# The structure of precipitous ideals

by

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**Abstract.** Kunen proved that every normal $\varkappa^+$-saturated ideal is strong. In fact, his proof is valid for normal precipitous ideals, where a $\varkappa$-complete ideal is precipitous iff the associated Boolean ultrapower is well-founded. We provide a characterization of those precipitous ideals which are strong. This characterization has several corollaries on the structure of precipitous ideals.

**1. Introduction.** One of the major results in the theory of saturated ideals is Kunen's theorem [1] that if $I$ is a normal $\varkappa^+$-saturated ideal on the cardinal $\varkappa$ then $I$ is strong, i.e., in $L[I]$, $I \cap L[I]$ is a prime ideal (and hence $\varkappa$ is measurable in $L[I]$ and $L[I]$ is the $\varkappa$-model $L[U]$). Obviously the normality condition is not a necessary one since any non-normal measure ultrafilter is strong. The purpose of this paper is to give a complete characterization of those ($\varkappa$-complete) $\varkappa^+$-saturated ideals on $\varkappa$ which are strong. In fact, the characterization is valid for a wider class of ideals, namely precipitous ideals.

As corollaries of the main theorem we derive several results on the structure of precipitous ideals. For example, all precipitous ideals admit a decomposition into strong ideals. We also show that in $L[U]$, the $\varkappa$-model, all precipitous ideals are atomic.

The paper is organized as follows. We discuss the basic properties of ideals at the end of this section. In Section 2 we review some of the theory of $\varkappa$-models. In Section 3 we prove the main theorem and in Section 4 various corollaries of it are derived. The proof of the main theorem uses techniques developed by Kunen [1] and Solovay [7], and a familiarity with these papers would be helpful.

Some of the results in this paper appeared in the author's doctoral dissertation [8] and were announced in [9]. The author wishes to express his gratitude to his adviser, James E. Baumgartner, for his constant encouragement and interest in this work.

We use standard set-theoretic notation. The term *ideal on* $\varkappa$ is reserved for a proper, non-principal, $\varkappa$-complete ideal on $\varkappa$, i.e., a collection $I \subseteq \mathscr{P}(\varkappa)$ such that $\{\alpha\} \in I$ if $\alpha<\varkappa$, $\varkappa$ is not in $I$, if $A \subseteq B \in I$ then $A \in I$ and if $\{A_\alpha\colon \alpha<\lambda\} \subseteq I$ and $\lambda<\varkappa$ then $\bigcup \{A_\alpha\colon \alpha<\lambda\} \in I$. An ideal on $\varkappa$ is *normal* if whenever $\{A_\alpha\colon \alpha<\varkappa\} \subseteq I$ then $\{\beta<\varkappa\colon \text{for some } \alpha<\beta,\ \beta \in A_\alpha\} \in I$. An *almost disjoint family* (with respect to an ideal $I$ on $\varkappa$) is a collection $\{A_\alpha: \alpha<\lambda\}\subseteq\mathscr{P}(\varkappa)-I$ such that if $\alpha<\beta<\lambda$ then $A_\alpha \cap B_\beta \in I$. An ideal $I$ is said to be *$\lambda$-saturated*, for a cardinal $\lambda$, if every almost disjoint family has cardinality less than $\lambda$. Equivalently, $I$ is $\lambda$-saturated if the associated Boolean algebra $\mathscr{P}(\varkappa)/I$ has the $\lambda$-chain condition. We use *sat* $I$ to denote the least cardinal $\lambda$ such that $I$ is $\lambda$-saturated. Also, we use $I^*$ to denote the filter on $\varkappa$ dual to $I$, i.e., $I^* = \{A\subseteq\varkappa: \varkappa-A \in I\}$. Likewise if $F$ is a filter on $\varkappa$, $F^*$ denotes the ideal dual to $F$.

DEFINITION 1.1. If $I$ is an ideal on $\varkappa$ and $A\in\mathscr{P}(\varkappa)-I$ then

$$I\restriction A = \{B\subseteq\varkappa: B\cap A\in I\},$$

i.e., $I\restriction A$ is the ideal on $\varkappa$ generated by $I$ together with $\varkappa-A$.

Note that $I\subseteq I\restriction A$ and if $I$ is normal or $\lambda$-saturated then so is $I\restriction A$.

Much of the theory of ideals is proved using the Boolean ultrapower construction introduced by Solovay in [7]. If $I$ is an ideal on $\varkappa$ let $\mathscr{B}(I)$ (or just $\mathscr{B}$ if it is clear which ideal is meant) be the Boolean completion of $\mathscr{P}(\varkappa)/I$. The canonical generic set, $G$, in $V^{\mathscr{B}(I)}$ is an ultrafilter on $\mathscr{P}(\varkappa)\cap V$ and may be used to form $\mathrm{Ult}(V, G)$, the ultrapower of $V$ using $G$-equivalence classes of functions from $\varkappa$ to $V$ which lie in $V$, within $V^{\mathscr{B}}$. Then in $V^{\mathscr{B}}$ there is a canonical elementary embedding $j\colon V\to\mathrm{Ult}(V, G)$.

DEFINITION 1.2. An ideal $I$ on $\varkappa$ is called *precipitous* if the associated Boolean ultrapower is well-founded, i.e., if $\|\mathrm{Ult}(V, G)$ is well-founded$\|^{\mathscr{B}} = 1$. If $I$ is precipitous we identify $\mathrm{Ult}(V, G)$ with its transitive collapse.

We refer the reader to Solovay [7] for the details of this construction. For more on precipitous ideals see Jech and Prikry [3]. Note that if $A\in\mathscr{P}(\varkappa)-I$ then there is a natural complete epimorphism $\pi\colon \mathscr{B}(I)\to\mathscr{B}(I\restriction A)$ (induced by $\pi([C]) = [C\cap A]$ if $C\subseteq\varkappa$) which induces $\pi_*\colon V^{\mathscr{B}(I)}\to V^{\mathscr{B}(I\restriction A)}$. Moreover, for $a\in V^{\mathscr{B}(I)}$ and $\varphi(x)$ a formula of set theory,

$$\pi(\|\varphi(a)\|^{\mathscr{B}(I)}) = \|\varphi(\pi_* a)\|^{\mathscr{B}(I\restriction A)}.$$

Thus if $\|\varphi(a)\|^{\mathscr{B}(I)}\geqslant[A]$ then $\|\varphi(\pi_* a)\|^{\mathscr{B}(I\restriction A)} = 1$. It follows that if $I$ is precipitous and $A\in\mathscr{P}(\varkappa)-I$ then $I\restriction A$ is precipitous.

The following theorem, due to Solovay [7], ties together the above notions.

THEOREM 1.3. (a) *If $\varkappa$ bears a $\varkappa^+$-saturated ideal then $\varkappa$ bears a normal $\varkappa^+$-saturated ideal.*

(b) *If $I$ is a $\varkappa^+$-saturated ideal on $\varkappa$ then $I$ is precipitous.*

On fact about this ultrapower construction which we shall use is the following: if $I$ is a precipitous ideal on $\varkappa$, $j\colon V\to\mathrm{Ult}(V, G)$ is as above and $\delta$ is a strong limit cardinal with $\mathrm{cf}\,\delta\neq\varkappa$ then $\|j(\check\delta) = \check\delta\|^{\mathscr{B}} = 1$.

Kunen [1] has shown that $\varkappa^+$-saturated ideals are related to large cardinals by proving that if $\varkappa$ bears a $\varkappa^+$-saturated ideal then $\varkappa$ is a measurable cardinal in an inner model. Jech (unpublished) proved that this conclusion follows from the weaker hypothesis that $\varkappa$ bears a precipitous ideal.



DEFINITION 1.4 (Kunen [1]). An ideal $I$ on $\varkappa$ is *strong* if, in $L[I]$, $I\cap L[I]$ is a prime ideal, i.e., $I\cap L[I]$ is dual to a measure ultrafilter in $L[I]$. (Note that this definition differs from Kunen's in that $I\cap L[I]$ is not required to be normal.)

THEOREM 1.5. (a) (Kunen [1]) *If $I$ is a normal $\varkappa^+$-saturated ideal on $\varkappa$ then $I$ is strong. Hence $L[I]$ is the $\varkappa$-model* (see Section 2).

(b) (Jech) *If $I$ is a precipitous ideal on $\varkappa$ then $\varkappa$ is measurable in an inner model. Hence the $\varkappa$-model exists.*

**2. A review of $\varkappa$-models.** Recall that if $M$ is a transitive model of ZFC containing all ordinals in which "the universe is constructible from a normal ultrafilter on $\varkappa$" is true, then $M$ is called a $\varkappa$-model. Silver [6] proved that the GCH holds in a $\varkappa$-model and Kunen [1] used the machinery of iterated ultrapowers to develop an extensive theory of such models. In this section we summarize the results about $\varkappa$-models which we shall need.

THEOREM 2.1 (Kunen [1]). *Let $M$ be a $\varkappa$-model. Then, in $M$, $\varkappa$ bears a unique normal ultrafilter $U$ and if $W$ is a measure ultrafilter on $\varkappa$ then $W$ is isomorphic to $U_n$, the $n$-fold power of $U$, for some finite $n$.*

The external structure of $\varkappa$-models is summarized in the following theorem, also due to Kunen. Recall that if $W$ is a measure ultrafilter on $\varkappa$ in $N$, a transitive model of ZFC, then $\mathrm{Ult}_\alpha(N, W)$ denotes the (transitive collapse of the) $\alpha$th iterated ultrapower of $N$ using $W$.

THEOREM 2.2. (a) *If a $\varkappa$-model exists it is unique.*

(b) *If $W$ is a measure ultrafilter on $\varkappa$ then $L[W]$ is the $\varkappa$-model.*

(c) *If $M$ is the $\varkappa$-model with normal ultrafilter $U$ and $N$ is the $\lambda$-model, where $\lambda\geqslant\varkappa$, then $N$ is $\mathrm{Ult}_\alpha(M, U)$ for some ordinal $\alpha$.*

Thus we shall use $L[U]$ to denote the $\varkappa$-model (assuming it exists) and $U$ to denote the unique normal ultrafilter on $\varkappa$ in $L[U]$. Furthermore, $i_\alpha$ will be used to denote the canonical elementary embedding from $L[U]$ to $\mathrm{Ult}_\alpha(L[U], U)$.

There is another characterization of measure ultrafilters in $L[U]$ which, though much less simple to state than that in Theorem 2.1, will be useful to us.

DEFINITION 2.3. Let $U$ be the normal ultrafilter on $\varkappa$ in $L[U]$ and let $\zeta$ be any ordinal with $\zeta\geqslant\varkappa$. Then $U^\zeta$ is the *measure ultrafilter* on $\varkappa$ in $L[U]$ defined as follows: choose $\alpha$ such that $i_\alpha(\varkappa)>\zeta$ and let $U^\zeta = \{x\in\mathscr{P}(\varkappa)\cap L[U]\colon \zeta\in i_\alpha(x)\}$. It is easy to see that this definition is independent of the choice of $\alpha$.

THEOREM 2.4 (Kunen [1], Paris [5]). *In $L[U]$ every measure ultrafilter on $\varkappa$ is equal to $U^\zeta$ for some $\zeta<i_\omega(\varkappa)$.*

This theorem allows us to define a function of ordinals which will be very important to us.

DEFINITION 2.5. Assuming the existence of the $\varkappa$-model, define a function $K\colon \{\alpha\colon \alpha\geqslant\varkappa\}\to i_\omega(\varkappa)$ by setting $K(\zeta)$ equal to the least ordinal such that $U^\zeta = U^{K(\zeta)}$.

THEOREM 2.4 shows that $K$ is well-defined. Note that $K(K(\zeta)) = K(\zeta)$.

We shall also need a technical lemma from [1] concerning embeddings of $\varkappa$-models which differ from the canonical embedding.

LEMMA 2.6. *Suppose* $k\colon L[U] \to \mathrm{Ult}_\alpha(L[U], U)$ *is an elementary embedding such that*

(i) $k(\alpha) = \alpha$ *if* $\alpha < \varkappa$, *and*

(ii) *if* $\delta > \alpha$ *is a strong limit cardinal of cofinality strictly greater than* $\varkappa$ (*in the sense of* $V$) *then* $k(\delta) = \delta$.

*Then for any* $x \in \mathscr{P}(\varkappa) \cap L[U]$, $k(x) = i_\alpha(x)$.

This lemma is proved by showing that all subsets of $\varkappa$ in $L[U]$ can be defined from ordinals which are fixed points of both $k$ and $i_\alpha$.

**3. The main theorem.** An important property of a normal precipitous ideal $I$ on $\varkappa$ is that in the ultrapower formed within $V^{\mathscr{B}}$ the ordinal $\varkappa$ is represented by the identity function on $\varkappa$, which we denote by id. More precisely, $\|[\mathrm{id}]_G = \check{\varkappa}\|^{\mathscr{B}} = 1$. The theorem below generalizes Theorem 1.5 by showing that a precipitous ideal is strong if the identity function represents a unique ordinal, not necessarily $\varkappa$. In order to obtain a true characterization, it turns out that we must consider $K([\mathrm{id}])$ rather than [id], where $K$ is as defined in Definition 2.5.

DEFINITION 3.1. If $I$ is a precipitous ideal on $\varkappa$ let

$$Z(I) = \{\zeta\colon \|K([\mathrm{id}]) = \check{\zeta}\|^{\mathscr{B}} > 0\}.$$

Note that $|Z(I)| < \mathrm{sat}\, I$, if $\zeta \in Z(I)$ then $K(\zeta) = \zeta$, and if $I$ is normal then $Z(I) = \{\varkappa\}$.

THEOREM 3.2. *For a precipitous ideal $I$ on $\varkappa$, $I$ is strong iff* $|Z(I)| = 1$. *Moreover, $\zeta$ is the least ordinal such that $I \cap L[U]$ is dual to $U^\zeta$ iff* $Z(I) = \{\zeta\}$.

Proof. Suppose $Z(I) = \{\zeta\}$. Form $\mathrm{Ult}(V, G)$ within $V^{\mathscr{B}}$, letting $j\colon V \to \mathrm{Ult}(V, G)$ denote the canonical embedding. Note that by Theorems 1.3 (a) and 1.5 the $\varkappa$-model $L[U]$ exists. We shall show that $L[I] \subseteq L[U]$ and that $I \cap L[U] = U^{\zeta *}$. This implies that $I$ is strong for if $x \in \mathscr{P}(\varkappa) \cap L[I]$ then $x \in L[U]$ whence either $x$ or $\varkappa - x$ is in $U^\zeta$. Thus either $x$ or $\varkappa - x$ is in $I$. Since $K(\zeta) = \zeta$, it remains only to show that $I \cap L[U] = U^{\zeta *}$ (for then $I \cap L[U] \in L[U]$ whence $L[I] \subseteq L[U]$).

Work within $V^{\mathscr{B}}$. Let $k = j \restriction L[U]$ and let $\sigma = j(\check{\varkappa})$. Then $k$ is an elementary embedding from $L[U]$ to $L[kU]$, the $\sigma$-model. Applying Theorem 2.2(c) shows that $L[kU]$ must be $\mathrm{Ult}_\beta(L[U], U)$ for some ($\mathscr{B}$-valued) ordinal $\beta$ such that $i_\beta(\check{\varkappa}) = \sigma$. Since $k$ is induced by the ultrapower embedding $j$, it follows that (see remark following Theorem 1.3) $k$ leaves fixed any cardinal $\lambda$ which, in $V$, is strong limit and not cofinal with $\varkappa$. Therefore, by Lemma 2.6, $k$ agrees with $i_\beta$ on $\mathscr{P}(\varkappa) \cap L[U]$.

Now, to show $I \cap L[U] = U^{\zeta *}$ consider $x \in \mathscr{P}(\varkappa) \cap L[U]$. Then $x \in I$ iff $[x] = \|[\mathrm{id}]_G \in k(\check{x})\|^{\mathscr{B}} = 0$ iff $\|[\mathrm{id}]_G \in i_\beta(\check{x})\| = 0$ iff

$$\|\check{x} \in \check{U}^{[\mathrm{id}]} = \check{U}^{K[\mathrm{id}]} = \check{U}^{\check{\zeta}} = (U^\zeta)^{\check{}}\| = 0$$



iff $x \notin U^\zeta$ iff $x \in U^{\zeta *}$. Thus $I \cap L[U] = U^{\zeta *}$ and the proof of one implication is complete.

For the converse, suppose $I$ is a strong precipitous ideal on $\varkappa$ and $\zeta$ is the least ordinal less than $i_\omega(\varkappa)$ such that $I \cap L[U]$ is dual to $U^\zeta$. Then $\|K[\mathrm{id}] = \check{\zeta}\|^{\mathscr{B}} = 1$ for suppose $\|K[\mathrm{id}] = \check{\delta}\|^{\mathscr{B}} = [A] > 0$. Then by the remarks following Definition 1.2 $\|K[\mathrm{id}] = \check{\delta}\|^{\mathscr{B}(I \restriction A)} = 1$ and so the implication above yields that $I \restriction A$ is strong and $I \restriction A \cap L[U]$ is dual to $U^\delta$. But $I \subseteq I \restriction A$ and, since $I \cap L[U]$ is a maximal ideal, $I \restriction A \cap L[U] = I \cap L[U]$. Hence, since $K(\delta) = \delta$, $\delta = \zeta$ and the theorem is proved.

We remark that $Z(I)$ can be defined without reference to Boolean-valued universes, though it is tedious to do so. With this in mind, Theorem 3.2 can be regarded as equating a metamathematical property of an ideal (strength) with a combinatorial property.

**4. The structure of precipitous ideals.** If $I$ is a precipitous ideal on $\varkappa$ then there is a maximal almost disjoint family $\{A_\alpha\colon \alpha < \lambda\}$ with respect to $I$ and an enumeration $\{\zeta_\alpha\colon \alpha < \lambda\}$ of $Z(I)$ (possibly with repetition) such that $\|K[\mathrm{id}]_G = \check{\zeta}_\alpha\|^{\mathscr{B}} \geqslant [A_\alpha]$. Hence by Theorem 3.2 each $I \restriction A_\alpha$ is a strong ideal. Thus we have proved the following corollary.

COROLLARY 4.1. *If $I$ is a precipitous ideal on $\varkappa$ then there is a maximal almost disjoint family $\{A_\alpha\colon \alpha < \lambda\}$ such that each $I \restriction A_\alpha$ is a strong ideal.*

Note that $I$ is recoverable from the $I \restriction A_\alpha$ since $x$ lies in $I$ iff $x$ lies in each $I \restriction A_\alpha$. We remark that one cannot choose the $A_\alpha$ so that each $I \restriction A_\alpha$ is normal unless $I$ itself is normal.

If $\varkappa$ is a strongly compact cardinal then any ($\varkappa$-complete) ideal on $\varkappa$ can be extended to a strong ideal, in fact to a prime ideal. The following weaker statement is true for all cardinals and is clear from the previous corollary since each $I \restriction A_\alpha$ is a strong extension of $I$.

COROLLARY 4.2. *If $I$ is a precipitous ideal then $I$ can be extended to a strong precipitous ideal.*

Theorem 3.2 provides some information about the structure of ideals in the $\varkappa$-model, $L[U]$. It is not hard to see, by standard methods, that the only cardinal in $L[U]$ which bears a precipitous ideal is the measurable cardinal $\varkappa$. The following corollary shows that even on $\varkappa$, the prime ideals are essentially the only precipitous ideals.

COROLLARY 4.3. *In $L[U]$, the $\varkappa$-model, all precipitous ideals $I$ on $\varkappa$ are atomic, i.e., there is a maximal almost disjoint family $\{A_\alpha\colon \alpha < \lambda\}$ such that each $I \restriction A_\alpha$ is prime.*

Proof. Work in $L[U]$. If $I$ is a precipitous ideal on $\varkappa$ choose $\{A_\alpha\colon \alpha < \lambda\}$ as in Corollary 4.1. Then $I \restriction A_\alpha \cap L[I \restriction A_\alpha]$ is prime so $I \restriction A_\alpha \cap L[I \restriction A_\alpha] = I \restriction A_\alpha \cap L[U] = I \restriction A_\alpha$ is prime.

Corollary 4.3 also holds in $L[U]$, the universe constructible from a coherent sequence of ultrafilters as defined by Mitchell [4]. That it does not hold of measurable cardinals in general has been shown by Baumgartner who proved the following theorem by the methods of Kunen–Paris [2].

THEOREM 4.4. *If it is consistent that a measurable cardinal exists then it is consistent that there is a measurable cardinal $\varkappa$ which bears an atomless $\varkappa^+$-saturated ideal $I$ ($I$ is atomless if for any $A \in \mathcal{P}(\varkappa) - I$, $I \upharpoonright A$ is not prime).*

Sketch of proof. Using Theorem 2.1 of [2], assume $M$ is a model of ZFC such that, in $M$, $D_1$ and $D_2$ are distinct normal ultrafilters on the measurable cardinal $\varkappa$. Choose $X \in D_2 - D_1$ such that for all $\alpha \in X$, $\alpha$ is a regular cardinal. For $i = 1, 2$ let $j_i$: $M \to \mathrm{Ult}(M, D_i)$ be the canonical embedding and let $P$ be the Easton partial ordering in $M$ for adding a single generic subset of each $\alpha$ in $X$. Then $j_1 P \cong P \times Q$ where $Q$ is $\varkappa^+$-closed and $j_2 P \cong P \times P_\varkappa \times R$ where $R$ is $\varkappa^+$-closed and $P_\varkappa$ is the partial ordering for adding a generic subset of $\varkappa$ to $\mathrm{Ult}(M, D_2)$. Now, if $G$ is $P \times Q \times R$-generic over $M$ then arguments as in [2] can be used to show that, in $M[G]$, $D_1$ extends to a normal ultrafilter on $\varkappa$ and $D_2$ extends to the dual of a non-atomic $\varkappa^+$-saturated ideal on $\varkappa$.

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# On Postnikov-true families of complexes and the Adams completion

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**Abstract.** Let $\mathcal{C}_r$ be the homotopy category of pointed $r$-connected CW-complexes, let $\mathcal{V}$ be a non-empty collection of objects of $\mathcal{C}_r$, and let $S(\mathcal{V})$ be the family of those morphisms $s$: $X \to Y$ in $\mathcal{C}_r$ such that $s^*$: $[Y, V] \to [X, V]$ is bijective for every $V$ in $\mathcal{V}$. In the case where $r \geqslant 1$, it is proved that the Adams $S(\mathcal{V})$-completion exists if, essentially, $\mathcal{V}$ has the property that, whenever $V$ belongs to $\mathcal{V}$, then so do the Eilenberg–MacLane spaces $K(\pi_n V, k)$, $k = n, n+1, n+2$; $n \geqslant r+1$. An extension of the result is obtained in the case where $r = 0$ and the objects of $\mathcal{V}$ are assumed to be nilpotent, by using the characterization of a nilpotent space in terms of the principal refinement of its Postnikov tower. It is pointed out that this framework is adequate to obtain the Sullivan $p$-profinite completion, where $p$ is a prime. Finally, one considers the general non-simply-connected case, where one does not insist that the objects of $\mathcal{V}$ be nilpotent. Here, non-simple obstruction theory is needed, and therefore the Eilenberg–MacLane spaces must be replaced by certain spaces $L(A, k)$, obtained by a significant modification from the spaces $\hat{K}(A, k)$ constructed by C. A. Robinson as representing objects for cohomology with local coefficients. The Sullivan $P$-profinite completion is obtained among the applications, where $P$ is an arbitrary family of primes.

**0. Introduction.** We consider, for a fixed $r$, the homotopy category $\mathcal{C}_r$ of pointed $r$-connected CW-complexes, and a non-empty collection $\mathcal{V}$ of objects of $\mathcal{C}_r$. With respect to $\mathcal{V}$ we form the family $S = S(\mathcal{V})$ of those morphisms $s$: $X \to Y$ in $\mathcal{C}_r$ with respect to which every $V$ in $\mathcal{V}$ is *left-closed*, that is, those morphisms $s$ such that

$$s^*\colon [Y, V] \to [X, V]$$

is bijective for every $V$ in $\mathcal{V}$. The family $\mathcal{S}$ is plainly saturated, and we may ask whether the (generalized) Adams $S$-completion [1, 3] exists (1).

We introduce a condition on $\mathcal{V}$, of a rather natural character, which comes close to guaranteeing the existence of the Adams completion and which is certainly verified in the two cases of principal importance — the $p$-profinite completion and the $P$-localization, where $p$ is a prime and $P$ is a (possibly empty) family of primes.

(1) In this case, the Adams completion has been called by Harvey Wolff the $\mathcal{V}$-*localization*.