Continuous functions on products of topological spaces

by

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Abstract. Let f be a continuous function from the product space $X = \prod \{X_i;\ i \in I\}$ onto the T_2-space Y. Well-known results of Miščenko, Engelking and others state that under certain conditions f depends only on few coordinates. In section 2 of the paper some generalizations of these theorems are given; e.g. if the cardinal $\alpha > \omega$ is a caliber for X and Y does not contain topologically the Cantor cube of weight α then f depends on $< \alpha$ coordinates (Corollary 3). In section 3 local-type variants are considered.

1. Let $\{X_i;\ i \in I\}$ be a family of topological spaces, $X = \prod \{X_i;\ i \in I\}$ the topological product and $f\colon X \to Y$ a mapping onto the T_2-space Y.

A well-known theorem of A. Miščenko [8] asserts that if α^+ is a caliber for X and $\psi Y \leqslant \alpha$ (¹) then f depends at most on α coordinates.

Our main aim in this paper is to prove a generalization of this theorem. Some applications, to derive a generalization of a theorem of R. Engelking [4] and a new result for dyadic compacta, will be given.

We shall use the usual set-theoretic notions; cardinals are identified with initial ordinals. All undefined terms can be found in [3] or [7].

In the sequel we shall need the following definitions and theorems.

Definition (N. Sanin [10]). A cardinality α is said to be a *caliber* for the topological space X if given any sequence $\{G_\xi;\ \xi < \alpha\}$ of non-empty open sets in X, there exists a set $A \subset \alpha$, $|A| = \alpha$ with $\bigcap \{G_\xi;\ \xi \in A\} \neq \emptyset$.

(¹) If R is a topological space, $x \in R$, $A \subset R$, $x \in \overline{A}$,

$$\begin{aligned}
\psi(x, R) &= \min\{|\mathcal{G}|;\ \mathcal{G} \text{ is an open system}, \textstyle\bigcap \mathcal{G} = \{x\}\},\\
\chi(x, R) &= \min\{|\mathcal{G}|;\ \mathcal{G} \text{ is a nbd-base of } x \text{ in } R\},\\
a(x, A) &= \min\{|B|;\ B \subset A,\ x \in \overline{B}\},\\
t(x, R) &= \sup\{a(x, A);\ A \subset R,\ x \in \overline{A}\},\\
t(R) &= \sup\{t(x, R);\ x \in R\}
\end{aligned}$$

(R) is the tightness of the space R.

THEOREM (N. Sanin [10]). *Suppose α is a cardinality,* $\mathrm{cf}(\alpha)>\omega$ *and* $\{X_i;\ i\in I\}$ *is a family of topological spaces,* $w(X_i)<\mathrm{cf}(\alpha)$ *for* $i\in I$; *then α is a caliber for the product space* $X=\prod\{X_i;\ i\in I\}$. ∎

THEOREM (A. Hajnal [6]). *Suppose H is a set and* $\varphi\colon H\to P(H)$ *is a set-mapping (that is, for each element x of H, $\varphi(x)$ is a subset of H). If there exists a cardinality β with* $|\varphi(x)|<\beta<|H|$ *for each* $x\in H$, *then there exists a set* $F\subset H$. $|F|=|H|$ *such that F is "free" with respect to φ, i.e.,* $\varphi(x)\cap F\subset\{x\}$ *for each* $x\in F$.

We shall use this theorem only in the special case when $\beta=\omega$; that was first proved by S. Piccard [9].

2. DEFINITION 1. Let $\{X_i\colon i\in I\}$ be a family of sets, $X=\prod\{X_i;\ i\in I\}$, $A\subset X$, f be a function from X to the set Y. The set $J\subset I$ *determines f in A* if $p\in A$, $q\in X$, $p|J=q|J$ implies $f(p)=f(q)$.

Denote by $\mathrm{ord}(f,A)=\min\{|J|;\ J\subset I,\ J \text{ determines } f \text{ in } A\}$. Note that if $A=X$ then J determines f in X iff f depends only on the coordinates in J.

Denote by D the two-point discrete space.

Now we can formulate our main result.

THEOREM 2. *Let* $\{X_i;\ i\in I\}$ *be a family of topological spaces,* $X=\prod\{X_i;\ i\in I\}$, $f\colon X\to Y$ *a mapping to the* T_2*-space Y,* $A\subset X$, $\alpha>\omega$ *a cardinality. If α is a caliber for the subspace A and* $\mathrm{ord}(f,A)\geqslant\alpha$, *then there exists a subspace* $C\subset X$, $C\cap A\neq\emptyset$ *such that C is homeomorphic with D^α and $f|C$ is a homeomorphism.*

Proof. An elementary open set in X is a set $U=\prod\{U_i;\ i\in I\}$ where $U_i\subset X_i$ is an open set in X_i for $i\in I$ and $I(U)=\{i\in I;\ U_i\neq X_i\}$ is finite.

Suppose $\alpha>\omega$ is a caliber for $A\subset X$ and $\mathrm{ord}(f,A)\geqslant\alpha$. We shall define by transfinite induction a sequence $\{\langle p_\xi,q_\xi,U_\xi^0,U_\xi^1\rangle;\ \xi<\alpha\}$.

Suppose $\varrho<\alpha$ and for $\xi<\varrho$ are defined p_ξ, q_ξ, U_ξ^0, U_ξ^1 with

(i) $p_\xi\in A$, $q_\xi\in X$, U_ξ^0 and U_ξ^1 are elementary open sets,

(ii) $p_\xi\in U_\xi^0$, $q_\xi\in U_\xi^1$, $I(U_\xi^0)=I(U_\xi^1)=I_\xi$,

(iii) if $\eta<\xi<\varrho$ then $p_\xi|I_\eta=q_\xi|I_\eta$,

(iv) if $i\in I$ and $p_\xi(i)=q_\xi(i)$, then $\pi_i(U_\xi^0)=\pi_i(U_\xi^1)$,

(v) $f(U_\xi^0)\cap f(U_\xi^1)=\emptyset$.

Put $J=\bigcup\{I_\xi;\ \xi<\varrho\}$; now $|J|\leqslant\varrho\cdot\omega<\alpha$.

By $\mathrm{ord}(f,A)\geqslant\alpha$ there exist two points p_ϱ, q_ϱ, $p_\varrho\in A$, $q_\varrho\in X$, $p_\varrho|J=q_\varrho|J$ with $f(p_\varrho)\neq f(q_\varrho)$.

The space Y is T_2 and f is continuous so there exist two elementary open sets U_ϱ^0, U_ϱ^1 with $p_\varrho\in U_\varrho^0$, $q_\varrho\in U_\varrho^1$, $f(U_\varrho^0)\cap f(U_\varrho^1)=\emptyset$. Evidently we can suppose that $I(U_\varrho^0)=I(U_\varrho^1)=I_\varrho$ and if $i\in I_\varrho$, $p_\varrho(i)=q_\varrho(i)$, then $\pi_i(U_\varrho^0)=\pi_i(U_\varrho^1)$.

It is now very easy to check that the sequence $\{\langle p_\xi,q_\xi,U_\xi^0,U_\xi^1\rangle,\ \xi<\varrho+1\}$ satisfies our conditions (i)–(v).

Using now that α is a caliber for A and $U_\xi\cap A\neq\emptyset$ for $\xi<\alpha$, we can assume that $p\in A\cap\bigcap\{U_\xi^0;\ \xi<\alpha\}$.

For each $\xi<\alpha$ put $J_\xi=\{i\in I;\ \pi_i(U_\xi^0)\neq\pi_i(U_\xi^1)\}$. Evidently, $J_\xi\subset I_\xi$. If $\eta<\xi<\alpha$, then, by (iii) and (iv), $J_\xi\cap I_\eta=\emptyset$; hence the sets $\{J_\xi;\ \xi<\alpha\}$ are pairwise disjoint. By (v), the sets J_ξ are non-empty ($\xi<\alpha$).

If $\xi<\alpha$, put $\varphi(\xi)=\{\eta<\alpha;\ J_\eta\cap I_\xi\neq\emptyset\}$.

The set I_ξ is finite and the J_η's are disjoint, hence $|\varphi(\xi)|<\omega<\alpha$ for each $\xi<\alpha$. Using now Hajnal's theorem on set-mapping, we can suppose that

$$(*)\qquad J_\xi\cap I_\eta\neq\emptyset\quad\text{iff}\quad \xi=\eta\ (\xi<\alpha,\eta<\alpha).$$

Let now g be the following function from D^α into X: if $x\in D^\alpha$, $i\in I$ put

$$g(x)(i)=\begin{cases}q_\xi(i) & \text{if } i\in J_\xi \text{ and } x(\xi)=1\,.\\ p(i) & \text{otherwise.}\end{cases}$$

The definition of g is meaningful because the sets J_ξ are disjoint. The function g is continuous; indeed, it is enough to prove that $\pi_i\circ g$ is continuous for each $i\in I$.

If $i\in I-J_\xi$ for each $\xi<\alpha$, then $\pi_i\circ g$ is constant; if $i\in J_\xi$ for an ordinal $\xi<\alpha$, then $\pi_i\circ g$ is constant on two complementary clopen sets of D^α.

We assert now that for each $x\in D^\alpha$ and $\xi<\alpha$ $g(x)\in U_\xi^{x(\xi)}$.

Note that if $i\in I-J_\xi$ or $\varepsilon=0$, then $p(i)\in\pi_i(U_\xi^\varepsilon)$. Indeed, $p\in U_\xi^0$ for each $\xi<\alpha$; on the other hand, if $i\in I-J_\xi$, then $p(i)\in\pi_i(U_\xi^0)=\pi_i(U_\xi^1)$ by the definition of J_ξ. Now, we have to prove that for each $i\in I$, $g(x)(i)\in\pi_i(U_\xi^{x(\xi)})$. This is evidently true if $i\notin I_\xi$, because, then $\pi_i(U_\xi^{x(\xi)})=X_i$. If $i\in I_\xi-J_\xi$, then, by $(*)$, $i\notin J_\eta$ for each $\eta<\alpha$ hence $g(x)(i)=p(i)\in\pi_i(U_\xi^{x(\xi)})$. Finally, let $i\in J_\xi$. If $x(\xi)=0$, $g(x)(i)=p(i)\in\pi_i(U_\xi^{x(\xi)})$; if $x(\xi)=1$, $g(x)(i)=q_\xi(i)$ and $q_\xi\in U_\xi^1$ so $g(x)(i)\in\pi_i(U_\xi^{x(\xi)})$.

Now, if $x,y\in D^\alpha$, $x\neq y$, then there exists a $\xi<\alpha$ with $x(\xi)\neq y(\xi)$. Hence $f(g(x))\in f(U_\xi^{x(\xi)})$, $f(g(y))\in f(U_\xi^{y(\xi)})$ and $f(U_\xi^{x(\xi)})\cap f(U_\xi^{y(\xi)})=\emptyset$ by (v), so $f(g(x))\neq f(g(y))$.

The space Y is T_2 consequently $h=f\circ g$ is a homeomorphic embedding of D^α into Y. Put $K=h(D^\alpha)\subset Y$, $C=g(D^\alpha)\subset X$. Now $h=f\circ g$ is a homeomorphic mapping from D^α onto K and g and f are continuous; so g is a homeomorphism of D^α onto C. ∎

Applying the theorem for the case $A=X$, we get the promised generalization of the theorem of A. Miščenko:

COROLLARY 3. *Let the cardinality $\alpha>\omega$ be a caliber for the product space* $X=\prod\{X_i;\ i\in I\}$, $f\colon X\to Y$ *a mapping into the* T_2*-space Y. If Y does not contain a topological copy of D^α, then f depends on $<\alpha$ coordinates.* ∎

If $A=\{p\}$, then each cardinality $\alpha>\omega$ is a caliber for A, hence

COROLLARY 4. *Let* $X=\prod\{X_i;\ i\in I\}$, $f\colon X\to Y$ *a mapping onto the* T_2*-space Y,* $p\in X$. *If* $\mathrm{ord}(f,p)\geqslant\alpha>\omega$, *then there exists a set C,* $p\in C\subset X$ *such that C is homeomorphic with D^α and $f|C$ is a homeomorphism.*

The following lemma will be useful for us.

LEMMA 5. *Let* $X=\prod\{X_i;\ i\in I\}$ *be the topological product of the spaces X_i,* $w(X_i)<\alpha$ *for each* $i\in I$, *where $\alpha>\omega$ is a regular cardinality. Suppose* $A\subset X$ *and for* *each $p \in A$ there exists a set $J = J(p) \subset I$, $|J| < \alpha$ such that $q \in X$, $p|J = q|J$ implies $q \in A$. Then α is a caliber for the subspace A.*

Proof. For each $i \in I$ choose a base $\mathfrak{B}$ in X_i, $|\mathfrak{B}_i| < \alpha$, $X_i \in \mathfrak{B}_i$. A set $U \subset X$ is a basic open set if $U = \prod \{U_i;\ i \in I\}$, $U_i \in \mathfrak{B}_i$ for $i \in I$ and $I(U) = \{i \in I;\ U_i \neq X_i\}$ is finite. It is enough to prove that if $\{U_s;\ s \in S\}$ is a family of basic open sets in X with $|S| = \alpha$ and $U_s \cap A \neq \emptyset$ for $s \in S$, then there exists a set $S' \subset S$, $|S'| = \alpha$ such that $\bigcap \{U_s;\ s \in S'\} \cap A \neq \emptyset$.

By the Erdös–Radó theorem (see e.g. [7] A.2) we can suppose that the system $\{I(U_s);\ s \in S\}$ is a quasi-disjoint family; i.e., that for $s, s' \in S$, $s \neq s'$ $I(U_s) \cap I(U_{s'}) = K$ holds. The set $K \subset I$ is finite and $|\mathfrak{B}_i| < \alpha$ for each $i \in I$, hence we can assume that for each $i \in K$ and for each $s \in S$ $\pi_i(U_s) = G_i$.

Choose now an arbitrary $s_0 \in S$ and $p_0 \in A \cap U_{s_0}$. In accordance with our conditions there exists a set $J \subset I$, $|J| < \alpha$ such that $q \in X$, $q|J = p_0|J$ implies $q \in A$. The members of the family $\{I(U_s) - K;\ s \in S\}$ are pairwise disjoint hence the set $S' = \{s \in S,\ I(U_s) \cap J \subset K\}$ has cardinality α. Select a point $q_s \in U_s$ for $s \in S'$ and let $q \in X$ be the point

$$q(i) = \begin{cases} q_s(i) & \text{if } i \in I(U_s) - K,\ s \in S', \\ p_0(i) & \text{otherwise.} \end{cases}$$

This is a meaningful definition because the sets $\{I(U_s) - K;\ ,s \in S'\}$ are pairwise disjoint. Now $q|J = p_0|J$ so $q \in A$; on the other hand, evidently $q \in \bigcap \{U_s;\ s \in S'\}$. ■

Now, let $X = \prod \{X_i;\ i \in I\}$ be a product of topological spaces, $w(X_i) < \alpha$ $(i \in I)$, $\alpha > \omega$ a regular cardinality and $f\colon X \to Y$ a mapping onto the T_2-space Y. Denote by H the set of those points $y \in Y$, for which it does not exist a compact set C, homeomorphic with D^α, $y \in C \subset Y$. (For example, if the topological character of y in Y is less than α, then $y \in H$.) Put $A = f^{-1}(H)$; evidently, it does not exist a set $C \subset X$, $C \cap A \neq \emptyset$, C homeomorphic with D^α such that $f|C$ is a homeomorphism. This shows, by Corollary 4, that for each $p \in A$ $\operatorname{ord}(f, p) < \alpha$ and hence the conditions of Lemma 5 are satisfied, thus α is a caliber for A. Applying now Theorem 2, we conclude that $\operatorname{ord}(f, A) < \alpha$ and so there exists a set $J \subset I$, $|J| < \alpha$ such that $p \in A$, $q \in X$, $p|J = q|J$ implies $f(q) = f(p) \in H$ hence $q \in A$. This shows that $A = \pi_J^{-1}(B)$ where π_J is the projection on the partial product

$$X_J = \prod \{X_i;\ i \in J\}$$

and $B \subset X_J$. Moreover, $f|A$ depends only on the coordinates in J.

Using that Y is a T_2-space we immediately obtain $f|\bar{A}$ depends only on the coordinates of J, too. Hence we can conclude that if A is dense in X, then f depends only on $<\alpha$ coordinates, sharpening thus a theorem of R. Engelking [4]:

THEOREM 6. *Let f be a mapping from the topological product $X = \prod \{X_i;\ i \in I\}$ of spaces X_i with $w(X_i) < \alpha$ to a T_2-space Y, where $\alpha > \omega$ is a regular cardinality. If f depends on $\geqslant \alpha$ coordinates then for each $Q \subset X$ dense set in X there exists a point $q \in Q$ and a space C homeomorphic with D^α such that $f(q) \in C \subset Y$.* ■

If the spaces X_i are compact spaces, then the conclusion is simpler.

THEOREM 7. *Let f be a mapping from the topological product $X = \prod \{X_i;\ i \in I\}$ of compact spaces with $w(X_i) < \alpha$ onto a T_2-space Y, where $\alpha > \omega$ is a regular cardinality. If $w(Y) \geqslant \alpha$, then for each $Q \subset Y$ dense set there exists a subspace $C \subset Y$ homeomorphic with D^α, $C \cap Q \neq \emptyset$.*

Proof. Indeed, if the above set H is dense in Y then $f(\bar{A}) = Y$ because f is now a closed mapping. But evidently $\bar{A} = \overline{\pi_J^{-1}(B)} = \pi_J^{-1}(\bar{B})$ and so $Y = (f \circ \pi_J^{-1})(\bar{B})$. Surely $w(\bar{B}) \leqslant w(X_J) \leqslant \sum \{w(X_i);\ i \in J\} < \alpha$ (α is regular) hence by a theorem of Arhangel'skiǐ (see e.g. [3] Corollary 2 to Theorem 3.1.11) $w(Y) < \alpha$ too. ■

This is a sharpening of a theorem of R. Engelking [5].

3. In this section we are investigating the local version of the issue of the preceding section.

Let $\alpha > \omega$ be a regular cardinality, $w(X_i) < \alpha$ $(i \in I)$, $X = \prod \{X_i;\ i \in I\}$, $f\colon X \to Y$ a continuous mapping onto the T_2-space Y.

Now, a typical "local problem" is the following: suppose $y \in Y$, $\chi(y, Y) \geqslant \alpha$; can we assert that there exists a set C, $y \in C \subset Y$, C homeomorphic with D^α?

The answer to this question is in general negative; indeed if $\alpha = \omega_1$, it is easy to construct a T_5 topological space Y with $|Y| = \omega$, $\chi(y, Y) = \omega_1$ for each $y \in Y$ (²); this space Y is the continuous image of the discrete countable space N, $w(N) = \omega < \omega_1$. What is more, the answer is negative even if we assume only that $\psi(y, Y) \geqslant \alpha = \omega_1$.

EXAMPLE 8. The basic set of our space Y will be $N^{\omega_1} \cup \{p\}$, $p \notin N^{\omega_1}$, N^{ω_1} is an open subspace of Y and $\psi(p, Y) = \omega_1$. The existence of such a space is guaranteed by the fact that N^{ω_1} is a zerodimensional non-Lindelöf space but it can also be constructed directly. For example, a neighbourhood-base of the point p is the system

$$\{U_T(p);\ T \subset \omega_1,\ |T| < \omega\}$$

where

$$U_T(p) = \{q \in N^{\omega_1};\ q(i) \neq q(j) \text{ if } i, j \in T,\ i \neq j\} \cup \{p\}.$$

Evidently Y is the continuous image of the product space N^{ω_1}, $w(N) = \omega < \omega_1$, $\psi(y, Y) = \omega_1$ for each $y \in Y$.

We assert that does not exist a set C homeomorphic with D^{ω_1}, $p \in C \subset Y$. Indeed, suppose on the contrary that $\varphi\colon D^{\omega_1} \to Y$ is an embedding, $\varphi(\mathbf{1}) = p$.

Put

$$\Sigma = \{q \in D^{\omega_1};\ |\{\xi < \omega_1;\ q(\xi) = 1\}| \leqslant \omega\},$$

$$A = f(\Sigma).$$

Evidently, $A \subset N^{\omega_1}$, $p \in \bar{A}$, moreover, if $B \subset A$, $|B| \leqslant \omega$, then $\bar{B} \subset A$, $\bar{B}$ is a compact

(²) For example, take for Y a countable dense subset of D^{ω_1}.

set. Now, for each $\xi<\omega_1$, $\pi_\xi(A)$ is a finite subset of N; indeed, otherwise we should choose a countable set $B\subset A$ with $\pi_\xi(B)$ infinite and this contradicts to the fact that $\pi_\xi(\overline{B})$ is compact in the discrete space N. Put $K=\prod\{\pi_\xi(A);\ \xi<\omega_1\}$. Now $A\subset K$, K is a compact set in N^{ω_1} hence $p\notin\overline{A}$, a contradiction.

Nevertheless, the following theorem can be asserted.

THEOREM 9. *Let f be a continuous function from the product space*

$$X=\prod\{X_i;\ i\in I\}$$

of spaces X_i with $w(X_i)<\alpha$ onto the T_2-space Y. If $\alpha>\omega$ is a regular cardinality and $y\in Y$ with $\psi(y,Y)\geqslant\alpha$, then there exists a set $A\subset Y$ such that $y\notin A$, $a(y,A)\geqslant\alpha$.

Proof. Put $F=f^{-1}(y)$, $G=X-F$. Denote by

$$\mathfrak{A}=\{A\subset G;\ \psi(y,f(A)\cup\{y\})\geqslant\alpha\}.$$

Evidently $G\in\mathfrak{A}$; moreover, if $A\in\mathfrak{A}$, $A=\bigcup\{A_\xi;\ \xi<\beta\}$ and $\beta<\alpha$ then there exists a $\xi<\beta$ with $A_\xi\in\mathfrak{A}$. Note that if $A\in\mathfrak{A}$ then $y\in\overline{f(A)}$. Choose now a base $\mathfrak{B}_i$ in X_i with $|\mathfrak{B}_i|<\alpha$, $X_i\in\mathfrak{B}_i$ $(i\in I)$. A basic open set in X is a set $U=\prod\{U_i;\ i\in I\}$ with $U_i\in\mathfrak{B}_i$ for $i\in I$ and $I(U)=\{i\in I;\ U_i\neq X_i\}$ finite.

Making use of the fact that Y is T_2 and f continuous we get a cover $\mathfrak{U}$ of G of basic open sets such that

$$\overline{f(U)}\subset Y-\{y\}\quad\text{for each } U\in\mathfrak{U}.$$

Denote by

$$\sigma=\{\mathfrak{B}\subset\mathfrak{U};\ \bigcup\{V;\ V\in\mathfrak{B}\}\in\mathfrak{U}\}.$$

Evidently $\mathfrak{U}\in\sigma$ and if

$$(*)\qquad \mathfrak{B}\in\sigma,\quad \mathfrak{B}=\bigcup\{\mathfrak{B}_\xi;\ \xi<\beta\}$$

and $\beta<\alpha$ then there exists a $\xi<\beta$ with $\mathfrak{B}_\xi\in\sigma$.

Put now

$$\mathfrak{U}_n=\{U\in\mathfrak{U};\ |I(U)|\leqslant n\}\qquad(n<\omega).$$

$\mathfrak{U}=\bigcup\{\mathfrak{U}_n;\ n<\omega$ and $\omega<\alpha$ implies $\mathfrak{U}_m\in\sigma$ for a suitable $m<\omega$.

Let now r be the maximal integer such that there exists a set $T\subset I$, $|T|=r$ with

$$\mathfrak{U}(T)=\{U\in\mathfrak{U}_m;\ T\subset I(U)\}\in\sigma\qquad(0\leqslant r\leqslant m).$$

Select $T_0\subset I$, $|T_0|=r$ with $\mathfrak{U}(T_0)\in\sigma$. Using now $(*)$ and that $|\mathfrak{B}_t|<\alpha$ for $t\in T_0$ we get a family $\mathfrak{B}\subset\mathfrak{U}(T_0)$, $\mathfrak{B}\in\sigma$ with $\pi_t(U)=B_t$ for each $U\in\mathfrak{B}$, $t\in T_0$.

If $i\in I-T_0$, then, by the maximality of r, $\mathfrak{B}_i=\{U\in\mathfrak{B};\ i\in I(U)\}\notin\sigma$. More generally, if $J\subset I-T_0$, $|J|<\alpha$, then, by $(*)$, if $\mathfrak{B}_J=\bigcup\{\mathfrak{B}_i;\ i\in J\}$, $\mathfrak{B}_J\notin\sigma$ hence there exists a set $U\in\mathfrak{B}$ with $I(U)\cap J=\emptyset$.

This shows that we can choose a family $\{U_\xi;\ \xi<\alpha\}\subset\mathfrak{B}$ with $I(U_\xi)\cap I(U_\eta)=\emptyset$ for $\xi<\eta<\alpha$.

Denote now by Σ the set of those points in X which are contained in all but finitely many U_ξ's. It is very easy to see that Σ is dense in the set

$$\bigcap\{\pi_t^{-1}(B_t);\ t\in T_0\}$$

hence $\overline{\bigcup\{U;\ U\in\mathfrak{B}\}}\subset\overline{\Sigma}$. This implies, of course, that if $A=f(\Sigma)$ then $y\in\overline{A}$. On the other hand, if $P\subset\Sigma$, $|P|<\alpha$ then there exists a $\xi<\alpha$ with $P\subset U_\xi$ hence $f(P)\subset f(U_\xi)$. But $U_\xi\in\mathfrak{B}\subset\mathfrak{U}$ hence $y\notin\overline{f(U_\xi)}$ so $y\notin\overline{f(P)}$. This shows that $a(y,A)\geqslant\alpha$. ■

Assuming that "almost all" factor spaces X_i are compact, we get a sharper result.

THEOREM 10. *Let f be a mapping from the topological product $\prod\{X_i;\ i\in I\}$ of spaces with $w(X_i)<\alpha$ onto the T_2-space Y where $\alpha>\omega$ is a regular cardinality. Suppose that the set $\{i\in I;\ X_i$ is not compact$\}$ has cardinality $<\alpha$ and $y\in Y$, $\psi(y,Y)\geqslant\alpha$; then there exists a space C, homeomorphic with D^α, $y\in C\subset Y$.*

Proof. Suppose it does not exist such a subset for the point $y\in Y$.

Denote by $A=f^{-1}(y)\subset X$. Exactly as in the proof of Theorem 6, we can prove that there exists a set $J\subset I$, $|J|<\alpha$ such that $A=\pi_J^{-1}(B)$ where $B\subset X_J$. Evidently we can also assume that if $i\in I-J$ then X_i is compact. Now A is a closed set in X and so B is also a closed set in X_J because $\pi_J^{-1}(B)=A=\overline{A}=\pi_J^{-1}(\overline{B})$ hence $B=\overline{B}$.

Now $w(X_J)<\alpha$ hence there exists a base $\mathfrak{B}$ of X_J, $|\mathfrak{B}|<\alpha$. For $B\in\mathfrak{B}$, put $F_B=\overline{f(\pi_J^{-1}(B))}\subset Y$ and put $\mathfrak{A}=\{B\in\mathfrak{B};\ y\notin F_B\}$.

It is enough to prove that $Y-\{y\}=\bigcup\{F_B;\ B\in\mathfrak{A}\}$ because then $\{y\}=\bigcap\{Y-F_B;\ B\in\mathfrak{A}\}$ and $\psi(y,Y)\leqslant|\mathfrak{A}|\leqslant|\mathfrak{B}|<\alpha$.

Let $x\in Y-\{y\}$ and choose a point $p\in f^{-1}(x)$.

Put $p'=\pi_J(p)\in X_J-B$. The set $C=\pi_J^{-1}(p^1)\subset X$ is compact and so $K=f(C)\subset Y$ is compact, too, $y\notin K$, $x\in K$. Select an open set $G\subset Y$, $K\subset G$, $y\notin\overline{G}$.

Using the compactness of the subspace C we get a $B\in\mathfrak{B}$ with $p'\in B$, $\pi_J^{-1}(B)\subset f^{-1}(G)$; now $C\subset\pi_J^{-1}(B)$.

So we proved $x\in F_B\subset\overline{G}\subset Y-\{y\}$ hence $B\in\mathfrak{A}$. ■

Using this theorem we can deduce a seemingly new result for dyadic compacta. (A dyadic compactum is a Housdorffic image of a product space D^I.)

THEOREM 11. *Let R be a dyadic compactum, $x\in R$, $\alpha>\omega$ a regular cardinality; then the following conditions are equivalent*

a) $\chi(x,R)\geqslant\alpha$,

b) *there exists a set $A\subset R$ with $x\in\overline{A}$, $a(x,A)\geqslant\alpha$,*

c) *there exists a subspace $C\subset R$, C homeomorphic with D^α, $x\in C$.*

Proof. a)$\to$c). This follows from Theorem 8 because in a compact space $\psi(x,R)=\chi(x,R)$ for each point.

c)$\to$b). Put $A=\{y\in D^\alpha;\ |\{\xi<\alpha;\ x(\xi)=y(\xi)\}|<\alpha$ for $x\in D^\alpha$. It is very easy to see that A works.

b)$\to$a). Trivial.

COROLLARY 12. *Let R be a dyadic compactum, $x \in R$; then $\chi(x, R) = t(x, R)$.* ■

The "global" version of Corollary 12 (i.e. that for a dyadic compactum R $\chi(R) = t(R)$ holds) is a well-known fact. By a theorem of A. Arhangel'skiĭ and V. Ponomariov [1] $t(R) = w(R)$ for a dyadic compactum R; moreover, if $w(R) = \alpha$ and $\mathrm{cf}(\alpha) > \omega$, then there exists a point $x \in R$ and a set $A \subset R$, such that $x \in \bar{A}$ and $a(x, A) \geqslant \alpha$. So it is natural to guess that Theorem 11 remains true if we suppose only that $\mathrm{cf}(\alpha) > \omega$. The following example is a counterexample to this conjecture; it is a modification of a construction due to B. Efimov [2].

EXAMPLE 13. Let $\alpha > \omega$ be a singular cardinality, $\beta = \mathrm{cf}(\alpha) < \alpha$. Select a sequence $\{\alpha_\xi;\ \xi < \beta\}$ with $\alpha = \sup\{\alpha_\xi;\ \xi < \beta\}$, $\beta < \alpha_\xi < \alpha_\eta < \alpha$ for $\xi < \eta < \beta$. Put

$$\Phi = \{p \in D^\alpha;\ p|\beta \equiv 0\},$$

$$\Phi_\xi = \{p \in D^\alpha;\ p(\xi) = 1,\ p|\alpha_\xi - \{\xi\} \equiv 0\} \quad (\xi < \beta),$$

$$F = \Phi \cup \bigcup \{\Phi_\xi;\ \xi < \beta\}.$$

Now $F \subset D^\alpha$ is a closed set. Indeed, if $p \in D^\alpha$ and for each $\xi < \beta$ $p(\xi) = 0$, then $p \in \Phi \subset F$. If there is an ordinal $\xi < \beta$ with $p(\xi) = 1$ and $p \notin \Phi_\xi$ then there exists an $\eta < \alpha_\xi$, $\eta \neq \xi$ with $p(\eta) = 1$; now $U = \pi_\xi^{-1}(1) \cap \pi_\eta^{-1}(1)$ is a neighbourhood of p and $U \cap F = \emptyset$.

Denote now by R the quotient space identifying the points of F; R is certainly a dyadic compactum; denote by x the point $\varphi(F)$ of R where φ is the quotient mapping.

If $\xi < \beta$, let $A_\xi = \{p \in D^\alpha;\ p(\xi) = 1,\ |\{\eta < \alpha;\ p(\eta) = 0\}| < \omega\}$. Evidently $\bar{A}_\xi \supset \Phi_\xi$ but if $B \subset A_\xi$, $|B| < \alpha_\xi$ then $\bar{B} \cap \Phi_\xi = \emptyset$. This shows that in R $x \in \bar{A}_\xi$ and $a(x, A_\xi) \geqslant \alpha_\xi$. Specially, $\chi(x, R) \geqslant \alpha_\xi$ for each $\xi < \beta$ hence $\chi(x, R) = \alpha$.

On the other hand, if $A \subset D^\alpha - F$, $\bar{A} \cap F \neq \emptyset$, let $p \in \bar{A} \cap F$. Suppose $p \in \Phi_\xi$, $\xi < \beta$. If $J \subset \alpha_\xi$ is a finite set, there exists a point $q_J \in A_\xi$ with $p|J = q_J|J$. Put $B = \{q_J;\ J \subset \alpha_\xi,\ |J| < \omega\}$, $C = \pi_{\alpha_\xi}(\bar{B}) \subset D^{\alpha_\xi}$.

Now, in D^{α_ξ}, $p' = \pi_{\alpha_\xi}(p)$ is in the closure of the compact set C hence $p' \in C$. This means that there exists a point $q \in \bar{B}$ with $q|\alpha_\xi = p|\alpha_\xi$ but then $q \in \Phi_\xi$ and hence $a(x, A) \leqslant \alpha_\xi < \alpha$ in R.

Quite similarly, if $p \in \Phi \cap \bar{A}$ then $a(p, A) \leqslant \beta < \alpha$ in R. Hence in R if $A \subset R$, $x \in \bar{A}$ then $a(x, A) < \alpha$. ■

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