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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Density-preserving homeomorphisms

by

Jerzy Niewiarowski (Łódź)

Abstract. The work consists of two parts. In the first part it is proved that the sum and the product of linear homeomorphisms preserving density points are homeomorphisms preserving density points and the inverse homeomorphism need not to preserve density points. The second part deals with homeomorphisms transforming a plane into a plane. This part includes the proof of the theorem on interval sets — an analogue of the theorem of Bruckner for linear homeomorphisms.

The notion of a homeomorphism preserving density points was introduced by Bruckner in [1]. Bruckner proved that if g is a homeomorphism of $[0, 1]$ onto itself, then a necessary and sufficient condition for $f \circ g$ to be approximately continuous for every approximately continuous real function f defined on $[0, 1]$ is that $h = g^{-1}$ preserves density points. He studied also the conditions, in terms of differentiability properties, for a homeomorphism to preserve density points.

In the first part of this work we shall study some properties of the class of linear homeomorphisms preserving density points. We shall prove that the sum and the product of such homeomorphisms (under some additional conditions) are also homeomorphisms preserving density points, but the inverse homeomorphisms need not preserve density points.

Recently, in [4], U. Wilczyńska has proved that the above cited theorem of Bruckner is true also in the case of functions of several variables. In the second part of this work we shall deal with homeomorphisms transforming a plane (or its subsets) into a plane. We shall prove an analogue of the theorem of Bruckner concerning so-called interval sets. All theorems and proofs will be formulated in the two-dimensional case only for the sake of simplicity. It is easy to see that the theorems are also true for every dimension n. We shall study the case of weak and strong density points.

We shall start with recalling the basic definitions in the formulation suitable for our purposes.

Definition 1. If $B \subset R$ is a measurable set, $p_0 \in R$ and P denotes the closed interval in R, then the number

$$D(p_0, B) = \lim_{\substack{d(P) \to 0 \\ p_0 \in P}} \frac{|B \cap P|_1}{|P|_1}$$

is called the *density of B at a point p_0* (if the limit exists).

The upper, lower and one-sided density is defined in a similar way with obvious modifications. In the case of non-measurability of B one can define outer density by using the outer Lebesgue measure.

DEFINITION 2. If $B\subset R^2$ is a measurable set, $p_0\in R^2$ and K denotes the square in R^2 (with the sides parallel to the axes of coordinates), then the number

$$D(p_0, B) = \lim_{\substack{d(K)\to 0\\ p_0\in K}} \frac{|B\cap K|_2}{|K|_2}$$

is called the *weak density of B at a point p_0* (if the limit exists).

DEFINITION 3. If $B\subset R^2$ is a measurable set, $p_0\in R^2$ and P denotes the closed interval in R^2, then the number

$$D_s(p_0, B) = \lim_{\substack{d(P)\to 0\\ p_0\in P}} \frac{|B\cap P|_2}{|P|_2}$$

is called the *strong density of B at a point p_0* (if the limit exists).

We shall say that p_0 is a point of density (of dispersion) of the set $B\subset R$ if and only if $D(p_0, B) = 1$ $\big(D(p_0, B) = 0\big)$. Similarly one can define the point of one-sided density or dispersion in the one-dimensional case and the point of weak or strong density of dispersion in the two-dimensional case.

DEFINITION 4. A homeomorphism $h\colon R \xrightarrow[\text{onto}]{} R$ is said to *preserve density points* provided for every measurable set $B\subset R$ $h(p)$ is a point of density of the set $h(B)$ whenever p is a point of density of B.

DEFINITION 5. A homeomorphism $h\colon R^2 \xrightarrow[\text{onto}]{} R^2$ is said to *preserve weak* (*strong*) *density points* provided for every measurable set $B\subset R^2$ $h(p)$ is a point of weak (strong) density of the set $h(B)$ whenever p is a point of weak (strong) density of B.

It is not difficult to see that a point p is a point of density (weak or strong) of a set B if and only if p is a point of dispersion of a complementary set CB. Hence a homeomorphism preserves density points if and only if it preserves points of dispersion. Moreover, sometimes it is much easier to prove that a homeomorphism preserves points of dispersion, and so the above remark is very useful.

I. Linear homeomorphisms. Recall that the set $A\subset R$ is called an *interval set at a point x_0* if and only if $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1}<a_n<b_n$ and $a_n\searrow x_0$, $b_n\searrow x_0$ or $b_n<a_n<b_{n+1}$ and $a_n\nearrow x_0$, $b_n\nearrow x_0$ (see [1]).

The following lemma will be very useful in the proofs of several theorems. We shall omit the easy proof of the lemma.

LEMMA 1. *Let $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ be an interval set at a point x_0 (from the right). A function*

$$W(x) = \frac{|A\cap [x_0, x]|_1}{x-x_0}$$

has a local minimum at every point $x = a_n$ and a local maximum at every point $x = b_n$.

THEOREM 1. *If f and g are increasing homeomorphisms preserving density points defined on the interval* $[0, 1]$, *then $h = f+g$ is a homeomorphism preserving density points.*

Proof. In virtue of Theorem 3 of [1] it suffices to prove that h preserves one-sided points of dispersion of interval sets.

Let x_0 be a point of right-hand dispersion of the interval set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $a_n\searrow x_0$, $b_n\searrow x_0$, $b_{n+1}<a_n<b_n$.

So we have

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} \big(f(b_k)-f(a_k)\big)}{f(b_n)-f(x_0)} = 0, \tag{1}$$

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} \big(g(b_k)-g(a_k)\big)}{g(b_n)-g(x_0)} = 0. \tag{2}$$

Let $\{k_n\}$ be an arbitrary sequence such that $k_n\to 0_+$. In virtue of Lemma 1 we have

$$\overline{\lim_{k_n\to 0+}} \frac{|h(A)\cap [h(x_0), h(x_0)+k_n]|_1}{k_n} \leqslant \overline{\lim_{n\to\infty}} \frac{|h(A)\cap [h(x_0), h(b_n)]|_1}{h(b_n)-h(x_0)}. \tag{3}$$

But

$$\frac{|h(A)\cap [h(x_0), h(b_n)]|_1}{h(b_n)-h(x_0)} = \frac{\sum_{k=n}^{\infty} \big(h(b_k)-h(a_k)\big)}{h(b_n)-h(x_0)} = \frac{\sum_{k=n}^{\infty} \big(f(b_k)-f(a_k)+g(b_k)-g(a_k)\big)}{f(b_n)-f(x_0)+g(b_n)-g(x_0)}$$
$$\leqslant \frac{\sum_{k=n}^{\infty} \big(f(b_k)-f(a_k)\big)}{f(b_n)-f(x_0)} + \frac{\sum_{k=n}^{\infty} \big(g(b_k)-g(a_k)\big)}{g(b_n)-g(x_0)}.$$

So from (1), (2) and (3) it follows that $h(x_0)$ is the point of right-hand dispersion of the set $h(A)$. The proof for left-hand points of dispersion is similar.

THEOREM 2. *If f and g are increasing homeomorphisms preserving density points and transforming the interval* $[0, 1]$ *onto itself, then the homeomorphism $h = f\cdot g$ preserves density points.*

Proof. We have

$$f(x)\cdot g(x) = e^{\ln(f(x)g(x))} = e^{\ln f(x)+\ln g(x)} \tag{4}$$

for $x>0$. It is not difficult to see that the function $\ln x$ preserves density points in the interval $(0, 1]$ (see [1], Corollary 2). Observe that a superposition of homeomorphisms preserving density points preserves density points. From (4) in virtue of Theorem 1 it follows that $h = f\cdot g$ preserves density points in $(0, 1]$.

We shall now prove that h preserves also density points at 0. Let 0 be a point of right-hand dispersion of the interval set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$. We have for every natural n

$$\frac{\sum_{k=n}^{\infty}(f(b_k)g(b_k)-f(a_k)g(a_k))}{f(b_n)g(b_n)} = \frac{\sum_{k=n}^{\infty} f(b_k)(g(b_k)-g(a_k))}{f(b_n)g(b_n)} + \frac{\sum_{k=n}^{\infty} g(a_k)(f(b_k)-f(a_k))}{f(b_n)g(b_n)}$$
$$\leqslant \frac{\sum_{k=n}^{\infty}(g(b_k)-g(a_k))}{g(b_n)} + \frac{\sum_{k=n}^{\infty}(f(b_k)-f(a_k))}{f(b_n)}.$$

From Lemma 1 we conclude that h preserves the right-hand point of dispersion at 0.

THEOREM 3. *There exists a homeomorphism f transforming the interval* $[0, 1]$ *onto the interval* $[0, a]$, *where* $a>0$, *preserving density points and such that the inverse homeomorphism does not preserve density points.*

Proof. Let $x_n = (2^{-n+1})$, $y_n = \sum_{i=n}^{\infty}(i^{-2})$. Let $f(x_n) = y_n$ and let f be a linear function in every interval $[x_{n+1}, x_n]$; moreover, let $f(0) = 0$. Denote $E_i = [x_{i+1}, x_i]$, $e_i = f(E_i) = [y_{i+1}, y_i]$. The function f is a homeomorphism transforming $[0, 1]$ onto $[0, y_1]$. We shall show that f preserves points of dispersion and f^{-1} does not preserve points of dispersion. From Corollary 1 in [1] it follows that f preserves density points in the interval $(0, 1]$. We must only prove that f preserves right-hand points of dispersion of every interval set at a point $x = 0$. Let $Z = \bigcup_{j=1}^{\infty} Z_j = \bigcup_{j=1}^{\infty} [x'_j, x''_j]$, be an interval set at a point 0. We can suppose (dividing, if necessary, some of the intervals Z_j) that every Z_j is included in a certain E_i. Denote by $Z_1^{(i)}, \ldots, Z_{j_i}^{(i)}$ all intervals of Z included in E_i. Let $Z^* = \bigcup_{i=1}^{\infty} Z_i^*$, where $Z_i^* = [x_{i+1}, x_i^*]$ and $|Z_i^*|_1 = \sum_{k=1}^{j_i} |Z_k^{(i)}|_1$ (for some i Z_i^* can be empty). If $x_k^{(i)''}$ denotes the right end point of $Z_k^{(i)}$, then for each i and for every $k\in[1, j_i]$ we have

$$\frac{|[0, x_k^{(i)''}]\cap Z|_1}{x_k^{(i)''}} \leqslant \frac{|[0, x_i^*]\cap Z^*|_1}{x_i^*} \tag{5}$$

and

$$\frac{|[0, x_i^*]\cap Z^*|_1}{x_i^*} \leqslant \frac{2|[0, x_{j_i}^{(i)''}]\cap Z|_1}{x_{j_i}^{(i)''}}. \tag{6}$$

Inequality (5) follows from Lemma 1. To prove (6) observe that $|[0, x_i^*]\cap Z^*|_1 = |[0, x_{j_i}^{(i)''}]\cap Z|_1$; so it suffices to show that $2x_i^* \geqslant x_{j_i}^{(i)''}$. We have $2x_{i+1} = x_i$ and $x_i^* > x_{i+1}$, $2x_i^* > x_i$, but $x_i > x_{j_i}^{(i)''}$, and so $2x_i^* \geqslant x_{j_i}^{(i)''}$.

If Z is an interval set such that $D_+(0, Z) = 0$, then we shall prove that also for the set Z^* constructed in the above way

$$D_+(0, Z^*) = 0. \tag{7}$$

From Lemma 1 it follows that the quotient for the set Z^* takes maximal values at points $x_i^{*''}$, and so (7) is an immediate corollary of (6).

Observe that $f(Z^*) = \bigcup_{i=1}^{\infty} f(Z_i^*)$, where

$$f(Z_i^*) = [f(x_i^{*'}), f(x_i^{*''})] = [y_{i+1}, f(x_i^{*''})] \subset e_i.$$

So the image of Z^* is an interval set $f(Z^*)$ having the same structure with respect to $\{e_i\}$ as has Z with respect to $\{E_i\}$.

Suppose that Z is an interval set at a point 0 such that $D_+(f(0), f(Z^*)) = 0$. Then from an inequality similar to (5) (for $y_k^{(i)''}$ instead of $x_k^{(i)''}$) we conclude that also $D_+(f(0), f(Z)) = 0$.

So we have proved that if f preserves right-hand points of dispersion of sets of the form Z^*, then f preserves also right-hand points of dispersion of arbitrary interval sets at 0. Then we shall consider only interval sets of type Z^*.

Let $A = \bigcup_{i=1}^{\infty} A_i$, be an interval set such that $A_i \subset E_i$ and the left-hand end point of A_i coincides with the left-hand end point of E_i. Let $a_i = |A_i|_1$. Suppose that $D_+(0, A) = 0$. We shall prove that $D_+(f(0), f(A)) = 0$. From the assumption it follows that $(\sum_{i=n}^{\infty} a_i) : (\sum_{i=n+1}^{\infty} 2^{-i} + a_n) \xrightarrow[n\to\infty]{} 0$, so $(\sum_{i=n}^{\infty} a_i) : (\sum_{i=n}^{\infty} 2^{-i}) \xrightarrow[n\to\infty]{} 0$. Put $\varepsilon_i = a_i\cdot 2^i$. We have

$$(\sum_{i=n}^{\infty} a_i) : (\sum_{i=n}^{\infty} 2^{-i}) \geqslant a_n : 2^{-n+1} = 2^{-1}\cdot(a_n\cdot 2^n) = 2^{-1}\cdot\varepsilon_n > 0,$$

so $\lim_{n\to\infty} \varepsilon_n = 0$.

Let ε be an arbitrary positive number. There exists a number N such that $\varepsilon_n < \varepsilon$ for $n>N$. So for $n>N$ we have

$$(|f(\bigcup_{i=n}^{\infty} A_i)|_1) : (|\bigcup_{i=n+1}^{\infty} e_i \cup f(A_n)|_1) = (\sum_{i=n}^{\infty} 2^i(i^2)^{-1}\cdot a_i) : (\sum_{i=n+1}^{\infty}(i^{-2}) + 2^n n^{-2}\cdot a_n)$$
$$\leqslant (\sum_{i=n}^{\infty}\varepsilon_i\cdot i^{-2}) : (\sum_{i=n+1}^{\infty} i^{-2}) \leqslant (\varepsilon\cdot\sum_{i=n}^{\infty} i^{-2}) : (\sum_{i=n+1}^{\infty} i^{-2})$$
$$\leqslant \varepsilon + \varepsilon(n+1)n^{-2} < 3\varepsilon,$$

because $(n+1)^{-1} < \sum_{i=n+1}^{\infty} i^{-2}$.

Then $D_+(f(0), f(A)) = 0$ and f preserves points of the right-hand dispersion at 0.

To prove that f^{-1} does not preserve density points we shall construct an increasing sequence of natural numbers such that for $e = \bigcup_{k=1}^{\infty} e_{n_k} = \bigcup_{k=1}^{\infty} [y_{n_{k+1}}, y_{n_k}]$ we have $D_+(0, e) = 0$.

Put $n_1 = 1$. Suppose that we have chosen n_k for $k \geqslant 1$. Let n_{k+1} be a natural number such that $\sum_{n=n_{k+1}}^{\infty} |e_n|_1 < |e_{n_k}|_1$ (obviously such a number does exist). Observe that $|e_{n_m}|_1 = n_m^{-2}$ and $|[0, y_{n_m}]|_1 = \sum_{i=n_m}^{\infty} i^{-2} > n_m^{-1}$. Hence

$$(|\bigcup_{k=m}^{\infty} e_{n_k}|_1) : (|[0, y_{n_m}]|_1) \leqslant (|e_{n_m}|_1 + |\bigcup_{k=m+1}^{\infty} e_{n_k}|_1) : (|[0, y_{n_m}]|_1)$$
$$\leqslant (2|e_{n_m}|_1) : (|[0, y_{n_m}]|_1) < 2n_m^{-1} < 2m^{-1},$$

so $D_+(0, e) = 0$. Simultaneously for $f^{-1}(e) = \bigcup_{k=1}^{\infty} E_{n_k}$ we have

$$(|\bigcup_{k=m}^{\infty} E_{n_k}|_1) : (|[0, x_{n_m}]|_1) \geqslant (|E_{n_m}|_1) : (\sum_{i=n_m}^{\infty} 2^{-i}) = 2^{-1},$$

so $\bar{D}_+(f^{-1}(0), f^{-1}(e)) \geqslant 2^{-1}$; hence f^{-1} does not preserve density points.

From the above proof it follows that there exists a homeomorphism fulfilling the Lipschitz condition and not preserving density points; obviously it is possible to give a simpler construction of such a homeomorphism.

II. Plane homeomorphisms.

Definition 6. We shall say that the function $f\colon R^2 \to R^2$ is *absolutely continuous* if and only if it is continuous and for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that, for every set $Z \subset R^2$, if $|Z|_2 < \delta$ then $|f(Z)|_2 < \varepsilon$.

Theorem 4. *If a homeomorphism* $h\colon K \xrightarrow[\text{onto}]{} h(K)$ (*where* K *is a square*) *preserves points of strong or weak density, then* h *is an absolutely continuous function.*

Proof. Suppose that h is not absolutely continuous. Then in virtue of Theorem 3 in [2] (p. 284) there exists a set Z such that $|Z|_2 = 0$ and $|h(Z)|_e > 0$. Let $q_0 \in h(Z)$ be a point of strong (weak) exterior density of $h(Z)$ and let $p_0 = h^{-1}(q_0)$. Obviously p_0 is a point of strong (weak) dispersion of Z and so h does not preserve density points.

Definition 7. A set A is called an *interval set at a point* p if and only if $A = \{p\} \cup \bigcup_{n=1}^{\infty} P_n$, where P_n are nondegenerate closed intervals with disjoint interiors such that for every $\varepsilon > 0$ the set $\{n\colon P_n - K(p, \varepsilon) \neq \emptyset\}$ (here $K(p, \varepsilon)$ denotes the square with centre at p and sides of length equal to ε) is finite, a point p does not belong to any P_n and $p \in \operatorname{Fr} A$.

Let us introduce the following operation: for every $C, D \subset R^2$ put $C \ominus D = \overline{\operatorname{Int}(C - D)}$. It is known (see for example [3], p. 59) that, if C and D are finite sums of closed rectangles with disjoint interiors, then $C \ominus D$ is also of this form. Using this fact one can prove without difficulty the following lemma:

Lemma 2. *If* A *is an interval set at a point* p *and* r *is an arbitrary positive number, then* $K(p, r) \ominus A$ *is also an interval set at a point* p.

Observe that the sets $K(p, r) - A$ and $K(p, r) \ominus A$ differ by the set of measure 0 and the homeomorphism in the following theorem is absolutely continuous; so the remark concerning the equivalence of preserving density and dispersion points is still valid.

Theorem 5. *Let* h *be an absolutely continuous homeomorphisms of* $[0, 1] \times [0, 1]$ *onto itself. A necessary and sufficient condition for* h *to preserve strong density points is that* h *should preserve points of strong density* (*or of strong dispersion*) *of every interval set.*

Proof. The necessity is obvious.

Sufficiency. Suppose that h does not preserve strong density points. There exists a set $S \subset [0, 1] \times [0, 1]$ and a point $p_0 = (x_0, y_0)$ such that p_0 is a point of strong dispersion of S and $h(p_0)$ is not a point of strong dispersion of $h(S)$.

Let D be a set consisting of all points of strong density of S belonging to S. Then $|D|_2 = |S|_2$. For every $p \in D$ there exists a square K_p such that $p \in \operatorname{Int} K_p$ and for every square K if $p \in K \subset K_p$, then the following inequality holds:

$$|D \cap K|_2 > 2^{-1} |K|_2 . \tag{8}$$

Let $X_0 = \{(x_0, y)\colon y \in R\}$, $Y_0 = \{(x, y_0)\colon x \in R\}$. For every $p \in D - (X_0 \cup Y_0)$ we shall construct a family of squares $\mathscr{K}_p$. Let $\varrho(p) = \min(\varrho(p, X_0), \varrho(p, Y_0))$ and let n be a smallest natural number such that $n^{-1} < \varrho(p)$. Put $\delta(p) = \varrho(p) - n^{-1}$. Let $\mathscr{K}_p = \{K\colon p \in K \subset K_p$ and $d(K) < \min(n^{-2}, \delta^2(p))\}$. At last let $\mathscr{K} = \bigcup_{p \in D - (X_0 \cup Y_0)} \mathscr{K}_p$.

The family $\mathscr{K}$ covers the set $D_0 = D - (X_0 \cup Y_0)$ in the sense of Vitali. So in virtue of the theorem of Vitali there exists a (finite or infinite) sequence of squares $\{K_i\}$ such that $|D_0 - \bigcup_i K_i|_2 = 0$ and $K_i \cap K_j = \emptyset$ for $i \neq j$.

Using the fact that $p_0 \notin K_i$ for every i and that h is an absolutely continuous function, one can easily prove the sequence $\{K_i\}$ must be infinite.

Let $A_0 = \bigcup_{i=1}^{\infty} K_i$. Suppose that P is a rectangle with side lengths a and b, $a \leqslant b$ such that $p_0 \in P$. Let $N = \{i\colon K_i \subset P\}$ and $M = \{i\colon K_i \cap CP \neq \emptyset$ and $K_i \cap P \neq \emptyset\}$. We have

$$\frac{|P \cap A_0|_2}{|P|_2} = \frac{|P \cap \bigcup_{i \in N} K_i|_2}{|P|_2} + \frac{|P \cap \bigcup_{i \in M} K_i|_2}{|P|_2} .$$

From (8) it follows that

$$\frac{|P\cap \bigcup_{i\in N} K_i|_2}{|P|_2} \leqslant \frac{2|P\cap \bigcup_{i\in N}(K_i\cap D)|_2}{|P|_2} \leqslant \frac{2|P\cap D|_2}{|P|_2}.$$

Suppose that n is a natural number such that $d(P)<n^{-1}$ and let m be a natural number for which $m^{-1}<a\leqslant(m-1)^{-1}$. Obviously $m\geqslant n$. Let

$$E=\{p\colon\ \varrho(p,\operatorname{Fr}P)<m^{-2}\}.$$

From the construction of $\mathscr{K}$ it follows that $\bigcup_{i\in M} K_i\subset E$. It is not difficult to see that $|E|_2\leqslant 4(a+b)m^{-2}\leqslant 8bm^{-2}$. Hence

$$\frac{|P\cap \bigcup_{i\in M} K_i|_2}{|P|_2} \leqslant \frac{8b}{m^2ab} = \frac{8}{am^2} < \frac{8}{m} \leqslant \frac{8}{n}.$$

$$\frac{|P\cap A_0|_2}{|P|_2} < \frac{2|P\cap D|_2}{|P|_2} + \frac{8}{n}.$$

Then p_0 is a point of strong dispersion of the set A_0.

Let $L_i=h(K_i)$, $B_0=h(A_0)=\bigcup_{i=1}^{\infty} L_i$. There exists a set $C\subset S$ such that $|C|_2=0$ and $S-C\subset A_0$; so $h(S-C)\subset h(A_0)$ and $|h(C)|_2=0$ (from the absolute continuity of h). Then $h(p_0)$ is not a point of strong dispersion of B_0. Let $\overline{D}_s(h(p_0),B_0)=\varepsilon$. There exists a descending sequence $\{P_n\}$ of closed rectangles such that $|B_0\cap P_n|_2>2^{-1}\varepsilon|P_n|_2$ and $h(p_0)\in \operatorname{Int}P_n$.

Let $R_n=P_n-P_{n+1}$ and $I_k^n=L_k\cap R_n$ for $n=1,2,\ldots$ and $k=1,2,\ldots$ Then $R_n\cap B_0=\bigcup_k I_k^n$, where I_k^n is a finite or infinite sequence of disjoint sets. In any case let $M_1^n,\ldots,M_{k_n}^n$ be a finite sequence chosen for each n from $\{I_k^n\}$ and such that

$$|\bigcup_{i=1}^{k_n} M_i^n|_2>2^{-1}|R_n\cap B_0|_2.$$

Denote $M=\bigcup_{r=1}^{\infty}\bigcup_{i=1}^{k_r} M_i^r$. Then $M\cap P_n=\bigcup_{r=n}^{\infty}\bigcup_{i=1}^{k_r} M_i^r$ and $|M\cap P_n|_2>2^{-1}|P_n\cap B_0|_2$, and so $|M\cap P_n|_2>\varepsilon\cdot 4^{-1}|P_n|_2$. Every set M_i^r is included in a certain L_k, let us denote this set by N_i^r. Let $N=\bigcup_{r=1}^{\infty}\bigcup_{i=1}^{kr} N_i^r$. We have $M\subset N$, so $\overline{D}_s(h(p_0),N)\geqslant 4^{-1}\varepsilon$. Let $\{N_i\}$ be a sequence of sets in which every N_i^r appears exactly once.

Let n_0 be a fixed natural number. Using the fact that h is a homeomorphism, it is not difficult to prove that there is only a finite number of sets N_i having points in common with CP_{n_0}.

Let $A=h^{-1}(N)=\bigcup_{i=1}^{\infty} h^{-1}(N_i)$. It is not difficult to see that A is an interval set at a point p. The function h does not preserve points of strong dispersion of interval sets, because $D_s(p_0,A)=0$ and $\overline{D}_s(h(p_0),h(A))>0$. The theorem is proved.

Similarly one can prove an analogous theorem concerning weak density.

Now we shall consider the case where the homeomorphism is of the form $H(x,y)=(h_1(x),h_2(y))$. At first we shall prove an auxiliary theorem concerning absolutely continuous functions.

DEFINITION 8. We shall say that the function $f\colon R^2\to R^2$ is *absolutely continuous in sense of Banach* if and only if for every positive number ε there exists a positive number δ such that for every elementary figure A (an elementary figure being a set which is a finite union of closed intervals with disjoint interiors) if $|A|_2<\delta$, then $|f(A)|_2<\varepsilon$.

THEOREM 6. *If h_1 and h_2 are homeomorphisms transforming the interval* $[0,1]$ *onto itself and h_1, h_2 are absolutely continuous, then the function $H(x,y)=(h_1(x),h_2(y))$ is a homeomorphism transforming the square* $[0,1]\times[0,1]$ *onto itself, and H is absolutely continuous in the sense of Banach.*

Proof. The fact that H is a homeomorphism is obvious. We shall prove the absolute continuity.

Let A be an elementary figure included in $[0,1]\times[0,1]$. Let $\varepsilon>0$ be an arbitrary number. From the absolute continuity of h_1 and h_2 it follows that there exist two numbers η_1 and η_2 such that for every elementary figure $B\subset R$ if $|B|_1<\eta_i$, then $|h_i(B)|_1<\varepsilon\cdot 2^{-1}$ for $i=1,2$.

Let $A_x=\{y\colon (x,y)\in A\}$. Put $E_1=\{x\colon |A_x|_1<\eta_2\}$, $E_2=\{x\colon |A_x|_1\geqslant\eta_2\}$, $A_1=A\cap(E_1\times R)$, $A_2=A-A_1$.

Then for every $x\in[0,1]$ $|(A_1)_x|_1<\eta_2$, and so $|H((A_1)_x)|_1<2^{-1}\varepsilon$. From the equality $H((A_1)_x)=(H(A_1))_{h_1(x)}$ and from the theorem of Fubini it follows that

$$(9)\qquad |H(A_1)|_2=\int_0^1 |(H(A_1))_z|_1\,dz<2^{-1}\varepsilon.$$

Now suppose that δ is a positive number such that $\delta<\eta_1\eta_2$. If $|A|_2<\delta$, then $|E_2|_1<\eta_1$. Indeed, if $|E_2|_1\geqslant\eta_1$, then we should have

$$|A|_2=\int_0^1|A_x|_1\,dx=\int_{E_1}|A_x|_1\,dx+\int_{E_2}|A_x|_1\,dx\geqslant\int_{E_1}|A_x|_1\,dx+\eta_1\eta_2\geqslant\delta,$$

a contradiction.

If we denote $(A_2)^y=\{x\colon (x,y)\in A_2\}$, then for every $y\in[0,1]$ we have $(A_2)^y\subset E_2$, so $|(A_2)^y|_1<\eta_1$. Hence $|(H(A_2))^{h_2(y)}|_1<2^{-1}\varepsilon$. From the last inequality it follows that

$$(10)\qquad |H(A_2)|_2=\int_0^1 |(H(A_2))^t|_1\,dt<2^{-1}\varepsilon.$$

From (9) and (10) we have $|H(A)|_2=|H(A_1)\cup H(A_2)|_2<\varepsilon\cdot 2^{-1}+\varepsilon\cdot 2^{-1}=\varepsilon$.

THEOREM 7. *Let $H(x,y)=(h_1(x),h_2(y))$ be a homeomorphism transforming*

$[0,1]\times[0,1]$ *onto itself. If* h_1 *and* h_2 *are absolutely continuous functions such that there exist two numbers* α *and* β *for which* $0<\beta\leqslant|h_1'(x)|\leqslant\alpha<\infty$ *and*

$$0<\beta\leqslant|h_2'(y)|\leqslant\alpha<\infty$$

almost everywhere, then H *preserves points of strong density.*

Proof. In virtue of Theorem 6 H is absolutely continuous in the sense of Banach. Suppose that p_0 is the point of strong dispersion of some measurable set S; so

$$\lim_{\substack{d(p)\to 0\\ p_0\in P}}\frac{|S\cap P|_2}{|P|_2}=0 \tag{11}$$

where P is a rectangle.

It is not difficult to verify that

$$\frac{\partial(h_1,h_2)}{\partial(x,y)}=h_1'(x)\cdot h_2'(y).$$

From Theorem 6 in [2], p. 414 it follows that

$$|H(S\cap P)|_2=\iint_{S\cap P}|h_1'(x)\cdot h_2'(y)|\,dx\,dy\leqslant\alpha^2|S\cap P|_2,$$

$$|H(P)|_2=\iint_{P}|h_1'(x)\cdot h_2'(y)|\,dx\,dy\geqslant\beta^2|P|_2.$$

Hence

$$\frac{|H(S)\cap H(P)|_2}{|H(P)|_2}\leqslant\frac{\alpha^2|S\cap P|_2}{\beta^2|P|_2}. \tag{12}$$

From the fact that H is a homeomorphism of the above form and from (11) and (12) we have

$$\lim_{\substack{d(H(P))\to 0\\ H(p_0)\in H(P)}}\frac{|H(S)\cap H(P)|_2}{|H(P)|_2}=0.$$

COROLLARY. *If* H *is a homeomorphism of the above form whose Jacobian is continuous and different from zero, then* H *preserves points of strong density.*

THEOREM 8. *Let* $H(x,y)=\big(h_1(x),h_2(y)\big)$ *be a homeomorphism transforming* $[0,1]\times[0,1]$ *onto itself. If the following condition is fulfilled:*

(*) *for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that for every measurable set* S *and for every interval* I *if* $\dfrac{|S\cap I|_2}{|I|_2}<\delta$, *then* $\dfrac{|H(S)\cap H(I)|_2}{|H(I)|_2}<\varepsilon$; *then* H *preserves points of strong density.*

Proof. Let p_0 be a point of strong dispersion of a measurable set S and let I be a rectangle including p_0. Let $\varepsilon>0$ be an arbitrary number. We choose such a number $\delta>0$ that the inequality in (*) is fulfilled, next we choose such a number $\eta>0$ that, if $d(I)<\eta$ (where $d(I)$ denotes the diameter of the interval I), then

$$\frac{|S\cap I|_2}{|I|_2}<\delta; \tag{13}$$

finally for η we choose such a number $\gamma>0$ that if $d(P)<\gamma$, then

$$d(H^{-1}(P))<\eta, \tag{14}$$

where P is a rectangle.

Consider a rectangle P such that $H(p_0)\in P$ and $d(P)<\gamma$. Let $I=H^{-1}(P)$. Then from (14) it follows that $d(I)<\eta$ and from (13) and (*) we obtain the inequality

$$\frac{|H(S)\cap H(I)|_2}{|H(I)|_2}<\varepsilon.$$

So $H(p_0)$ is a point of strong dispersion of the set $H(S)$.

Observe that Theorems 7 and 8 are also true for the case of weak density. The proofs are quite similar.

It is nearly obvious that the rotation of a plane around the origin is a homeomorphism preserving points of weak density. We shall show that if the angle of rotation is different from $k\cdot(2^{-1}\pi)$, then this transformation does not preserve strong density points. Let $A=\{(x,y)\colon y\geqslant x^2 \text{ or } y\leqslant -x^2\}$. It is easy to see that the origin is not the point of strong density of A. Let h be a rotation at an angle different from any multiple of $2^{-1}\pi$. Put $B=h^{-1}(A)$. Then it is not difficult to see that the origin is the point of strong density of B. So h does not preserve strong density points, because $h(B)=A$.

The question if there exists a homeomorphism preserving strong density points and not preserving weak density points remains an open problem.

References

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