able cardinals in general has been shown by Baumgartner who proved the following theorem by the methods of Kunen–Paris [2].

THEOREM 4.4. *If it is consistent that a measurable cardinal exists then it is consistent that there is a measurable cardinal $\varkappa$ which bears an atomless $\varkappa^+$-saturated ideal I (I is atomless if for any $A \in \mathscr{P}(\varkappa)-I$, $I \upharpoonright A$ is not prime).*

Sketch of proof. Using Theorem 2.1 of [2], assume M is a model of ZFC such that, in M, D_1 and D_2 are distinct normal ultrafilters on the measurable cardinal $\varkappa$. Choose $X \in D_2 - D_1$ such that for all $\alpha \in X$, α is a regular cardinal. For $i = 1, 2$ let j_i: $M \to \mathrm{Ult}(M, D_i)$ be the canonical embedding and let P be the Easton partial ordering in M for adding a single generic subset of each α in X. Then $j_1 P \cong P \times Q$ where Q is $\varkappa^+$-closed and $j_2 P \cong P \times P_\varkappa \times R$ where R is $\varkappa^+$-closed and $P_\varkappa$ is the partial ordering for adding a generic subset of $\varkappa$ to $\mathrm{Ult}(M, D_2)$. Now, if G is $P \times Q \times R$-generic over M then arguments as in [2] can be used to show that, in $M[G]$, D_1 extends to a normal ultrafilter on $\varkappa$ and D_2 extends to the dual of a non-atomic $\varkappa^+$-saturated ideal on $\varkappa$.

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On Postnikov-true families of complexes and the Adams completion

by

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Abstract. Let $\mathcal{C}_r$ be the homotopy category of pointed r-connected CW-complexes, let $\mathfrak{V}$ be a non-empty collection of objects of $\mathcal{C}_r$, and let $S(\mathfrak{V})$ be the family of those morphisms s: $X \to Y$ in $\mathcal{C}_r$ such that s^*: $[Y, V] \to [X, V]$ is bijective for every V in $\mathfrak{V}$. In the case where $r \geqslant 1$, it is proved that the Adams $S(\mathfrak{V})$-completion exists if, essentially, $\mathfrak{V}$ has the property that, whenever V belongs to $\mathfrak{V}$, then so do the Eilenberg–MacLane spaces $K(\pi_n V, k)$, $k = n, n+1, n+2$; $n \geqslant r+1$. An extension of the result is obtained in the case where $r = 0$ and the objects of $\mathfrak{V}$ are assumed to be nilpotent, by using the characterization of a nilpotent space in terms of the principal refinement of its Postnikov tower. It is pointed out that this framework is adequate to obtain the Sullivan p-profinite completion, where p is a prime. Finally, one considers the general non-simply-connected case, where one does not insist that the objects of $\mathfrak{V}$ be nilpotent. Here, non-simple obstruction theory is needed, and therefore the Eilenberg–MacLane spaces must be replaced by certain spaces $L(A, k)$, obtained by a significant modification from the spaces $\hat{K}(A, k)$ constructed by C. A. Robinson as representing objects for cohomology with local coefficients. The Sullivan P-profinite completion is obtained among the applications, where P is an arbitrary family of primes.

0. Introduction. We consider, for a fixed r, the homotopy category $\mathscr{C}_r$ of pointed r-connected CW-complexes, and a non-empty collection $\mathscr{V}$ of objects of $\mathscr{C}_r$. With respect to $\mathscr{V}$ we form the family $S = S(\mathscr{V})$ of those morphisms s: $X \to Y$ in $\mathscr{C}_r$ with respect to which every V in $\mathscr{V}$ is *left-closed*, that is, those morphisms s such that

$$s^*\colon [Y, V] \to [X, V]$$

is bijective for every V in $\mathscr{V}$. The family $\mathscr{S}$ is plainly saturated, and we may ask whether the (generalized) Adams S-completion [1, 3] exists (¹).

We introduce a condition on $\mathscr{V}$, of a rather natural character, which comes close to guaranteeing the existence of the Adams completion and which is certainly verified in the two cases of principal importance — the p-profinite completion and the P-localization, where p is a prime and P is a (possibly empty) family of primes.

(¹) In this case, the Adams completion has been called by Harvey Wolff the $\mathfrak{V}$-*localization*.

We first discuss, in Section 1, the case $r \geqslant 1$. We then say that $\mathscr{V}$ is *Postnikov-true* if, whenever V belongs to $\mathscr{V}$ then so do the Eilenberg–MacLane spaces

$$K(\pi_n V, k), \quad k = n, n+1, n+2; \; n \geqslant r+1 .$$

(Of course, in actual examples, we would expect to find $K(\pi_n V, k)$ in $\mathscr{V}$ for all $k \geqslant r+1$, but only the given values of k actually enter into the argument.) This condition turns out to be sufficient to guarantee that S admits a calculus of left fractions (provided always that $r \geqslant 1$) and we have only to add a solution set condition to be able to apply the main result of [1] or [3] to prove the existence of the Adams S-completion. We also discuss the saturation of $\mathscr{V}$ and apply the ideas of this paper to the consideration of the Kan extension of a cohomology theory, defined on the full subcategory of $\mathscr{C}_r$ generated by $\mathscr{V}$, to the whole of $\mathscr{C}_r$ or to the saturation of $\mathscr{V}$.

In Sections 2 and 3 we consider the modifications needed in the non-simply-connected case. In Section 2 we obtain an extension of Theorem 1.4 by insisting that the objects of $\mathscr{V}$ be nilpotent [9] and using the characterization of a nilpotent space in terms of the principal refinement of its Postnikov tower. However, we meet a real difficulty in attempting to generalize the last part of Section 1, in which we consider Kan extensions of cohomology theories, since the category of nilpotent spaces does not admit mapping cones and is therefore not admissible for a cohomology theory. On the other hand, the class of nilpotent spaces forms a sort of maximal class of spaces for which ordinary obstruction theory suffices for maps into spaces belonging to the class, so that the proof of Theorem 1.4 generalizes easily. It is interesting to note that, as pointed out in Section 2, this framework is adequate in order to obtain the Sullivan p-profinite completion.

In Section 3, we consider again the non-simply-connected case, but this time we do not insist that the objects of $\mathscr{V}$ be nilpotent. Consequently we need to use non-simple obstruction theory, and therefore we must replace the Eilenberg–MacLane spaces in the definition of the Postnikov-true families by certain spaces $L = L(A, k)$, where A is a given π-module, closely related to the spaces $\hat{K}(A, k)$ constructed by Robinson in [12] as representing objects for cohomology with local coefficients. As an application we then obtain the Sullivan P-profinite completion. In an appendix we show how we can make some small technical improvements in our main theorems; and we use some elementary abstract nonsense to derive further examples of the Adams S-completion from those already cited in the text.

1. The existence of Adams completions: the simply-connected case. Let $\mathscr{C}_r$ be the homotopy category of r-connected (pointed) CW-complexes, $r \geqslant 0$, and let $\mathscr{V}$ be a collection of objects of $\mathscr{C}_r$ which we assume to include the singleton; we will feel free to identify $\mathscr{V}$ with the full subcategory of $\mathscr{C}_r$ generated by the objects of $\mathscr{V}$. Let $S = S(\mathscr{V})$ be the family of morphisms $s\colon X \to Y$ in $\mathscr{C}_r$ such that the induced map

$$s^* = [s, V]\colon [Y, V] \to [X, V]$$

is bijective for all V in $\mathscr{V}$. We are principally interested in the question of when the (generalized) Adams S-completion of objects of $\mathscr{C}_r$ exists. However, we begin with some very elementary observations.

PROPOSITION 1.1. *Let $X \xrightarrow{s} Y \xrightarrow{t} Z$ in $\mathscr{C}_r$ with $ts = u$. Then if any two of s, t, u belong to S, so does the third.*

PROPOSITION 1.2. *Let $K\colon \mathscr{V} \to \mathscr{C}_r$ be the full embedding. Then if $\mathscr{S}$ is the shape category of K with shape functor $T\colon \mathscr{C}_r \to \mathscr{S}$, S is precisely the family of morphisms of $\mathscr{C}_r$ rendered invertible by T.*

Proof. We need only recall [6] the definitions

$$\mathscr{S}(X, Y) = \mathrm{Nat}([Y, K-], [X, K-]), \quad Ts = [s, K-]\colon [Y, K-] \to [X, K-] .$$

PROPOSITION 1.3. *Let $F\colon \mathscr{V} \to \mathscr{D}$ be a functor and let $\bar{F}\colon \mathscr{C}_r \to \mathscr{D}$ be the right Kan extension of F along K. Then $\bar{F}(s)$ is an equivalence for all s in S.*

Proof. According to Theorem 1.4 of [6], the right Kan extension of any functor on $\mathscr{V}$ is shape-invariant.

We now move towards the principal results of this section. We say that $\mathscr{V}$ is *Postnikov-true* if $r \geqslant 1$ and, whenever V belongs to $\mathscr{V}$, then the Eilenberg–MacLane spaces

(1.1) $$K(\pi_n V, k), \quad k = n, n+1, n+2; n \geqslant r+1$$

also belong to $\mathscr{V}$. Notice that $\mathscr{V}$ is Postnikov-true if it is just the family of spaces $\mathscr{V}$ in $\mathscr{C}_r$ whose non-zero homotopy groups belong to a given family $\mathscr{A}$ of abelian groups. We prove

THEOREM 1.4. *If $\mathscr{V}$ is Postnikov-true and if $S = S(\mathscr{V})$, then S admits a calculus of left fractions.*

Proof. We apply the criterion of Theorem 3.1 of [4], and so must prove the following weak push-out property for S. Namely, we assume given a diagram in $\mathscr{C}_r$

(1.2) $$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\sigma}\downarrow & & \\ Z & & \end{array}$$

with σ in S, and prove that (1.2) may be embedded in a weak push-out diagram in $\mathscr{C}_r$,

(1.3) $$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ Z & \xrightarrow[\psi]{} & W \end{array}$$

with τ in S. In fact, we construct (1.3) as the weak push-out in $\mathscr{C}_0$ and then show that W is in $\mathscr{C}_r$ and τ is in S.

Thus let (1.3) be the weak push-out of (1.2) in $\mathscr{C}_0$. Since $X, Y, Z \in \mathscr{C}_r$, $r \geqslant 1$, it follows that W is simply-connected. Since $H_i X = H_i Y = H_i Z = 0$, $1 \leqslant i \leqslant r$, it follows from the Mayer–Vietoris sequence that $H_i W = 0$, $1 \leqslant i \leqslant r$, so that $W \in \mathscr{C}_r$.

It remains to show that τ is in S. Let $n \geqslant r+1$ and let $\mathscr{G}_n$ be the collection of abelian groups $\pi_n V$ as V ranges over $\mathscr{V}$. Since $\mathscr{V}$ is Postnikov-true and $\sigma \in S$ it follows that

$$\sigma^*\colon [Z, K(G, k)] \cong [X, K(G, k)], \qquad G \in \mathscr{G}_n,\ k = n, n+1, n+2.$$

Thus the relative cohomology groups

$$H^j(\sigma; G), \quad j = n, n+1,\ G \in \mathscr{G}_n,$$

vanish; but, by the construction of the weak push-out in $\mathscr{C}_0$ it follows that

$$H^i(\sigma; G) \cong H^i(\tau; G), \quad \text{all } i, G.$$

Thus,

$$(1.4) \qquad H^j(\tau; G) = 0, \quad j = n, n+1,\ G \in \mathscr{G}_n.$$

Now let $V \in \mathscr{V}$, and consider $\tau^*\colon [W, V] \to [Y, V]$. Given $y \in [Y, V]$, the obstructions to the existence and uniqueness of a counterimage of y under τ^* lie in $H^j(\tau; \pi_n V)$, $j = n, n+1$; $n = r+1, r+2, \ldots$ From (1.4) it is immediately seen that these cohomology groups vanish, so that $\tau^*\colon [W, V] \cong [Y, V]$ and $\tau \in S$ as asserted.

THEOREM 1.5. *If $\mathscr{V}$ is Postnikov-true and if $S = S(\mathscr{V})$, then the Adams S-completion Y_S of a given object Y of $\mathscr{C}_r$ exists, provided that there exists a small subset Σ_Y of the set $S_Y = \{s \in S \mid s\colon Y \to Y'\}$ with the property that, for each $s \in S_Y$, there exists $\sigma \in \Sigma_Y$ and $u \in \mathscr{C}_r$ with $us = \sigma$.*

Proof. By Theorem 1.4, S admits a calculus of left fractions. Now suppose $s_i\colon X_i \to Y_i$ lies in S for each $i \in I$. Then $\vee X_i$, $\vee Y_i \in \mathscr{C}_r$ if X_i, $Y_i \in \mathscr{C}_r$, $i \in I$ and

$$\vee s_i\colon \vee X_i \to \vee Y_i$$

obviously lies in S. Thus we obtain the result by applying the Theorem in [3] or Theorem 4.6 in [1].

EXAMPLE 1. We can take $\mathscr{V}$ to be the category of r-connected spaces, $r \geqslant 1$, whose homotopy groups belong to a given class of abelian groups. In this case we get an application of Theorem 1.4; but we cannot infer the existence of an Adams completion in this generality, since we have no reason to suppose that the solution set condition (the hypothesis of Theorem 1.5) will be satisfied. We will give an interesting special case where we can draw this inference, but will defer consideration of this to an appendix, since it will be easier in this case to make the argument depend on the discussions in Section 3.

EXAMPLE 2. We can take $\mathscr{V}$ to be the category of r-connected spaces, $r \geqslant 1$, whose homotopy groups are P-local, where P is a family of primes. Thus the spaces of $\mathscr{V}$ are P-local in the sense, for example, of [9]. In this case the family S is precisely the family of *P-equivalences* in $\mathscr{C}_r$ [9]. For if $s\colon X \to Y$ is a P-equivalence and V is P-local then $s^*\colon [Y, V] \cong [X, V]$. Conversely if $s^*\colon [Y, V] \cong [X, V]$ for all P-local V in $\mathscr{C}_r$, $r \geqslant 1$, then $s_P^*\colon [Y_P, V] \cong [X_P, V]$, where $s_P\colon X_P \to Y_P$ is the P-localization of s. It follows immediately that s_P is a (homotopy) equivalence, so that s is a P-equivalence. But then [4] Y_S is just the P-localization Y_P of Y.

It is natural to consider the question of which objects of $\mathscr{C}_r$ are S-complete. Obviously, a satisfactory situation is one in which all objects of $\mathscr{V}$ are S-complete, but we would expect, in general, to find S-complete objects outside $\mathscr{V}$.

THEOREM 1.6. *If $\mathscr{V}$ is Postnikov-true and if $S = S(\mathscr{V})$ then $Z \in \mathscr{C}_r$ is S-complete if and only if the induced map $s^*\colon [Y, Z] \to [X, Z]$ is bijective for all s in S.*

Proof. If Z is S-complete we have, according to the definition of the Adams S-completion, for each $s\colon X \to Y$ in S a commutative diagram whose horizontal arrows are bijections,

$$\begin{array}{ccc} \mathscr{C}_r[S^{-1}](Y, Z) & \cong & [Y, Z] \\ \downarrow{\scriptstyle s^\dagger} & & \downarrow{\scriptstyle s^*} \\ \mathscr{C}_r[S^{-1}](X, Z) & \cong & [X, Z] \end{array}$$

But since $s \in S$, s is invertible in $\mathscr{C}_r[S^{-1}]$ and hence $s^\dagger$ is bijective; so therefore is s^*.

Conversely, let $Z \in \mathscr{C}_r$ be such that s^* is bijective for all s in S. It follows that if $s\colon Z \to U$ lies in S, then there exists a unique $t\colon U \to Z$ with $ts = 1$. By Proposition 1.1, t also lies in S. In view of Theorem 1.4 we may apply the criterion of [4] to infer that 1_Z is the Adams S-completion of Z.

Remark. Plainly it was only the converse which required Theorem 1.4 and hence the hypothesis that $\mathscr{V}$ be Postnikov-true. The inference that if Z is S-complete then $s^*\colon [Y, Z] \to [X, Z]$ is bijective for all s in S is a quite general categorical fact.

COROLLARY 1.7. *If $\mathscr{V}$ is Postnikov-true then all objects of $\mathscr{V}$ are S-complete, where $S = S(\mathscr{V})$.*

We may rephrase Theorem 1.6 by saying that the collection of S-complete spaces of $\mathscr{C}_r$ coincides with the *saturation* of $\mathscr{V}$. To illustrate these ideas let us revert to Example 2. With the precise definitions of that example it is plain that $\mathscr{V}$ is already saturated, since the S-complete spaces are precisely the P-local spaces. However it is obvious that we could replace $\mathscr{V}$ by the smaller family $\mathscr{V}_0$ consisting of Eilenberg–MacLane spaces $K(G; k)$ with G P-local and $k \geqslant r+1$; we would then obtain the same family S and $\mathscr{V}$ would be the saturation of $\mathscr{V}_0$. It would even suffice to replace $\mathscr{V}$ by $\mathscr{V}_1$ consisting of the spaces $K(\mathbf{Z}_P, k)$, $k \geqslant r+1$.

Let us now further suppose — examples will be given below — that $\mathscr{V}$ is an admissible category for a cohomology theory $h_0\colon \mathscr{V} \to Ab$. Then we may form the Kan extension $h_1\colon \mathscr{C}_r \to Ab$ of h_0 along the embedding $\mathscr{V} \subseteq \mathscr{C}_r$ and, as pointed out in Proposition 1.3, $h_1(s)$ will be an equivalence for all $s \in S$. In particular, if $e: Y \to Y_S$ is the S-completion, then

$$h_1(e): h_1(Y_S) \cong h_1(Y).$$

Since Y_S is S-complete this suggests, in the light of Theorem 1.6, that it would be interesting to study the Kan extension of h_0 along the embedding of $\mathscr{V}$ in the full subcategory of $\mathscr{C}_r$ generated by the saturation of $\mathscr{V}$.

EXAMPLE 1. Let $\mathscr{V}$ be the full subcategory of $\mathscr{C}_r$ whose objects have homotopy groups in some acyclic Serre class C. Then if $h_0: \mathscr{V} \to Ab$ is a cohomology theory (obtained, for example, by restricting a cohomology theory defined on $\mathscr{C}_r$), we may form the Kan extension $h_1: \mathscr{C}_r \to Ab$, which will again be a cohomology theory (see Example 3.12 of [5]). It follows that $h_1(s)$ is an isomorphism for all $s \in S = S(\mathscr{V})$.

EXAMPLE 2. Let $\mathscr{V}$ be the full subcategory of $\mathscr{C}_r$ whose objects are the P-local complexes for some family of primes P. Since the property of being P-local is preserved under mapping cones, (weak) pullbacks and passage to the universal cover of the loop-space, it follows from Corollary 3.9 of [5] that the Kan extension h_1 of a cohomology theory $h_0: \mathscr{V} \to Ab$ is again a cohomology theory. Then $h_1(s)$ is an isomorphism if $s \in S = S(\mathscr{V})$, that is, if s is a P-equivalence. It may be shown that $h_1 X = h_0 X_P$.

2. The nilpotent case. The preliminary results (Propositions 1.1, 1.2, 1.3) of the preceding section were explicitly valid also in the case $r = 0$, that is, when we were considering the homotopy category $\mathscr{C}_0$ of connected (pointed) CW-complexes. However in order to obtain an analog of Theorem 1.4 in this case we must make an appropriate definition of a *Postnikov-true* family $\mathscr{V}$. If we wish to use only ordinary obstruction theory, then we must insist that the objects of $\mathscr{V}$ be nilpotent. Recall that, if $V \in \mathscr{C}_0$ then $\pi_1 V$ acts on $\pi_n V$ and we may form the *lower central series* of $\pi_n V$,

(2.1) $$\pi_n V = \Gamma^1 \pi_n V, \Gamma^2 \pi_n V, \ldots,$$

as in [9]. If $n = 1$, then (2.1) is just the usual lower central series of the group $\pi_1 V$. Moreover, V is *nilpotent* if and only if, for each $n \geq 1$, there exists $c = c(n)$ such that $\Gamma^{c+1} \pi_n V$ is the trivial subgroup of $\pi_n V$. Equivalently, V is nilpotent if and only if, for each $n \geq 1$, the nth stage of the Postnikov system of V,

(2.2) $$V_n \xrightarrow{p_n} V_{n-1},$$

admits a *refinement* (or factorization)

$$V_n = U_c \xrightarrow{q_c} U_{c-1} \xrightarrow{q_{c-1}} \ldots \xrightarrow{q_2} U_1 \xrightarrow{q_1} U_0 = V_{n-1},$$

where the fibration $q_i: U_i \to U_{i-1}$ is induced by a map

$$g_i: U_{i-1} \to K(\Gamma^i \pi_n V / \Gamma^{i+1} \pi_n V, n+1).$$

DEFINITION 2.1. We say that the collection $\mathscr{V}$ of objects of $\mathscr{C}_0$ is *Postnikov-true* if, whenever V belongs to $\mathscr{V}$, then V is nilpotent and the Eilenberg–MacLane spaces

(2.3) $$K(\Gamma^i \pi_n V / \Gamma^{i+1} \pi_n V, k), \quad k = n, n+1, n+2;\ n \geq 1$$

also belong to $\mathscr{V}$.

Remark. This definition is quite compatible with that given in Section 1; for if V is 1-connected then V is nilpotent and $\Gamma^1 \pi_n V = \pi_n V, \Gamma^2 \pi_n V = \{0\}$, and so the spaces (2.3) consist of the spaces (1.1), together with the singleton.

With this definition of a Postnikov-true family $\mathscr{V}$, Theorem 1.4 — and hence also Theorem 1.5 — remain valid in the case $r = 0$. Notice, however, that the argument of Theorem 1.4 does not permit us to replace $\mathscr{C}_0$ by $\mathscr{N}$, the category of nilpotent spaces, since we do not have a weak push-out construction in $\mathscr{N}$.

EXAMPLE 1. We can take $\mathscr{V}$ to be the collection of nilpotent spaces whose homotopy groups belong to a given generalized Serre class of groups in the sense of [9], p. 43.

EXAMPLE 2. We can take $\mathscr{V}$ to be the collection of spaces whose homotopy groups are finite p-groups. Indeed, these spaces are automatically nilpotent; explicitly, we have [1].

PROPOSITION 2.1. *If V is a connected complex whose homotopy groups are finite p-groups, then V is nilpotent.*

Proof. Certainly $\pi_1 V$, being a finite p-group, is nilpotent. Given the action of $\pi_1 V$ on $\pi_n V$, form the semi-direct product G_n. Then G_n is a finite p-group and hence nilpotent. Thus (Theorem 2.7 of [8]) the action of $\pi_1 V$ on $\pi_n V$ is nilpotent.

With this choice of $\mathscr{V}$, if Y is of finite type, then Y_S exists and is the Sullivan p-profinite completion of Y.

EXAMPLE 3. We can take $\mathscr{V}$ to be the collection of P-local nilpotent spaces. Then the family S consists of those maps s which induce an isomorphism in homology with Z_P-coefficients, that is, a P-isomorphism in integral homology. This implies that Y_S is the Bousfield localization [2] of Y with respect to the homology theory $H_*(-; Z_P)$. In particular, if Y is itself nilpotent then Y_S is just Y_P, the P-localization of Y [9].

Theorem 1.6 and Corollary 1.7 (and the subsequent remarks) again extend in straightforward manner to the non-simply-connected case. However, a delicate problem arises with respect to the examples which close Section 1. For the category $\mathscr{V}$, consisting of certain nilpotent spaces, will not be admissible for a cohomology theory since it will not admit, in general, mapping cones. It may be possible to circumvent this difficulty by considering cohomology theories on $\mathscr{C}_0$ and their restrictions to $\mathscr{V}$.

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(1) This proposition is surely well-known, but we give a very simple proof.

3. The non-nilpotent case. In order to obtain an analog of Theorem 1.4 for $\mathscr{C}_0$ in the case where we do not insist that the objects of $\mathscr{V}$ be nilpotent, we need to use non-simple obstruction theory. Thus the obstructions to extensions and homotopies of maps will lie in cohomology groups with *local* (*twisted*) coefficients. In order to carry out this generalization we will have need of Robinson's construction [12] of a representing object for cohomology groups with local coefficients, so we now review this construction, and introduce a significant modification.

Let A be an abelian group and $\operatorname{Aut}A$ its group of automorphisms. Let $K(A, k)$ be an Eilenberg–MacLane complex which is a topological group on which $\operatorname{Aut}A$ acts cellularly. Let $Q = K(\operatorname{Aut}A, 1)$, so that the universal cover $\tilde{Q}$ is a contractible complex on which $\operatorname{Aut}A$ acts freely and cellularly. The diagonal action of $\operatorname{Aut}A$ on $K(A, k)\times\tilde{Q}$ is free and cellular, and we denote the quotient space by $\hat{K}(A, k)$. The projection $K(A, k)\times Q\to\tilde{Q}$ induces a fibre map $q\colon \hat{K}(A, k)\to Q$ with fibre $K(A, k)$ and the inclusion $\tilde{Q}\to K(A, k)\times\tilde{Q}$, given by $x\mapsto(0, x)$, $x\in\tilde{Q}$ induces a standard section $l\colon Q\to\hat{K}(A, k)$ of the bundle map q. We identify Q with its image under l.

Let (W, Y) be a CW-pair such that W, Y are connected and fix a base point in Y. It is well-known that, for any group G, there is a natural bijection

(3.1) $$[W, K(G, 1)]\cong\operatorname{Hom}(\pi_1 W, G).$$

Let $\chi\colon \pi_1 W\to\operatorname{Aut}A$ be a homomorphism defining a local system of groups on W, and let $e = e(\chi)\colon W\to K(\operatorname{Aut}A, 1)$ belong to the homotopy class corresponding to χ under the bijection (3.1). Then Robinson shows that, for cohomology groups of (W, Y) with coefficients in the local system χ, we have a set-bijection

(3.2) $$H^k(W, Y; \chi)\cong[W, Y;\ \hat{K}(A, k), Q]_e,$$

where, on the right, we designate the set of fibrewise homotopy classes of maps $f\colon W, Y\to\hat{K}(A, k), Q$ such that $qf = e\colon W\to Q$.

Suppose now that A is, in fact, a π-module and that $\varepsilon\colon \pi\to\operatorname{Aut}A$ describes the module structure. Then we may choose a map $u\colon K(\pi, 1)\to K(\operatorname{Aut}A, 1)$ inducing ε and use u to obtain an induced fibre map over $K(\pi, 1)$,

(3.3) $$\begin{array}{ccccc} K(A, k) & \longrightarrow & L & \xrightarrow{r} & K(\pi, 1)\\ \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u}\\ K(A, k) & \longrightarrow & \hat{K}(A, k) & \xrightarrow{q} & Q\end{array}$$

Moreover, the canonical section $l\colon Q\to\hat{K}(A, k)$ induces a canonical section $m\colon K(\pi, 1)\to L$ such that $vm = lu$; we, likewise, use m to embed $K(\pi, 1)$ in L. Let us suppose further that $\chi\colon \pi_1 W\to\operatorname{Aut}A$ factors as $\pi_1 W\xrightarrow{\alpha}\pi\xrightarrow{\varepsilon}\operatorname{Aut}A$ and let α correspond to the homotopy class of $d\colon W\to K(\pi, 1)$; we may, of course, suppose that $e = ud$. It is then evident from (3.2) that

(3.4) $$H^k(W, Y;\ \varepsilon\alpha)\cong[W, Y;\ L, K(\pi, 1)]_d.$$

It is this modification of Robinson's construction which we will exploit in the sequel. Indeed, if no confusion is to be feared, we will regard A as a fixed π-module and suppress the symbol ε from (3.4), writing

(3.5) $$H^k(W, Y; \alpha)\cong[W, Y;\ L, K(\pi, 1)]_d,$$

where $\alpha\colon \pi_1 W\to\pi$ and the homotopy class of $d\colon W\to K(\pi, 1)$ corresponds to α under [1] (3.1). This will especially be the case when V is a connected space, $A = \pi_n V$, $n\geqslant 2$, and $\pi = \pi_1 V$.

We next need two elementary lemmas.

Lemma 3.1. *If $g\colon Z\to W$ induces a bijection $[W, L]\cong[Z, L]$, then it induces a bijection $[W, K(\pi, 1)]\cong[Z, K(\pi, 1)]$.*

Proof. This follows immediately from the fact that L dominates $K(\pi, 1)$.

Lemma 3.2. *If $g\colon Z\to W$ induces a bijection $[W, L]\cong[Z, L]$, then it induces an isomorphism*
$$g^*\colon H^k(W; \alpha)\cong H^k(Z; \alpha\circ\pi_1 g), \quad \textit{for every } \alpha\colon \pi_1 W\to\pi.$$

Proof. The set $[W, L]$ may be identified [2] with the disjoint union of the sets $[W, L]_d$ as d ranges over a set of representatives of the elements of $[W, K(\pi, 1)]$. But now Lemma 3.1 tells us that, under the hypotheses of the lemma, dg will then range over a set of representatives of the elements of $[Z, K(\pi, 1)]$. Thus the bijection $[W, L]\cong[Z, L]$ breaks down into a collection of bijections $[W, L]_d\cong[Z, L]_{dg}$, and the lemma follows from (3.5).

We are now ready to make the appropriate modification of the concept of a Postnikov-true family in the case of a collection of non-nilpotent (that is, not necessarily nilpotent) spaces $\mathscr{V}$. For any connected space V, write $L(\pi_n V, k)$ for the space L of (3.3) with $A = \pi_n V$, $n\geqslant 2$, $\pi = \pi_1 V$; thus
$$K(\pi_n V, k)\to L(\pi_n V, k)\xrightarrow{r} K(\pi_1 V, 1).$$

Definition 3.1. We say that the collection $\mathscr{V}$ of objects of $\mathscr{C}_0$ is *twisted-Postnikov-true* if, whenever V belongs to $\mathscr{V}$, so do the (twisted Eilenberg–MacLane) spaces
$$L(\pi_n V, k), \qquad k = n, n+1, n+2;\ n\geqslant 2.$$

We now prove the main result of this section — the analog of Theorem 1.4.

Theorem 3.3. *If $\mathscr{V}$ is twisted-Postnikov-true, and if $S = S(\mathscr{V})$, then S admits a calculus of left fractions.*

Proof. As in the proof of Theorem 1.4 we assume given a diagram in $\mathscr{C}_0$,
$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y\\ \downarrow{\scriptstyle\sigma} & & \\ Z & & \end{array}$$

---

[1] Note that (3.5) generalizes (3.2), which is the case $\varepsilon = 1$.

[2] As in the proof of Lemma 1.5 of [12].

with σ in S, construct the weak push-out in $\mathscr{C}_0$,

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ Z & \xrightarrow[\psi]{} & W \end{array} \tag{3.6}$$

and must show that τ is in S. We will feel free, where convenient, to assume all the maps in (3.6) to be (cofibration) inclusions, and to use the abbreviated notations of the proof of Theorem 1.4 to refer to the cohomology groups of σ and τ.

We want to show that, if $V \in \mathscr{V}$, then

$$\tau^*\colon [W, V] \to [Y, V]$$

is bijective. Now, the obstructions to the existence and uniqueness of a counterimage under τ^* of an element of $[Y, V]$ will, with the exception of the lowest dimensions, all be in the cohomology groups of τ with (local) coefficients $\pi_n V$,

$$H^k(\tau; \alpha), \quad k = n, n+1; \ n \geqslant 2,$$

where $\alpha\colon \pi_1 W \to \pi_1 V$ is induced by some map $W^n \to V$ for the existence problem and some map $W \to V$ for the uniqueness problem (see [11] or [10], p. 308). We now show that

$$\psi^*\colon H^k(\tau; \alpha) \cong H^k(\sigma; \alpha \circ \pi_1 \psi). \tag{3.7}$$

In view of the independent interest of (3.7), we prefer to state it in appropriate generality as follows.

PROPOSITION 3.4. *Given any weak push-out* (3.6) *in* $\mathscr{C}_0$ *and a* π*-module* A *and given any homomorphism* $\alpha\colon \pi_1 W \to \pi$ *defining a local system of groups on* W, *then* ψ *induces an isomorphism*

$$\psi^*\colon H^k(\tau; \alpha) \cong H^k(\sigma; \alpha \circ \pi_1 \psi).$$

Proof. By invoking (3.4) we see that we must establish a set bijection

$$\psi^*\colon [W, Y; L, K(\pi, 1)]_d \cong [Z, X; L, K(\pi, 1)]_{d\psi},$$

where $d\colon W \to K(\pi, 1)$ is in the class corresponding to α. Now (with all maps in (3.6) inclusions) W is just the union of Y and Z with X amalgamated. Given $f\colon Z, X \to L, K(\pi, 1)$ with $rf = d|Z$ we set $h = d|Y\colon Y \to K(\pi, 1)$, then $f|X = rf|X = d|X = h|X$ so that we may define $g\colon W \to L$ by $g|Y = h$, $g|Z = f$ and thus ψ^* is surjective.

Given $g, g'\colon W, Y \to L, K(\pi, 1)$ with $rg = rg' = d$ and a fibrewise homotopy $g|Z \overset{H}{\simeq} g'|Z$ rel X, we set up a fibrewise homotopy $g \overset{G}{\simeq} g'$ rel Y by defining $G|Y = \text{const}$, $G|Z = H$, thus showing that ψ^* is injective.

Thus (3.7) is established. Now we know that

$$\sigma^*\colon [Z, L(\pi_n V, k)] \cong [X, L(\pi_n V, k)], \quad k = n, n+1, n+2; \ n \geqslant 2.$$

Thus we infer from Lemma 3.2 that

$$\sigma^*\colon H^k(Z; \beta) \cong H^k(X; \beta \circ \pi_1 \sigma), \quad k = n, n+1, n+2; \ n \geqslant 2, \tag{3.8}$$

for any $\beta\colon \pi_1 Z \to \pi_1 V$. Appeal to the exact cohomology sequence [12] of σ,

$$\ldots \to H^i(Z; \beta) \xrightarrow{\sigma^*} H^i(X; \beta \circ \pi_1 \sigma) \to H^i(\sigma; \beta) \to H^{i+1}(Z; \beta) \xrightarrow{\sigma^*} H^{i+1}(X; \beta \circ \pi_1 \sigma) \to \ldots$$

then shows that

$$H^k(\sigma; \beta) = 0, \quad k = n, n+1; \ n \geqslant 2. \tag{3.9}$$

From (3.7) and (3.9) we infer that

$$H^k(\tau; \alpha) = 0, \quad k = n, n+1; \ n \geqslant 2,$$

so that the obstructions, in higher dimensions, to the existence and uniqueness of a counterimage under $\tau^*\colon [W, V] \to [Y, V]$ of an arbitrary element of $[Y, V]$ all lie in trivial cohomology groups.

It remains to consider the low dimensions. These require a separate discussion. However, our discussion will show that the low-dimensional obstructions will vanish *without any condition whatsoever on the family* $\mathscr{V}$. Thus, according to [11], p. 42, a map $h\colon Y \to V$ is extendable to $Y \cup W^2$ if and only if there exists a homomorphism

$$\theta\colon \pi_1 W \to \pi_1 V$$

such that $\theta \circ \pi_1 \tau = \pi_1 h$. Now, since σ belongs to S, there exists $k\colon Z \to V$ such that $k\sigma = h\varphi$. The maps k and h clearly combine to yield a map $W \to V$ inducing an appropriate θ. The subtler point is the following: again according to [11], p. 42, if two maps $h_0, h_1\colon W \to V$ agree on Y, then $h_0|Y \cup W^1 \simeq h_1|Y \cup W^1$ rel Y if and only if $\pi_1 h_0 = \pi_1 h_1$. Now since σ belongs to S and $h_0 \psi \sigma = h_1 \psi \sigma$ it follows that $h_0 \psi \simeq h_1 \psi$. We now apply Van Kampen's Theorem (see, for instance, [7] or [10], p. 360) to infer from

$$\pi_1(h_0)\pi_1(\psi) = \pi_1(h_0 \psi) = \pi_1(h_1 \psi) = \pi_1(h_1)\pi_1(\psi),$$
$$\pi_1(h_0)\pi_1(\tau) = \pi_1(h_0 \tau) = \pi_1(h_1 \tau) = \pi_1(h_1)\pi_1(\tau),$$

that $\pi_1 h_0 = \pi_1 h_1$, as required. This completes the proof of Theorem 3.3.

The analogs of Theorem 1.5, 1.6 and Corollary 1.7 clearly remain valid in the case $r = 0$.

EXAMPLE 1. We can take $\mathscr{V}$ to be the collection of connected complexes whose higher homotopy groups belong to any family of abelian groups. In this case, as in Example 1 of Section 1, we have no guarantee that the solution set condition is satisfied, but we are able to apply Theorem 3.3. We can modify this example by also requiring the fundamental group to belong to a given family.

EXAMPLE 2. Let P be a family of primes and let $\mathscr{V}$ be the collection of connected complexes whose homotopy groups (including the fundamental group) are finite

P-groups. With this choice of $\mathscr{V}$, the Adams S-completion of Y exists, provided Y is of finite type, and is the Sullivan P-profinite completion of Y. This example, in the case that P is the family of all primes, so that Y_S is the Sullivan profinite completion of Y when Y is of finite type, is mentioned without proof by Adams in [1].

4. Appendix: Improvements and further examples.

4.1. Our arguments in Sections 1, 2, 3, to establish the principal theorem (Theorems 1.4, 3.3), have all been based on the weak push-out diagrams (1.3), (3.6). Now it is easy to see that the weak push-out always has the property that if $\sigma^*\colon [Z, V] \to [X, V]$ is surjective, so is $\tau^*\colon [W, V] \to [Y, V]$. Thus, in reality, the subtlety of the argument has always been concerned with the *injectivity* of τ^*, that is, with the obstructions to (extending) homotopies. We preferred to lump together obstructions to extensions and obstructions to homotopies to give a more unified treatment, since we were not too much concerned with finding best possible conditions under which our theorems were valid (see the parenthetical remark in the Introduction). However, by concentrating on the injectivity of τ^* we are able to improve our results by *eliminating the case* $k = n+2$ in our definition of a (twisted) Postnikov-true collection of complexes.

4.2. A perhaps more significant improvement may be made by observing that the argument which concluded the proof of Theorem 3.3, to handle the low-dimensional obstructions, could just as well have been applied in Section 2. Thus, in Definition 2.1, we may *eliminate the case* $n = 1$ and retain our conclusion — the validity of the analog of Theorem 1.4 for $r = 0$ in the case of collections $\mathscr{V}$ of nilpotent complexes.

4.3. Suppose that we are given a homotopy category $\mathscr{C}$ and a collection $\mathscr{V}$, giving rise to a family $S(\mathscr{V})$ of morphisms of $\mathscr{C}$; and that we are further given a full subcategory $\mathscr{C}'$ of $\mathscr{C}$, a collection $\mathscr{V}'$, and a corresponding family $S'(\mathscr{V}')$, such that

$$S'(\mathscr{V}') = S(\mathscr{V}) \cap \mathscr{C}'. \tag{4.1}$$

Suppose finally that $Y \in \mathscr{C}'$ and admits, as an object of $\mathscr{C}$, an Adams S-completion Y_S which also belongs to $\mathscr{C}'$. It is then plain that, provided S' admits a calculus of left fractions, Y_S is also the Adams S'-completion of Y. For, by [4], we have only to check the existence of $e\colon Y \to Y_S$ in S' such that, for any $s'\colon Y \to Z'$ in S', there exists a unique $t'\colon Z' \to Y_S$ in S' with $t's' = e$. Now since Y_S is the Adams S-completion of Y, we know that there exists $e\colon Y \to Y_S$ in S such that, for any $s\colon Y \to Z$ in S, there exists a unique $t\colon Z \to Y_S$ in S with $ts = e$. But since Y, Y_S are in $\mathscr{C}'$ so is e, so that, by (4.1), e is in S'; and if s' is in S' then, again by (4.1), s' is in S, and so there exists a unique t' in S with $t's' = e$. However, again by (4.1), this t', and, indeed, any other t' satisfying $t's' = e$, will be in S', thus establishing the criterion for the Adams S'-completion.

This argument enables us to derive further examples from those already given. In particular, we may take Example 2 of Section 3, so that $\mathscr{C} = \mathscr{C}_0$, $\mathscr{V}$ = collection of connected complexes whose homotopy groups are finite P-groups. We then

take $\mathscr{C}' = \mathscr{C}_r$, $r \geqslant 1$, $\mathscr{V}'$ = collection of r-connected complexes whose homotopy groups are finite P-groups. As already observed in Example 1 of Section 1, we may apply Theorem 1.4 to this case. Moreover if Y is of finite type and in $\mathscr{C}_r$, then the Sullivan P-profinite completion of Y is also in $\mathscr{C}_r$. Thus to apply our previous remark, we only have to verify (4.1) in this case. Since $\mathscr{V}' \subseteq \mathscr{V}$ it is obvious that $S(\mathscr{V}) \cap \mathscr{C}' \subseteq S'(\mathscr{V}')$. Conversely, suppose that $g\colon X \to Y$ is in $S'(\mathscr{V}')$ and let $V \in \mathscr{V}$. Let V' be the r-connected cover of V. Then $V' \in \mathscr{V}'$ and the projection $p\colon V' \to V$ induces $p_*\colon [Z, V'] \cong [Z, V]$ for all Z in $\mathscr{C}'$. Since X, Y are in $\mathscr{C}'$ and g is in $S'(\mathscr{V}')$ we may immediately infer that $g^*\colon [Y, V] \cong [X, V]$, so that $g \in S(\mathscr{V})$ and (4.1) is verified. We conclude:

EXAMPLE 4.1. Let P be a family of primes and let $\mathscr{V}$ be the collection of r-connected complexes, $r \geqslant 0$, whose homotopy groups are finite P-groups. We consider $\mathscr{V}$ as a subclass of $\mathscr{C}_r$ and construct $S = S(\mathscr{V})$. Then the Adams S-completion of $Y \in \mathscr{C}_r$ exists, provided Y is of finite type, and is the Sullivan P-profinite completion of Y.

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