S_1$ with a point, where $x \in J$.

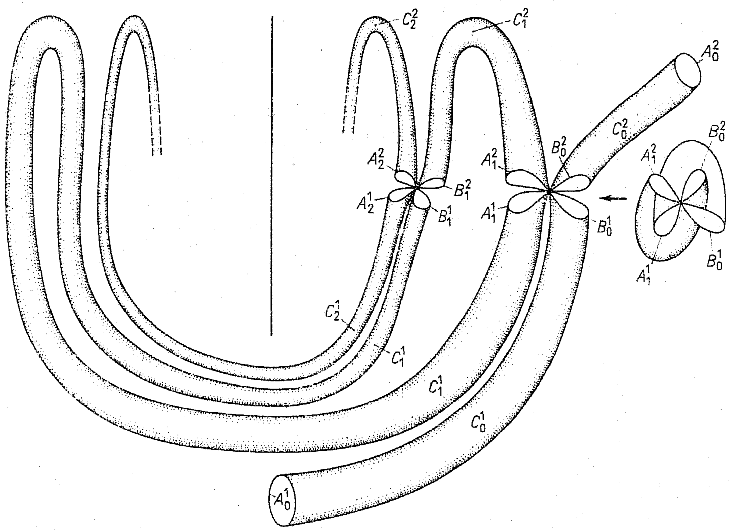

Fig. 4

By [4, 4.1 and 4.3] we get $F_\nu \in$ QANR.

Let D' be a disc in E^3 such that A_0^2 is its boundary and $D' \cap Y = A_0^2$. Let $\hat{Y}$ be the union of $Y \cup D'$ and of the two bounded components of $E^3 - (Y \cup D')$. It is easy to see that Y is an ANR-space. There is a neighbourhood V of $X - J$ in $\hat{Y} - J$ such that X is a retract of $\overline{V}$ and the boundary of V in $\hat{Y} - J$ consists of two components, K_1 and K_2, such that $\overline{K}_\nu$ is homeomorphic to F_ν $(\nu = 1, 2)$. By [4, 4.3], $K_1 \cup K_2 \cup J \in$ QANR. By [4, 2.2], there are a neighbourhood V_1 of $K_1 \cup K_2 \cup J$ in $\hat{Y}$ and a quasi-deformation H of V_1 onto $K_1 \cup K_2 \cup J$ in $\hat{Y}$. It is easy to see that $H_1\colon (V_1 \cup V) \times [0, \infty) \to \hat{Y}$ defined by the formula

$$H_1(v, t) = \begin{cases} v & \text{for } v \in V, \\ H(v, t) & \text{for } v \in V_1 - V \end{cases}$$

is a quasi-deformation of V_1 to $\overline{V}$ in $\hat{Y}$. Hence $\overline{V} \in$ QANR. Since X is a retract of $\overline{V}$, $X \in$ QANR (see [4, 4.7]). Again by [4, 4.3] we infer that $Y \in$ QANR.

By [4, 3.1] we get the following

3.2. COROLLARY. *X has the fixed point property.*

For a mapping $f\colon Y\to Y$, $\Lambda(f)$ denote the Lefschetz number of f, i.e. $\Lambda(f) = \sum_{j\geqslant 0}(-1)^j\operatorname{Tr}(f_{*j})$, where $\operatorname{Tr}(f_{*j})$ is the trace of the homomorphism f_{*j} induced by f on the jth (Vietoris or Čech) homology group of Y, over the rationals. Again by [4, 3.1] we get the following

3.3. COROLLARY. *Y satisfies the Lefschetz fixed point theorem, i.e. each mapping $f\colon Y\to Y$ with $\Lambda(f)\neq 0$ has a fixed point.*

4. Auxiliary lemmas. In this section we prove the most significant Lemma 4.4 of this paper. As a corollary we conclude that A_0^1 is not contractible in X. The following three lemmas are needed in the proof of 4.4.

4.1. LEMMA. *Let δ be a positive real number. For $\nu = 1, 2$ and for $n = 1, 2, \ldots$ let Q_n^ν be an arc with endpoints c_n^ν and d_n^ν, contained in the plane. Suppose that the following conditions are fulfilled for all $n\geqslant 1$:*

(i) $(Q_n^1\cup Q_n^2)\cap\bigcup_{j\neq n}(Q_j^1\cup Q_j^2) = \emptyset$,

(ii) $\operatorname{dist}(c_n^1, Q_n^2)>\delta$, $\operatorname{dist}(d_n^1, Q_n^2)>\delta$, $\operatorname{dist}(c_n^2, Q_n^1)>\delta$ *and* $\operatorname{dist}(d_n^2, Q_n^1)>\delta$,

(iii) $Q_n^1\cap Q_n^2\neq\emptyset$ *and*

(iv) *there is a simple closed curve $K_n\subset E^2$ such that*

a. *$K_n\cap Q_n^\nu$ consists of two points, e_n^ν and $e_n^{\nu+2}$, for $\nu = 1, 2$,*

b. *points e_n^1 and e_n^3 separate K_n between e_n^2 and e_n^4, and*

c. *sets c_n^ν, d_n^ν and $Q_n^1\cap Q_n^2$ are contained in three distinct components of $Q_n^\nu-\{e_n^\nu, e_n^{\nu+2}\}$ for $\nu = 1, 2$.*

Then the set $\bigcup_{n=1}^{\infty} Q_n^1$ is unbounded.

Proof. We claim that if C is a continuum such that $c_n^1, d_n^1\in C$ and $Q_n^2\cap C = \emptyset$ for some n, then $C\cup Q_n^1$ separates the plane between c_n^2 and d_n^2.

Suppose that this is not true. Let $L\subset Q_n^2-\{e_n^2, e_n^4\}$ be an arc containing $Q_n^1\cap Q_n^2$. Since sets $C\cup Q_n^1$ and $Q_n^1\cup L$ do not separate the plane between c_n^2 and d_n^2, and $(C\cup Q_n^1)\cap(Q_n^1\cup L) = Q_n^1$, the set $C\cup Q_n^1\cup L$ does not separate the plane between c_n^2 and d_n^2, either (see [3, § 61, I, Th. 7]). Let g be the canonical mapping of E^2 onto the quotient space of E^2 decomposed onto L and single points. By [3, § 61, IV Th 8], $g(E^2)$ is homeomorphic to the plane.

Let $Q\subset g(Q_n^1)$ be an arc with endpoints $g(c_n^1)$ and $g(d_n^1)$. Observe that $g(Q_n^2)\cap Q = g(L)$ is a single point and, by (iv), Q cuts sufficiently small neighbourhoods of $g(L)$ onto two components intersecting $g(Q_n^2)$. Thus points $g(c_n^2)$ and $g(d_n^2)$ belong to two different components of $g(E^2)-(Q\cup g(C))$, but this is impossible, because $Q_n^1\cup C\cup L$ does not separate the plane between these points. This contradiction proves the claim.

Now, suppose that the lemma fails. Choosing a convergent subsequence of (Q_n^1) in the hyperspace of a disc containing $\bigcup_{n=1}^{\infty} Q_n^1$, one can assume without loss of generality that $Q_n^1\subset B(Q_j^1, \frac{1}{2}\delta)$ for all n and j, where $B(Q_j^1, \frac{1}{2}\delta)$ denotes $\frac{1}{2}\delta$-ball around Q_j^1. By (ii) we get

1. $\left(B(c_n^2, \frac{1}{2}\delta)\cup B(d_n^2, \frac{1}{2}\delta)\right)\cap\bigcup_{j=1}^{\infty} Q_j^1 = \emptyset$

Similary, choosing convergent subsequences of (c_n^1) and (d_n^1), one can assume that there are two $\frac{1}{2}\delta$-balls B_1 and B_2 such that $c_n^1\in B_1$ and $d_n^1\in B_2$ for all n. By (ii) we infer

2. $Q_n^2\cap(B_1\cup B_2) = \emptyset$.

Let G_1 be a component of $E^2-(B_1\cup B_2\cup Q_1^1\cup Q_2^1)$ which contains infinitely many Q_n^2's (see 2 and (i)). By the claim, 2 and (i) there is another component, G_1', of $E^2-(B_1\cup B_2\cup Q_1^1\cup Q_2^1)$ which contains c_1^2 or d_1^2. By 1, G_1' contains a $\frac{1}{2}\delta$-ball.

Now, take n_1 such that $Q_{n_1}^2\subset G_1$. Let G_2 be a component of $G_1-Q_{n_1}^1$ which contains infinitely many Q_n^2's. By the claim, 2, (ii) and (iii) there is another component, G_2', of $G_1-Q_{n_1}^1$ which contains $c_{n_1}^2$ or $d_{n_1}^2$. By 2, G_2' contains a $\frac{1}{2}\delta$-ball. Repeating the argument, we construct an infinite sequence of mutually disjoint $\frac{1}{2}\delta$-balls lying in a bounded plane region. This contradiction completes the proof.

Let us prove the following

4.2. LEMMA. *Let F be a perforated disc with k holes, and let $f\colon F\to X$ be a mapping. Then there is an $n_0\in N$ such that for each $n\geqslant n_0$ and for each simple closed curve S contained in $f^{-1}(C_n^4)$, the mapping $f_{|S}$ is homotopic in C_n^4 to a constant map.*

Proof. By the compactness of F there is a $\delta>0$ such that if $\varrho(x_1, x_2)<\delta$ then $\varrho(f(x_1), f(x_2))<\frac{1}{3}$ for each $x_1, x_2\in F$. Let m be a natural number such that there is no family of m mutually disjoint δ-balls contained in F. Suppose that there are $k+m$ naturals $n_1, \ldots, n_{k+m}$ such that $f^{-1}(C_{n_j}^4)$ contains a simple closed curve R_j, with $f_{|R_j}$ not homotopic in $C_{n_j}^4$ to a constant map for $j = 1, 2, \ldots, k+m$. The curves $R_1, \ldots, R_{k+m}$ are mutually disjoint. One can easily see that there are least m components $G_1, \ldots, G_m$ of $F-\bigcup_{j=1}^{k+m} R_j$ such that the boundary (with respect to the plane) of each of them is contained in $\bigcup_{j=1}^{k+m} R_j$. Observe that $f(G_j)-\{(x, y, z)\in E^3;\ |x|>\frac{5}{6}\}$ is nonvoid. Hence each set G_j contains a ϱ-ball. This contradiction completes the proof.

4.3. LEMMA. *Let F be a perforated disc and let $\tilde{f}\colon F\to X$ be a mapping such that $\tilde{f}(\dot{F})\subset A_0^1$. Then there exist a mapping f and a sequence $F_0, F_1, \ldots$ of subsets of F such that*

(i) $f_{|\dot{F}} = \tilde{f}_{|\dot{F}}$,

(ii) *f is homotopic in X to $\tilde{f}$ relatively to $\dot{F}$,*

(iii) *each F_j ($j = 0, 1, \ldots$) is the union of a finite number of mutually disjoint perforated discs,*

(iv) $F_n \cap F_j$ *is the union of a finite number of mutually disjoint arcs and simple closed curves for* $n \neq j$,

(v) $f(F_{3j}) \subset C_j^1$, $f(F_{3j+1}) \subset C_j^2$ *and* $f(F_{3j+2}) \subset C_j^3$ *for* $j = 0, 1, \ldots$,

(vi) $f^{-1}(C^1 \cup C^2) = \bigcup_{n=0}^{\infty} F_n$.

Proof. Let us adopt the following notation: $C_j^\nu = T_{3j+\nu-1}$ for $j = 0, 1, \ldots$ and $\nu = 1, 2, 3$.

There is a sequence $U_0, U_1, \ldots$ of neighbourhoods of (respectively) $T_0, T_1, \ldots$ in X such that

1. $\overline{U}_n \cap \overline{U}_k = \emptyset$ for $|n-k| \geqslant 5$ and
2. $\overline{U}_n \cap A_0^1 = \emptyset$ for $n \geqslant 1$.

Observe that for each $n = 0, 1, \ldots$, there are a neighbourhood V_n of T_n in U_n and a homotopy h_n: $X \times [0, 1] \to X$ such that

3. $h_n(\cdot, 1)$ is a retraction of V_n onto T_n,
4. $h_n(x, t) = x$ for $x \in T_n \cup (X - U_n)$ and $t \in [0, 1]$,
5. $h_n(x, 0) = x$ for $x \in X$,
6. $h_n(x, t) \in T_k$ for $k = 0, 1, \ldots$, $t \in [0, 1]$ and $x \in T_k$,
7. $h_n(x, t) \in X - \bigcup_{k=0}^{\infty} T_k$ for $x \in X - \bigcup_{k=0}^{\infty} T_k$ and $t \in [0, 1)$,
8. $h_n(x, 1) \in T_n \cup (X - \bigcup_{k=0}^{\infty} T_k)$ for $x \in X - \bigcup_{k=0}^{\infty} T_k$ and
9. $\varrho(h_n(x, t), x) < 2^{-n}$ for $x \in X$ and $t \in [0, 1]$.

The proof of existence of such a homotopy is omitted here, but can easily be carried out by using the fact that T_n is topologically a polyhedron having nice polyhedral neighbourhoods.

There is a set $F_0 \subset \tilde{f}^{-1}(V_0)$ such that

10. both F_0 and $\overline{F - F_0}$ are the unions of a finite number of mutually disjoint perforated discs and
11. $\tilde{f}^{-1}(T_0) \subset F_0$.

Define g_0: $F \times [\frac{1}{2}, 1] \to X$ by the formula

$$g_0(x, t) = h_0(f(x), 2(1-t)[1+d(x, F_0)]^{-1}),$$

where $d(x, F_0)$ denotes the distance between x and F_0. Observe that by 5

12. $g_0(x, 1) = \tilde{f}(x)$ for $x \in F$, by 4
13. $g_0(x, t) = \tilde{f}(x)$ for $x \in A_0^1$ and $t \in [\frac{1}{2}, 1]$, and by 3, 7 and 8

14. $\{x \in F;\ g_0(x, \frac{1}{2}) \in T_0\} = F_0$.

We shall construct a sequence of mappings $g_0, g_1, \ldots$ and a sequence $F_0, F_1, \ldots$ of subsets of F satisfying the following conditions:

15. g_n: $F \times \left[\frac{1}{n+2}, 1\right] \to X$ is continuous for $n = 0, 1, \ldots$,
16. $g_n(x, t) = g_k(x, t)$ for $x \in F$, $k, n = 0, 1, \ldots$ and $t \in \left[\frac{1}{n+2}, 1\right] \cap \left[\frac{1}{k+2}, 1\right]$,
17. $\operatorname{diam}\left(g_n\left(\{x\} \times \left[\frac{1}{n+2}, \frac{1}{n+1}\right]\right)\right) < 2^{-n}$ for $n = 0, 1, \ldots$,
18. each set F_n and $\overline{F - \bigcup_{j=0}^{n} F_j}$ $(n = 0, 1, \ldots)$ is the union of a finite number of mutually disjoint perforated discs and $F_n \cap \overline{F - \bigcup_{j=0}^{n+1} F_j}$ is the union of a finite number of mutually disjoint arcs and simple closed curves,
19. $F_n \subset \overline{F - \bigcup_{j=0}^{n-1} F_j}$ for $n = 1, 2, \ldots$,
20. $\left\{x \in F;\ g_n\left(x, \frac{1}{n+2}\right) \in \bigcup_{j=0}^{n} T_j\right\} = \bigcup_{j=0}^{n} F_j$ for $n = 0, 1, \ldots$,
21. $g_n(F_k \times \{t\}) \subset T_k$ for $k, n = 0, 1, \ldots$, $k \leqslant n$ and $t \in \left[\frac{1}{n+2}, \frac{1}{k+2}\right]$,
22. if $n \geqslant k+5$, $t \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]$ and $g_n\left(x, \frac{1}{n+1}\right) \in U_k$, then $g_n(x, t) = g_n\left(x, \frac{1}{n+1}\right)$, and
23. $g_n(x, t) = \tilde{f}(x)$ for $n = 0, 1, \ldots$, $x \in \dot{F}$ and $t \in \left[\frac{1}{n+2}, 1\right]$.

By conditions 10–14, the mapping g_0 and the set F_0 satisfy 15–23. Now suppose that $g_0, \ldots, g_{n-1}$ and $F_0, \ldots, F_{n-1}$ have been constructed. To finish the construction it remains to construct g_n and F_n. By 18 and 20 for $n-1$, there is a set F_n contained in

$$\overline{F - \bigcup_{j=0}^{n-1} F_j} \cap \left\{x \in F;\ g_{n-1}\left(x, \frac{1}{n+1}\right) \in V_n\right\},$$

which satisfies the condition 18 for n and is such that

24. $\left\{x \in F;\ g_{n-1}\left(x, \frac{1}{n+1}\right) \in \bigcup_{j=0}^{n} T_j\right\} \subset \bigcup_{j=0}^{n} F_j$.

Define $g_n\colon F\times\left[\frac{1}{n+2}, 1\right]\to X$ by the following formula:

$$g_n(x,t)=\begin{cases} g_{n-1}(x,t) & \text{for } t\in\left[\frac{1}{n+1},1\right],\\ h_n\!\left(g_{n-1}\!\left(x,\frac{1}{n+1}\right)\right),\ (n+2)[1-t(n+1)][1+d(x,F_n)]^{-1} & \\ & \text{for } t\in\left[\frac{1}{n+2},\frac{1}{n+1}\right],\end{cases}$$

where $d(x, F_n)$ denotes the distance between x and F_n.

By 5 g_n is a well-defined continuous mapping. Condition 16 immediately follows from the formula. By 9 we get 17. Equality 20 follows from 3, 6, 7, 8 and 24. By 6 we get 21, 22 follows from 1 and 4 and finally by 2 and 4 we get 23. The construction is completed.

The mapping $g\colon F\times[0,1]\to X$, such that $g(x,t)=g_n(x,t)$ for $n=0,1,\dots$ and $t\in\left[\frac{1}{n+2},1\right]$, is continuous (see 15, 16 and 17). Define $f(x)=g(x,0)$. The mapping f is homotopic to $\tilde{f}$ with homotopy g not moving points of $\dot{F}$ (comp. 23). By 21 $f(F_k)\subset T_k$. The arbitrary $x\in F$ such that $f(x)\in C^1\cup C^2$. There is an integer k such that $f(x)\in T_k$. There is an $n\geqslant k+5$ such that $g_n\left(x,\frac{1}{n+1}\right)\in U_k$. By 22 we get $g(x,t)=g_n(x,t)=g_n\left(x,\frac{1}{n+1}\right)$ for $t\in\left[\frac{1}{n+2},\frac{1}{n+1}\right]$. By 16 and again by 22 we have $g(x,t)=g_m(x,t)=g_n\left(x,\frac{1}{n+1}\right)$ for $m\geqslant n$ and $t\in\left[\frac{1}{m+2},\frac{1}{n+1}\right]$. Thus $g_n\left(x,\frac{1}{n+2}\right)=g(x,0)=f(x)\in T_k$. Therefore by 20 we get $x\in\bigcup_{j=0}^{n}F_j$. Hence $f^{-1}(C^1\cup C^2)=\bigcup_{n=0}^{\infty}F_n$, which completes the proof of the lemma.

4.4. Lemma. *Let F be a perforated disc and let f_0 be a mapping of $\dot{F}$ into A_0^1 such that $w(f_0, F, A_0^1)\neq 0$. Then there is no continuous mapping $f\colon F\to X$ such that $f_{|\dot{F}}=f_0$.*

Proof. Suppose that the lemma fails. Then there are a mapping $f\colon F\to X$ and a sequence $F_0, F_1, \dots$ of subsets of F such that $f_{|\dot{F}}=f_0$ and conditions (iii)–(vi) of Lemma 4.3 are fulfilled.

Observe that

1. $f(\dot{F}_{3j})\subset A_j^1\cup B_j^1\cup L_j^1$,

2. $f(\dot{F}_{3j+1})\subset A_j^2\cup B_j^2\cup L_j^2$ and

3. $f(\dot{F}_{3j+2})\subset B_j^1\cup B_j^2\cup A_{j+1}^1\cup A_{j+1}^2$.

Since for each $\nu=1,2$ there is a mapping from C_j^ν onto an oriented circle which maps L_j^ν onto a single point and which is an orientation preserving homeomorphism onto A_j^ν and B_j^ν, by 1, 2 and Proposition 2.3 we infer that

4. $w(f, F_{3j}, A_j^1)+w(f, F_3, B_j^1)=0$ and

5. $w(f, F_{3j+1}, A_j^2)+w(f, F_{3j+1}, B_j^2)=0$ for $j=0,1,\dots$

Similarly, since for each $\nu=1,2$ there is a mapping from C_j^3 onto an oriented circle which maps $B_j^\mu\cup A_{j+1}^\mu$ ($\mu=1,2$ and $\mu\neq\nu$) otno a single point and which is an orientation preserving homeomorphism on B_j^ν and A_{j+1}^ν, by 3 and 2.3 we infer that

6. $w(f, F_{3j+2}, B_j^\nu)+w(f, F_{3j+2}, A_{j+1}^\nu)=0$ for $\nu=1,2$ and $j=0,1,\dots$

Observe that

7. $f^{-1}(B_j^\nu-\{p_j\})\cap\dot{F}_{3j+\nu-1}=f^{-1}(B_j^\nu-\{p_j\})\cap\dot{F}_{3j+2}$ and

8. $f^{-1}(A_{j+1}^\nu-\{p_j\})\cap\dot{F}_{3j+2}=f^{-1}(A_{j+1}^\nu-\{p_j\})\cap\dot{F}_{3j+2+\nu}$ for $\nu=1,2$ and $n=0,1,\dots$

By 7, condition (iv) of 4.3 and 2.4 we get

9. $w(f, F_{3j+\nu-1}, B_j^\nu)=-w(f, F_{3j+2}, B_j^\nu)$ for $\nu=1,2$ and $n=0,1,\dots$

By 8, condition (iv) of 4.3 and 2.4 we get

10. $w(f, F_{3j+2}, A_{j+1}^\nu)=-w(f, F_{3j+2+\nu}, A_{j+1}^\nu)$ for $\nu=1,2$ and $n=0,1,\dots$

Since $f^{-1}(A_0^1)\subset F_0-\bigcup_{n=1}^{\infty}F_n$, we have $f^{-1}(A_0^1-L_0^1)\cap\dot{F}_0=\dot{F}-f^{-1}(L_0^1)$ $(f(\dot{F})=f_0(\dot{F})\subset A_0^1)$, and by 2.2 we infer

11. $w(f, F_0, A_0^1)=w(f_0, F, A_0^1)$.

Since $f^{-1}(A_0^2)\subset F_1-(F_0\cup\bigcup_{n-2}^{\infty}F_n)$, we have $f^{-1}(A_0^2)\cap\dot{F}_1\subset f^{-1}(L_0^2)$ and

12. $w(f, F_1, A_0^2)=0$.

Combining 4, 5, 6, 9, 10, 11 and 12, one can easily get

13. $w(f, F_{3j+2}, B_j^1)=w(f_0, F, A_0^1)$,

14. $w(f, F_{3j+2}, A_{j+1}^1)=-w(f_0, F, A_0^1)$ and

15. $w(f, F_{3j+2}, B_j^2)=w(f, F_{3j+2}, A_{j+1}^2)=0$ for $j=0,1,\dots$

By 4.2 there is an integer $n_0>2$ such that, for each $n\geqslant n_0$ and for each simple closed curve $S\subset f^{-1}(C_n^4)$, $f_{|S}$ is homotopic in C_n^4 to a constant map.

Let δ be a positive real such that

16. if $\varrho(x_1, x_2)<\delta$, then $\varrho(f(x_1), f(x_2))<\frac{1}{8}$ for $x_1, x_2\in F$.

For each $n=0,1,\dots$, let H_n be the union of F_{3n+2} and all components of F_{3n}, F_{3n+1}, F_{3n+3} and F_{3n+4} contained in $f^{-1}(\{(x,y,z)\in X;\ |x|<\frac{1}{4}\})$. By (iii) and (iv) of 4.3, H_n is the union of a finite number of mutually disjoint perforated discs.

We claim that

17. there is an $n_1\geqslant n_0$ such that for each $n\geqslant n_1$ there is a component K_n of

$F-H_n$ such that, for each component E of $F_{3n}, F_{3n+1}, F_{3n+3}$ or F_{3n+4} intersecting H_n, we have $E-f^{-1}(C_n^4)\subset K_n$.

Suppose, conversely, that there is an infinite sequence $m_1, m_2, \ldots$ such that for each $j = 1, 2, \ldots$ there are E_j' and E_j'' components of $F_{3m_j}, F_{3m_j+1}, F_{3m_j+3}$ or F_{3m_j+4}, such that the sets $E_j' \cap H_{m_j}$ and $E_j'' \cap H_{m_j}$ are both nonvoid and the sets $E_j'-f^{-1}(C_{m_j}^4)$ and $E''-f^{-1}(C_{m_j}^4)$ are contained in two different components of $F-H_{m_j}$. Since $H_{m_1}, H_{m_2}, \ldots$ are mutually disjoint, one can easily see that for each $k = 1, 2, \ldots$, the set $F-\bigcup_{j=1}^{k} H_{m_j}$ has at least $k+1$ components intersecting the set $f^{-1}(\{(x, y, z)\in X;\ |x|>\frac{1}{2}\})$. Hence F contains $k+1$ mutually disjoint δ-balls (see 16). Since the choice of k is free, this is a contradiction, which proves 17.

Take an arbitrary $n\geqslant n_1$.

Let H be the union of H_n and of all components of $F_{3n}, F_{3n+1}, F_{3n+3}$ and F_{3n+4} contained in $f^{-1}(C_n^4)-K_n$. Let H^ν $(\nu = 1, 2)$ be the union of F_{3n+2} and of all components of $F_{3n+\nu-1}$ and $F_{3n+\nu+2}$ contained in H. By (iii) and (iv) of 4.3, each of the sets H, H^1 and H^2 is the union of a finite number of mutually disjoint perforated discs. Observe that $H = H^1 \cup H^2$, $f(\dot{H}^\nu)\subset B_n^\nu \cup A_{n+1}^\nu \cup L_n^\nu \cup L_{n+1}^\nu$ and

18. $\dot{H}^\nu \cap f^{-1}((B_n^\nu \cup A_{n+1}^\nu)-\{p_n\}) = \dot{H} \cap f^{-1}((B_n^\nu \cup A_{n+1}^\nu)-\{p_n\})$.

Note also that each component of $H-f^{-1}(L_n^1 \cup L_n^2 \cup L_{n+1}^1 \cup L_{n+1}^2)$ which is mapped by f onto $B_n^1-\{p_n\}$, $B_n^2-\{p_n\}$, $A_{n+1}^1-\{p_n\}$ or $A_{n+1}^2-\{p_n\}$ is contained in $\overline{K}_n$.

By 13, 14, 15, the choice of n $(n\geqslant n_0)$, 2.3 and 2.4, we get

$$w(f, H, B_n^1) = -w(f, H, A_{n+1}^1) \neq 0 \quad \text{and} \quad w(f, H, B_n^2) = w(f, H, A_{n+1}^2) = 0.$$

Therefore, there is a component K of $\dot{H}$ such that

19. $w(f, K, A_{n+1}^2) \neq w(f, K, B_n^1)$.

Observe that $K\subset\overline{K}_n$. Let H^0 be a component of H which contains K.

Let $I(S)$ be a collection of all components I of $\dot{H}^0-f^{-1}(L_n^1 \cup L_n^2 \cup L_{n+1}^1 \cup L_{n+1}^2)$ such that $f(I) = S-\{p_n\}$, where $S = B_n^1, B_n^2, A_{n+1}^1$ or A_{n+1}^2. Denote by $I^{+1}(S)$ a subcollection of those elements of $I(S)$ which are mapped by f onto $S-\{p_n\}$ so that their orientations determined by H^0 and S are preserved. Denote also $I^{-1}(S) = I(S)-I^{+1}(S)$.

Observe that each element of $I(B_n^1) \cup I(B_n^2) \cup I(A_{n+1}^1) \cup I(A_{n+1}^2)$ is contained in K. Arrange the elements of the collection $I(B_n^1) \cup I(B_n^2) \cup I(A_{n+1}^1) \cup I(A_{n+1}^2)$ into a sequence $I_1, I_2, \ldots, I_k$ such that the elements occur cyclically on K according to the orientation of K.

Observe that, for each $j = 1, \ldots, k$, if $I_j\in I(B_n^1)$ $(I_j\in I(B_n^2), I_j\in I(A_{n+1}^1)$ or $I_j\in I(A_{n+1}^2))$, then there is a component R_j of F_{3n} $(F_{3n+1}, F_{3n+3}$ or, respectively, $F_{3n+4})$ such that

20. $I_j\subset R_j$ and $f(R_j)-\{(x, y, z)\in X;\ |x|<\frac{1}{4}\} \neq \emptyset$.

Let $M_1, M_2, \ldots, M_{l_0}$ be components of H^1 which meet $\bigcup \{I_j\in I(B_n^1) \cup I(A_{n+1}^1)\}$ and let $M_{l_0+1}, M_{l_0+2}, \ldots, M_{l_1}$ be components of H^2 which meet $\bigcup \{I_j\in I(B_n^2) \cup I(A_{n+1}^2)\}$. Since $f_{|M_l}$ is homotopic in C_n^4 to a constant map $(n\geqslant n_1\geqslant n_0$, comp. 17), by 2.3 we get

21. $w(f, M_l, B_n^\nu)+w(f, M_l, A_{n+1}^\nu) = 0$ for $\nu = 1, 2$ and $l = 1, 2, \ldots, l_1$.

Denote $N_l = \{j;\ I_j\subset M_l\}$ for $l = 1, \ldots, l_1$. By 18, the sets $N_1, N_2, \ldots, N_{l_1}$ form a decomposition of $\{1, 2, \ldots, k\}$ into mutually disjoint subsets.

Let G denote the fundamental group of C_n^4. G is a free group with generators a and b corresponding to the oriented simple closed curves A_{n+1}^1 and B_n^2, respectively. The elements $a_1 = bab^{-1}$ and $b_1 = aba^{-1}$ correspond to the oriented simple closed curves B_n^1 and A_{n+1}^2, respectively. Let $\varkappa$ be a function from $\{1, 2, \ldots, k\}$ into $T = \{a, a^{-1}, a_1, a_1^{-1}, b, b^{-1}, b_1, b_1^{-1}\}$ defined as follows:

$$\varkappa(j) = \begin{cases} a^\alpha & \text{if} \quad I_j\in I^\alpha(A_{n+1}^1), \\ a_1^\alpha & \text{if} \quad I_j\in I^\alpha(B_n^1), \\ b^\alpha & \text{if} \quad I_j\in I^\alpha(B_n^2), \\ b_1^\alpha & \text{if} \quad I_j\in I_j^\alpha(A_{n+1}^2) \end{cases}$$

where $\alpha = \pm 1$.

Since $f_{|K}$ is homotopic in C_n^4 to a constant map, $\varkappa(1)\varkappa(2)\ldots\varkappa(k)$ is the identity of G. Observe that $w(f, K, B_n^1) = \sum_{j=1}^{k} i(a_1, \varkappa(j))$ and $w(f, K, A_{n+1}^2) = \sum_{j=1}^{k} i(b_1, \varkappa(j))$. Thus by 19 and 1.3 we infer that $\tau(\varkappa) \neq 0$. By 18 we have $w(f, M_l, A_{n+1}^1) = \sum_{j\in N_l} i(a, \varkappa(j))$ and $w(f, M_l, B_n^1) = \sum_{j\in N_l} i(a_1, \varkappa(j))$ for $l = 1, 2, \ldots, l_0$, hence by 21 we get $\sum_{j\in N_l} i_a(\varkappa(j)) = 0$ for $l = 1, 2, \ldots, l_0$.

Similarly, by 18 we have $w(f, M_l, B_n^2) = \sum_{j\in N_l} i(b, \varkappa(j))$ and $w(f, M_l, A_{n+1}^2) = \sum_{j\in N} i(b_1, \varkappa(j))$ for $l = l_0+1, \ldots, l_1$; hence by 21 we get $\sum_{j\in N_l} i_b(\varkappa(j)) = 0$ for $l = l_0+1, \ldots, l_1$. Now, applying 1.4, we infer that there are natural numbers s_1, s_2, j_1, j_2, j_3 and j_4 such that $1\leqslant s_1\leqslant l_0 < s_2\leqslant l_1$, $1\leqslant j_1<j_2<j_3<j_4\leqslant k$, $j_1, j_3\in N_{s_1}$ and $j_2, j_4\in N_{s_2}$ (or $j_1, j_3\in N_{s_2}$ and $j_2, j_4\in N_{s_1}$, but the proof in this case is the same).

Let e_n^μ be a point of I_{j_μ} $(\mu = 1, 2, 3, 4)$. Points e_n^1 and e_n^3 cut K between e_n^2 and e_n^4. Let $e_n^\nu e_n^{\nu+2}$ be an arc contained in M_{s_ν} such that $K \cap e_n^\nu e_n^{\nu+2} = \{e_n^\nu, e_n^{\nu+2}\}$ for $\nu = 1, 2$. Since $e_n^1e_n^3 \cup e_n^2e_n^4\subset H^0$, H^0 is a perforated disc and $K\subset\dot{H}^0$, the set $e_n^1e_n^3 \cap e_n^2e_n^4$ is nonvoid. There are mutually disjoint arcs $e_n^1c_n^1$, $e_n^2c_n^2$, $e_n^3d_n^1$ and $e_n^4d_n^2$ with endpoints, respectively, e_n^1 and c_n^1, e_n^2 and c_n^2, e_n^3 and d_n^1, and e_n^4 and d_n^2, such that

$$e_n^1c_n^1\subset R_{j_1}, \quad e_n^2c_n^2\subset R_{j_2}, \quad e_n^3d_n^1\subset R_{j_3}, \quad e_n^4d_n^2\subset R_{j_4},$$

$$(e_n^1c_n^1 \cup e_n^2c_n^2 \cup e_n^3d_n^1 \cup e_n^4d_n^2) \cap K = \{e_n^1, e_n^2, e_n^3, e_n^4\},$$

$$\{f(c_n^1), f(d_n^1)\}\subset\{(x, y, z)\in X;\ x<-\tfrac{1}{4}\}$$

and

$$\{f(c_n^2), f(d_n^2)\}\subset\{(x, y, z)\in X;\ x>\tfrac{1}{4}\} \quad \text{(comp. 20)}.$$

Let Q_n^1 be the arc $e_n^1c_n^1 \cup e_n^1e_n^3 \cup e_n^3d_n^1$ and let Q_n^2 be the arc $e_n^2c_n^2 \cup e_n^2d_n^4 \cup e_n^4d_n^2$. Observe that Q_n^1 and Q_n^2 satisfy conditions (ii), (iii) and (iv) of 4.1 (for δ see 16). Note also that $Q_{n'}^1$, $Q_{n'}^2$, $Q_{n''}^1$ and $Q_{n''}^2$ constructed in that manner satisfy the condition

$$(Q_{n'}^1 \cup Q_{n'}^2) \cap (Q_{n''}^1 \cup Q_{n''}^2) = \emptyset \quad \text{for} \quad |n'-n''| \geqslant 2 .$$

Hence 4.1 give a contradiction, which completes the proof of the lemma.

4.4. COROLLARY. *A_0^1 is not contractible in X.*

5. The main theorem. First let us prove the following

5.1. LEMMA. *Each continuous mapping f from the two-dimensional sphere S^2 in Y induces the trivial morphism on the (Čech or Vietoris) homology groups.*

Proof. Denote by S the equator of S^2. S decomposes S^2 into two discs S_+^2 and S_-^2. Let g be a mapping from Y onto S^2 such that $g(D) \subset S_-^2$, $g(X) \subset S_+^2$ and $g_{|A_0^1}$ is a homeomorphism onto S. Observe that g induces an isomorphism on the homology groups. Since D has a polyhedral neighbourhood in Y, one can assume that $f^{-1}(D)$ is the union of a finite number of mutually disjoint perforated discs. Denote by Q the closure of $S^2 - f^{-1}(D)$. Note that Q is also the union of a finite number of mutually disjoint perforated discs. The boundary of Q is mapped by f into A_0^1.

Let F be an arbitrary component of Q. From 4.4 we infer that $w(f, \dot{F}, A_0^1) = 0$. Observe that $w(gf, \dot{F}, S) = w(f, \dot{F}, A_0^1)$ (S is oriented by the homeomorphism $g_{|A_0^1}$). Hence $w(gf, \dot{F}, S) = 0$. Thus by 2.5, $gf_{|F}$ is homotopic in S_+^2 to a mapping with values in S with a homotopy not moving points of $\dot{F}$. Consequently $g \cdot f$ is homotopic to a constant map. But g induces an isomorphism on homology groups of Y and S^2; therefore f induces the trivial morphism.

5.2. THEOREM. *There is a locally connected continuum contained in E^3 which separates E^3 and has the fixed point property.*

Proof. To prove the theorem, it suffices to show that Y has the fixed point property.

Let f be a mapping from Y into itself. If $f(J) \subset J$ there is a fixed point. Now, suppose that there is a $y_0 \in J$ such that $f(y_0) \notin J$. Since Y is locally contractible at each point of the set $Y - J$, consequently Y is locally contractible at $f(y_0)$. Let W be a neighbourhood of $f(y_0)$ contractible in Y. There is a neighbourhood V of y_0 in Y such that $f(V) \subset W$.

Now, we construct an auxiliary continuum Q. Let F be a disc with the boundary S. Fix a point $s \in S$. Suppose that $F \times [0, \infty)$ is embedded in $E^3 - J$ such that (s, t) is equal $\varphi_1(t)$ for $t \in [0, \infty)$ and $\operatorname{diam}(F \times \{t\})$ converges to zero as t approaches infinity. Denote by Q the continuum $(F \times \{0\}] \cup (S \times [0, \infty)) \cup J$. There is a mapping $g\colon Q \to Y$ inducing an isomorphism of the Vietoris homology groups of Q and Y and such that

(i) $g(F \times \{0\}) = D$,

(ii) $g(S \times [0, \infty)) = C^1$,

(iii) $g(y) = y$ for $y \in J$ and

(iv) $\operatorname{diam} g(S \times \{t\})$ converges to zero as t approaches infinity.

There is a strictly monotone sequence of possitive reals $t_1, t_2, \ldots$ converging to infinity and such that $g(S \times \{t_j\}) \subset V$ for $j = 1, 2, \ldots$ (Notice that this is the unique reason for φ_1 being so complicated). Denote $Q_1 = Q \cup \bigcup_{j=1}^{\infty} F \times \{t_j\}$, $t_0 = 0$ and

$$K_j = (F \times \{t_j\}) \cup (S \times [t_j, t_{j+1}]) \cup (F \times \{t_{j+1}\}) \quad \text{for} \quad j = 0, 1, \ldots$$

Also, denote by p the inclusion of Q into Q_1. Note that p_* (the homomorphism induced by p) is a monomorphism of the homology groups of Q into the homology groups of Q_1.

Since $fg(S \times \{t_j\}) \subset W$ for $j = 1, 2, \ldots$, there is a mapping $h\colon Q_1 \to Y$ such that $h_{|Q} = fg$, i.e. the diagram

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Q_1 & \xrightarrow{h} & Y \end{array} \quad \text{is commutative} .$$

For each $j = 0, 1, \ldots$, let $\gamma_j = \{\gamma_j^i\}$ be a two-dimensional true cycle (see 2, p. 40) with a carrier K_j and a majorant 2^{-i} such that

1. γ_j represents a generator of the two-dimensional homology group of K_j over the rationals, and

2. $\sum_{j=0}^{n} \gamma_j$ represents a generator of the two-dimensional homology group of $(F \times \{0\}) \cup (S \times [0, t_{n+1}]) \cup (F \times \{t_{n+1}\})$ over the rationals, for $n = 0, 1, \ldots$

By 5.1, the true cycle $h(\gamma_j) = \{h(\gamma_j^i)\}$ is homologous in Y to zero. Thus there is an infinite chain $\alpha_j = \{\alpha_j^i\}$ with a majorant $\{\varepsilon_j^i\}$ and a carrier Y, sucht hat $\partial\alpha_j = h(\gamma_j)$. Let $i_0, i_1, i_2, \ldots$ be a sequence of natural numbers such that the sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$, where $\varepsilon_n = \max\{\varepsilon_0^{i_n}, \varepsilon_1^{i_n}, \ldots, \varepsilon_n^{i_n}\}$, converges to zero.

Put $\beta_n = \sum_{j=0}^{n} \alpha_j^{i_n}$. Note that $\beta = \{\beta_n\}$ is an infinite chain with a majorant $\{\varepsilon_n\}$ and a carrier Y.

Form another infinite chain $\varkappa = \{\varkappa_n\}$ putting $\varkappa_n = \sum_{j=0}^{n} \gamma_j^{i_n}$. Observe that $\varkappa$ is a true cycle which represents a generator of $p_*(H_2(Q))$. Since

$$\partial(\beta_n) = \sum_{j=0}^{n} \partial\alpha_j^{i_n} = \sum_{j=0}^{n} h(\gamma_j^{i_n}) = h(\varkappa_n),$$

we have $\partial\beta = h(\varkappa)$; in other words $h(\varkappa)$ is homologous to zero. Thus $h_*(p_*(H_2(Q))) = 0$. Since the diagram

$$\begin{array}{ccc} H_2(Q) & \xrightarrow{g_*} & H_2(Y) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_2(Q_1) & \xrightarrow{h_*} & H_2(Y) \end{array} \quad \text{is commutative}$$

and g_* is an isomorphism, f induces the trivial morphism on the two-dimensional homology group of Y. Since Y has the same homologies as the two-dimensional sphere, the Lefschetz number $\Lambda(f) = 1$, then by 3.3, f has a fixed point, which completes the proof.

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On the decidability of the theory of linear orderings with generalized quantifiers

by

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Abstract. $LO(Q_0, Q_1, \ldots, Q_m)$ be the theory of linear orderings with the additional quantifiers $Q_0, \ldots, Q_m$. Under various hypotheses on set theory it is proved that $LO(Q_0, \ldots, Q_m)$ is always decidable. This generalizes the result of the author for $LO(Q_1)$. The proof uses methods from Leonhard and Läuchli. The theorems can be generalized to arbitrary finite sets of regular cardinality quantifiers.

A. Ehrenfeucht proved in [1] that the elementary theory LO of linear orderings is decidable. In [4] H. Läuchli and J. Leonhard established the same result using games. Let us extend the elementary language of linear order by adding the generalized quantifiers $Q_0, Q_1, \ldots, Q_m$ to it.

We interpret the quantifier Q_k as: "there exist at least ω_k-many". Generalized quantifiers were introduced by A. Mostowski [6].

Let $LO(Q_0, \ldots, Q_m)$ be the theory of linear orderings with these additional quantifiers. Then we will prove that $LO(Q_0, Q_1, \ldots, Q_m)$ is decidable. This generalizes the result of H. P. Tuschik [9] for $LO(Q_1)$. As a corollary we infer that $LO(Q_i\colon i<\omega)$ is decidable.

§ 1. Let L be the first order language with identity and one binary predicate $<$. $L^m(Q)$ arises from L by adding the quantifiers $Q_0, \ldots, Q_m$. LO is the following theory:

(1) $\neg x<x$,

(2) $x<y \wedge y<z \rightarrow x<z$,

(3) $x = y \vee x<y \vee y<x$.

We use some definitions from [4] and [9]: $x<y(\operatorname{mod} A)$ denotes the order relation of an ordered set A, $|A|$ denoted the field of A. B is said to be a *segment* of A if B is a substructure of A and if $x<y(\operatorname{mod} B)$ and $x<z<y(\operatorname{mod} A)$ implies $z\in B$. Some special segments are the open interval $(x, y) = \{z\in |A|\colon x<z<y(\operatorname{mod} A)\}$, the left-open and right closed interval $(x, y] = \{z\in |A|\colon x<z\leqslant y(\operatorname{mod} A)\}$, the left-closed and right-open interval $[x, y) = \{z\in |A|\colon x\leqslant z<y(\operatorname{mod} A)\}$ and the closed interval $[x, y] = \{z\in |A|\colon x\leqslant z\leqslant y(\operatorname{mod} A)\}$. A map $f\colon A\rightarrow B$ of an ordered set A