On lattices of continuous order-preserving functions

by

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Abstract. Let $\boldsymbol{D}$ be the category of distributive lattices and lattice homomorphisms, and $\boldsymbol{D}_{01}$ that of distributive lattices with universal bounds and homomorphisms preserving these bounds.

For $L \in \boldsymbol{D}_{01}$ let PL be the Priestley dual space of L (cf. H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. 24 (1972), pp. 507–530), and let $CI\,(PL)$ the lattice of all continuous order-preserving functions from PL into the reals (under the pointwise order). We show that $L \mapsto CI\,(PL)$ defines, in a natural way, a covariant functor $F\colon \boldsymbol{D}_{01} \mapsto \boldsymbol{D}$. For any $L \in \boldsymbol{D}_{01}$, L is a regular sublattice of FL, and for L_1, L_2 in $\boldsymbol{D}_{01}$ and f in $\mathrm{Hom}(L_1, L_2)$, Ff is an extension of f which preserves injectivity or surjectivity of f.

For any space X, let $C(X)$ be the *lattice* of all continuous real-valued functions defined on X. Our main results are. 1) FL is conditionally complete iff L is such. This generalizes a result of Stone stating that a Boolean algebra B with dual space X is complete iff $C(X)$ is conditionally complete. (cf. M. H. Stone, *Boundedness properties of function lattices*, Canad. J. Math 1 (1949), pp. 176–186). 2) FL is DuBois–Raymond separable iff L is such (L is DuBois–Raymond separable iff $a_1 < a_2 < a_3 < \ldots < b_3 < b_2 < b_1$, $a_i, b_j \in L$ implies the existence of $c \in L$ such that $a_i \leqslant c \leqslant b_j$. See, e.g., R. C. Walker, *The Stone-Čech Compactification*, Springer 1974). Our result generalizes that of Seever's stating that a Boolean algebra B with dual space X is DuBois–Raymond separable iff $C(X)$ is such (cf. G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. 133 (1968), pp. 267–280). Finally, we consider the question in which cases $CI\,(PL)$ is a regular sublattice of $C(PL)$. Some open questions are sketched.

0. Introduction. This paper was inspired by the following result contained in Stone's 1949 paper [9]: Let B be a Boolean algebra and $S(B)$ its dual space, and consider the *lattice* of all continuous, real-valued functions defined on $S(B)$ with the pointwise order, which we denote by $C(S(B))$. Then $C(S(B))$ is conditionally complete if and only if so is B. There is a certain lack of symmetry in this situation: One starts with a Boolean algebra and ends up with a distributive lattice. Using the duality theory for distributive lattices as developed by H. Priestley in [6] and [7], we are able to show that Stone's theorem holds — in an (almost) symmetric form — in a much wider setting: Instead of a Boolean algebra, we may take an arbitrary

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distributive lattice with universal bounds as our starting point. In fact, we show that the map F sending any such lattice to the lattice of all order-preserving continuous real-valued functions on its (Priestley) dual space is functorial, and we derive some of the basic properties of this functor.

In more detail, the paper is organized as follows: Section 1 establishes the notation and terminology, which roughly follows Balbes-Dwinger [1] for lattice-theoretic and Kelley [3] for topological concepts. Section 2 contains the basic facts from Priestley duality theory, expressed in a purely topological way. The key fact is that every distributive lattice having universal bounds is isomorphic to the lattice of all clopen increasing subsets of a uniquely determined compact totally order disconnected ordered space. In Section 3, we consider the map F defined above, establish its functorial properties and give an intrinsic description of a real-valued continuous order-preserving function on a Priestley space. Section 4 is devoted to the proof of the theorem mentioned in the first paragraph: L is conditionally complete iff so is $F(L)$. Section 5 deals with the following concept: A poset is called Dubois–Reymond separable iff whenever a strictly decreasing sequence sits on top of a strictly increasing one, then the two sequences are separated by at least one element of the poset. Seever [8] proved that a Boolean algebra B is DuBois–Reymond separable iff $C(S(B))$ is. We show that this is still valid when B is replaced by an arbitrary distributive lattice L with universal bounds and $C(S(B))$ by $F(L)$. We should like to point out that Stone's as well as Seever's result easily follow from our theorems by restricting the order relations of the representation spaces under consideration to the trivial ones. Finally, Section 6 deals with the question under what conditions the lattice of order-preserving continuous real-valued functions on a dual space is a regular sublattice of the lattice of *all* continuous real-valued functions. Some open problems are mentioned in Sections 5 and 6.

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1. Preliminaries. Unless otherwise mentioned, L (sometimes with subscripts) always signifies a distributive lattice with a greatest and a smallest element. $\boldsymbol{D}_{01}$ denotes the category of all such lattices together with lattice homomorphisms preserving zeros and units, while $\boldsymbol{D}$ is the category of all distributive lattices and lattice homomorphisms.

An *ordered space* is triple $(X, \tau, \leqslant)$ consisting of a set X, a topology τ on X (identified with the set of all τ-open subsets of X) and a (partial) order relation $\leqslant$ on X. A subset $A \subseteq X$ is called *increasing* or an *upper end* iff $x \in A$, $y \in X$ and $x \leqslant y$ together imply $y \in A$. *Decreasing* subsets or *lower ends* are defined dually. Note that $A \subseteq X$ is increasing iff $X \setminus A$ is decreasing. A function f between ordered spaces is called *increasing* or *order-preserving* iff $x \leqslant y$ implies $fx \leqslant fy$ for all x, y in the domain of f (*decreasing* or *order-reversing* iff $fx \geqslant fy$ holds for all x, y with $x \leqslant y$). We denote by $\boldsymbol{U}(X)$ the collection of all open increasing and by $\boldsymbol{L}(X)$ that of all open decreasing subsets of $X = (X, \tau, \leqslant)$, writing simply $\boldsymbol{U}$ and $\boldsymbol{L}$ if no confusion is likely to arise. $\boldsymbol{U}$ and $\boldsymbol{L}$ obviously define topologies on X — coarser than τ — which we call the *upper* and *lower topologies* of X, respectively. For $A \subseteq X$, $\mathrm{cl}A$, $\boldsymbol{U}$-$\mathrm{cl}A$ and $\boldsymbol{L}$-$\mathrm{cl}A$ denote the closure of A with respect to τ, the upper and the lower topology on X, respectively, and similarly for other topological operations. Note that for any $A \subseteq X$, $\boldsymbol{L}$-$\mathrm{cl}A$ is the smallest closed upper end containing A, and dually $\boldsymbol{U}$-$\mathrm{cl}A$ the smallest closed lower end over A. Similarly for $\boldsymbol{L}$-$\mathrm{int}A$ and for $\boldsymbol{U}$-$\mathrm{int}A$. An ordered space is *totally order disconnected* iff for every two elements such that not $x \leqslant y$ there exist a clopen upper end J and a clopen lower end D satisfying $J \cap D = \emptyset$, $x \in J$ and $y \in D$. We let $\boldsymbol{CU}$ and $\boldsymbol{CL}$ stand for the sets of all clopen members of $\boldsymbol{U}$ and $\boldsymbol{L}$, respectively. We denote by $\boldsymbol{TOD}$ the category of all compact totally order-disconnected ordered spaces together with continuous order-preserving maps. The following lemma summarizes some of the properties of the spaces in $\boldsymbol{TOD}$.

LEMMA 1. *Let* $X \in \boldsymbol{TOD}$. *Then*:

(i) $\boldsymbol{CU}$ *and* $\boldsymbol{CL}$ *are open bases for* $\boldsymbol{U}$ *and* $\boldsymbol{L}$, *respectively.*

(ii) *If* $F \subseteq X$ *is closed, the smallest upper and lower ends containing* F *are closed.*

(iii) *Let* F_1, F_2 *be closed, disjoint,* F_1 *increasing and* F_2 *decreasing. Then there exist disjoint sets* $U_1 \in \boldsymbol{U}$, $U_2 \in \boldsymbol{L}$ *such that* $F_1 \subseteq U_1$ *and* $F_2 \subseteq U_2$.

Proofs may be found in Priestley [5], [6] or in Nachbin [4]. Finally, $\boldsymbol{R}$ always stands for the set of real numbers.

2. Priestley duality. The representation theory for distributive lattices as developed in Priestley [6], [7] may be phrased — to the extent we are going to use it — as follows:

THEOREM 2. $\boldsymbol{D}_{01}$ *is isomorphic with the dual of* $\boldsymbol{TOD}$.

The underlying contravariant functor P: $\boldsymbol{D}_{01} \to \boldsymbol{TOD}$ may be described as follows: Any $L \in \boldsymbol{D}_{01}$ is (lattice-) isomorphic with the lattice of all clopen upper ends of a uniquely determined space $PL \in \boldsymbol{TOD}$, in short, $L \cong \boldsymbol{CU}(PL)$. For $f \in \mathrm{Hom}(L_1, L_2)$, $Pf \in \mathrm{Hom}(PL_2, PL_1)$ may be given by

$$(*)\qquad Pf(x_2) = \bigcap \{C \in \boldsymbol{CU}(PL_1),\ x_2 \in f'C\} \setminus \bigcup \{C \in \boldsymbol{CU}(PL_1),\ x_2 \bar{\in} f'C\} \quad \text{for all } x_2 \in PL_2$$

where f': $\boldsymbol{CU}(PL_1) \to \boldsymbol{CU}(PL_2)$ is the lattice homomorphism induced by f and the isomorphisms $L_i \cong \boldsymbol{CU}(PL_i)$ $(i = 1, 2)$.

This description is a topological version of that given in [6] or [7]. As an illustration, we show in detail that Pf is indeed well-defined: Let C_i $(1 \leqslant i \leqslant n)$, C_j^* $(1 \leqslant j \leqslant m)$ belong to $\boldsymbol{CU}(PL_1)$ such that $x_2 \in f'C_i$ and $x_2 \bar{\in} f'C_j^*$. Hence $C = C_1 \cap \ldots \cap C_n$ and $C^* = C_1^* \cup \ldots \cup C_m^*$ belong to $\boldsymbol{CU}(PL_1)$ and, f' being a lattice homomorphism, $x_2 \in f'C$ and $x_2 \bar{\in} f'C^*$. This implies $C \setminus C^* \neq \emptyset$, for otherwise $C \subseteq C^*$, whence $f'C \subseteq f'C^*$ and $x_2 \in f'C^*$, a contradiction. Let R be the right-hand side of $(*)$. So R has the finite intersection property and, consequently, is nonempty, since PL_1 is compact. It remains to show that R is a singleton. Let

$y, z \in R$ and not $y \leqslant z$. By total order disconnectedness, there exists $C_0 \in \boldsymbol{CU}(PL_1)$ such that $y \in C_0$, $z \bar{\in} C_0$. Now if $x_2 \in f'C_0$, then $C_0 \supseteq R$ and hence $z \in C_0$. If $x_2 \bar{\in} f'C_0$, $C_0 \cap R = \emptyset$ and hence $y \bar{\in} C_0$. These contradictions show that (not $y \leqslant z$) is not possible, whence by symmetry $y = z$. ∎

It is easy to show that Pf is continuous and order-preserving. Moreover, if f is one-to-one (onto), then Pf is onto (one-to-one), see (7).

THEOREM 3. *L is complete iff in PL the following condition holds: For all $U \in \boldsymbol{U}$, $\boldsymbol{L}$-cl$\,U \in \boldsymbol{U}$, and for all $L \in \boldsymbol{L}$, $\boldsymbol{U}$-cl$\,L \in \boldsymbol{L}$.*

Proof ([7], p. 521). This condition is called extremal order disconnectedness. ∎

3. Lattices of continuous increasing real-valued functions. For any ordered space X, the set $CI(X)$ of all continuous increasing real-valued functions on X is a distributive lattice under the pointwise order. Obviously, $CI(X)$ is a sublattice of the lattice $C(X)$ of all continuous real-valued functions on X. However, $CI(X)$ is not a subring of the *ring* $C(X)$, since the difference of two increasing functions need not be increasing.

Let $X_1, X_2 \in \boldsymbol{TOD}$ and $h \in \mathrm{Hom}(X_2, X_1)$. Define $CI(h)\colon CI(X_1) \to CI(X_2)$ by $CI(h)(f) := f \circ h$ for any $f \in CI(X_1)$. Being the composition of two continuous increasing mappings, this clearly defines a member of $CI(X_2)$.

LEMMA 4. *CI is a contravariant functor from $\boldsymbol{TOD}$ to $\boldsymbol{D}$. If h is one-to-one, $CI(h)$ is onto, and if h is onto, $CI(h)$ is one-to-one.*

Proof. The first part is immediate. So let $h \in \mathrm{Hom}(X_2, X_1)$ be one-to-one, and consider an arbitrary function $g \in CI(X_2)$. We have to construct $f \in CI(X_1)$ such that $g = f \circ h$. Since h is one-to-one, hX_2 is a closed homeomorphic copy of X_2 contained in X_1. Consequently, $g \circ h^{-1}\colon hX_2 \to \boldsymbol{R}$ is continuous and increasing. By the Nachbin–Tietze extension theorem (see Nachbin [4], p. 48, or Priestley [5]), there exists $f \in CI(X_1)$ extending $g \circ h^{-1}$. Clearly, $f \circ h = g$, so $CI(h)$ is onto.

Now let $h \in \mathrm{Hom}(X_2, X_1)$ be onto, and consider $f, f' \in CI(X_1)$, $f \neq f'$. Hence there is an $x_1 \in X_1$ such that $fx_1 \neq f'x_1$. h being onto, we may find $x_2 \in X_2$ such that $x_1 = hx_2$. Thus $CI(h)(f)(x_2) = (f \circ h)(x_2) = fx_1 \neq f'x_1 = (f'h)(x_2) = CI(h)(f)(x_2)$. So $CI(h)(f) \neq CI(h)(f')$ and $CI(h)$ is one-to-one. ∎

For the rest of the paper, we shall be concerned with the properties of the composite functor $F\colon \boldsymbol{D}_{01} \to \boldsymbol{D}$ defined by $F = CI \circ P$. The next theorem sums up some of the basic properties of F:

THEOREM 5. (i) *F is covariant functor from $\boldsymbol{D}_{01}$ to $\boldsymbol{D}$.*

(ii) *For any $L \in \boldsymbol{D}_{01}$, L is a sublattice of FL: moreover, the embedding $L \to FL$ is regular.*

Let $f \in \mathrm{Hom}(L_1, L_2)$. Then:

(iii) *If f is one-to-one (onto), then Ff is one-to-one (onto).*

(iv) *Ff is an extension of f.*

(i) and (iii) are clear from the preceding discussion. Proofs for (ii) and (iv) are based on the following two lemmata, which describe the members of $CI(X)$ ($X \in \boldsymbol{TOD}$) intrinsically in purely topological terms.

For any real-valued function f on a space X and for $\alpha \in \boldsymbol{R}$, define $P_f(\alpha) = \mathrm{pos}(f-\alpha) = \{x \in X; fx > \alpha\}$. We have the following

LEMMA 6. *Let $X \in \boldsymbol{TOD}$ and $D \subseteq \boldsymbol{R}$ be dense. Assume $\{W_\lambda;\ \lambda \in D\}$ is a family of open upper ends of X such that*

(i) $\bigcap W_\lambda = \emptyset$,

(ii) $\bigcup W_\lambda = X$,

(iii) *$\lambda_1, \lambda_2 \in D$, $\lambda_1 < \lambda_2$ imply $W_{\lambda_1} \supseteq \mathrm{cl}\, W_{\lambda_2}$.*

Then $f\colon X \to \boldsymbol{R}$ defined by $fx := \sup\{\lambda \in D,\ x \in W_\lambda\}$ is continuous and increasing. If additionally

(iv) *$W_\lambda = \bigcup \{W_\mu\colon \mu > \lambda\}$ for all $\lambda \in D$,*

then $W_\lambda = P_f(\lambda)$ and (iii) holds in the stronger form $\lambda_1, \lambda_2 \in D$, $\lambda_1 < \lambda_2$ imply $W_{\lambda_1} \supseteq \boldsymbol{L}$-cl$\,W_{\lambda_2}$.

Conversely, if $g\colon X \to \boldsymbol{R}$ is continuous and increasing, the family $\{P_g(\lambda), \lambda \in D\}$ has the properties (i)–(iv).

Proof. Assume $\{W_\lambda, \lambda \in D\}$ has the properties (i)–(iii). (i) and (ii) guarantee that f is well-defined.

1) f is increasing: Let $x, y \in Y$, $x \leqslant y$ and $fx = \alpha$. This implies $x \in W_\lambda$ for all $\lambda \in D$, $\lambda < \alpha$. Since the sets W_λ are increasing, $y \in W_\lambda$ for all $\lambda \in D$, $\lambda < \alpha$, and so $fy \geqslant \alpha = fx$.

2) f is continuous: Consider the open interval $(\varrho, \to) \subseteq \boldsymbol{R}$. $f^{-1}(\varrho, \to) = \{x \in X;\ \sup\{\lambda \in D, x \in W_\lambda\} > \varrho\} = \{x \in X;\ x \in W_\lambda \text{ for some } \lambda > \varrho\} = \bigcup \{W_\lambda\colon \lambda > \varrho\}$ is open as the union of open sets. Similarly for the closed interval $[\varrho, \to)\colon f^{-1}[\varrho, \to) = \{x \in X;\ \sup\{\lambda \in D;\ x \in W_\lambda\} \geqslant \varrho\} = \{x \in X;\ x \in W_\lambda \text{ for all } \lambda < \varrho\} = \bigcap \{W_\lambda; \lambda < \varrho\}$. Now if $\lambda < \varrho$, choose $\lambda' \in D$ such that $\lambda < \lambda' < \varrho$; then by (iii) $W_\lambda \supseteq \mathrm{cl}\, W_{\lambda'}$; hence $\bigcap \{W_\lambda;\ \lambda < \varrho\} = \bigcap \{\mathrm{cl}\, W_\lambda;\ \lambda < \varrho\}$, closed as an intersection of closed sets. So $f^{-1}(\to, \varrho)$ is open and f is continuous. Assume now that $\{W_\lambda\}$ also satisfies (iv). Then:

3) $W_\lambda = P_f(\lambda)$: Let $\lambda \in D$ and $x \in P_f(\lambda)$. Hence $fx =: \alpha > \lambda$ and $x \in W_\mu$ for all $\mu < \alpha$, especially, $x \in W_\lambda$. Conversely, if $x \in W_\lambda = \bigcup \{W_\mu;\ \mu > \lambda\}$, then $x \in W_\mu$, for some $\mu' > \lambda$, that is, $fx \geqslant \mu' > \lambda$ and $x \in P_f(\lambda)$.

4) Assume $\lambda_1 < \lambda_2$. Since $W_\lambda = P_f(\lambda)$ for all $\lambda \in D$, we infer that

$$W_{\lambda_1} = P_f(\lambda_1) \supseteq f^{-1}[\tfrac{1}{2}(\lambda_1 + \lambda_2), \to) =: Q \supseteq P_f(\lambda_2) = W_{\lambda_2}.$$

f is continuous and increasing, so Q is closed and increasing, whence $Q \supseteq \boldsymbol{L}$-cl$\,W_{\lambda_2}$ and $W_{\lambda_1} \supseteq \boldsymbol{L}$-cl$\,W_{\lambda_2}$.

As for the converse, it is easy to check that for any $g \in CI(X)$, $\{P_g(\lambda);\ \lambda \in D\}$ satisfies (i)–(iv). ∎

In view of Lemma 6 we may identify members of $CI(X)$ with systems of sets

$\{W_\lambda\}$ satisfying (i)–(iv) above. The next (obvious) lemma characterizes the order relation on $CI(X)$ in the same terms:

LEMMA 7. *Let $f, g \in CI(X)$. Then $f \geqslant g$ iff $P_f(\lambda) \supseteq P_g(\lambda)$ for all $\lambda \in D$, where D is any dense subset of $\boldsymbol{R}$.*

Proof of Theorem 5. (ii) Identify L with $\boldsymbol{CU}(PL) =: \boldsymbol{CU}$, let $C \in \boldsymbol{CU}$ and define a family of sets $\{W_\alpha^C;\ \alpha \in \boldsymbol{R}\}$ by the following:

$$W_\alpha^C = \begin{cases} \emptyset & \text{for} \quad \alpha \geqslant 1, \\ C & \text{for} \quad 1 > \alpha \geqslant 0, \\ PL & \text{for} \quad 0 > \alpha. \end{cases}$$

By Lemma 6, $\{W_\alpha^C\}$ defines a function in $CI(PL)$, and it is easy to see that the rule $C \to \{W_\alpha^C;\ \alpha \in \boldsymbol{R}\}$ defines a lattice embedding $j\colon L \cong \boldsymbol{CU}(PL) \to CI(PL)$.

It remains to prove that j is regular. Let $\{C_i;\ i \in I\} \subseteq \boldsymbol{CU}$, and assume $C_0 = \sup\{C_i\}$ in $\boldsymbol{CU}$ for some $C_0 \in \boldsymbol{CU}$. This implies $C_0 = \boldsymbol{L}\text{-cl} \bigcup C_i$:

Clearly, $C_0 \supseteq \bigcup C_i$ and hence $C_0 \supseteq \boldsymbol{L}\text{-cl} \bigcup C_i$ since C_0 is closed and increasing. By Lemma 1(i), $\boldsymbol{L}\text{-cl} \bigcup C_i = \bigcap \{C_k;\ C_k \in \boldsymbol{CU}$ and $C_k \supseteq \boldsymbol{L}\text{-cl} \bigcup C_i\}$. But $C_k \supseteq \bigcup C_i$ for each such C_k whence $C_k \supseteq C_0$ and $\boldsymbol{L}\text{-cl} \bigcup C_i \supseteq C_0$, thus proving our claim.

Assume now that $\{P_g(\alpha);\ \alpha \in \boldsymbol{R}\}$ represents a function $g \in CI(PL)$ majorizing $jC_i = \{W_\alpha^C i;\ \alpha \in \boldsymbol{R}\}$ for all $i \in I$. This implies, for any $\alpha < 1$, that $P_g(\alpha) \supseteq C_i$. Fix α and choose α' such that $\alpha < \alpha' < 1$. By Lemma 6, $P_g(\alpha) \supseteq \boldsymbol{L}\text{-cl} P_g(\alpha') \supseteq P_g(\alpha') \supseteq \bigcup C_i$. Hence $C_0 = \boldsymbol{L}\text{-cl} \bigcup C_i \subseteq \boldsymbol{L}\text{-cl} P_g(\alpha') \subseteq P_g(\alpha)$ and by Lemma 7 g is seen to be greater than or equal to jC_0. In other words, j preserves the existing suprema. A dual proof works for infima, so (ii) is proved.

(iv) Let $f\colon \boldsymbol{CU}(PL_1) \to \boldsymbol{CU}(PL_2)$ be a lattice homomorphism. We have to show that $Ff(jC) = j(fC)$ for any member C of $\boldsymbol{CU}(PL_1)$, where we denote by j both of the canonical embeddings $\boldsymbol{CU}(PL_i)$ into $CI(PL_i)$ (defined as in the proof of (ii)).

If $h\colon PL_2 \to PL_1$ is continuous and increasing, $CI(h)$ sends any $g \in CI(PL_1)$ to $g \circ h$; in other words, $\{P_g(\alpha);\ \alpha \in \boldsymbol{R}\}$ is sent to $\{h^{-1}P_g(\alpha);\ \alpha \in \boldsymbol{R}\}$. Now let $C \in \boldsymbol{CU}(PL_1)$ and assume $h = Pf$. Then $h^{-1}C = fC$, as easily follows from the description of Pf given in Section 2. Now for $C \in \boldsymbol{CU}(PL_1)$, $jC = \{W_\alpha^C;\ \alpha \in \boldsymbol{R}\}$, and by the definition of j, $W_\alpha^C \in \boldsymbol{CU}(PL_1)$ for every $\alpha \in \boldsymbol{R}$. Putting everything together, we see that $\{W_\alpha^C;\ \alpha \in \boldsymbol{R}\}$ is sent to $\{fW_\alpha^C;\ \alpha \in \boldsymbol{R}\}$ by Ff, and the latter family of sets obviously coincides with $j(fC)$ by the definition of j. This completes the proof of Theorem 5. ■

From this point on, we forget about the functorial nature of F and restrict ourselves to the study of some of the properties of the object map defined by F. Any $L \in \boldsymbol{D}_{01}$ will be freely identified with $\boldsymbol{CU}(PL)$, and similarly functions in $CI(PL)$ with their corresponding set families as described by Lemma 6.

4. Completeness. In this section, we prove the following theorem which generalizes the classical result of Stone in [9]:

THEOREM 8. *FL is conditionally complete if and only if so is L.*

Proof. Of course, conditional completeness coincides with completeness in L, L having universal bounds.

I. If part. Assume L is complete. Consider functions f_i $(i \in I)$, $f_0 \in FL$ such that $f_i \leqslant f_0$ for all $i \in I$. We write $P_i(\alpha)$ instead of $P_{f_i}(\alpha)$ $(\alpha \in \boldsymbol{R})$, and identify f_i with $\{P_i(\alpha);\ \alpha \in \boldsymbol{R}\}$. It has to be shown that $\{f_i\}$ has a supremum in FL. Define

$$Q(\alpha) := \bigcup P_i(\alpha) \quad (i \in I) \quad \text{and} \quad R(\alpha) := \bigcup \{\boldsymbol{L}\text{-cl}\, Q(\beta);\ \beta > \alpha\}$$

for any $\alpha \in \boldsymbol{R}$.

1) $R(\alpha)$ is an open upper end. $R(\alpha)$ is obviously increasing as a union of increasing sets. Moreover, $\boldsymbol{L}\text{-cl}\, Q(\beta)$ is open since L is complete (Theorem 3), so $R(\alpha)$ is open.

2) $\bigcap \{R(\alpha);\ \alpha \in \boldsymbol{R}\} = \emptyset$. It is easy to see that

$$\bigcap \{R(\alpha);\ \alpha \in \boldsymbol{R}\} = \bigcap \{\boldsymbol{L}\text{-cl}\, Q(\alpha);\ \alpha \in \boldsymbol{R}\}.$$

Assume $\alpha_1 < \alpha_2$. Then

$$P_0(\alpha_1) \supseteq \{x \in X;\ f_0 x \geqslant \tfrac{1}{2}(\alpha_1 + \alpha_2)\} =: M \supseteq P_0(\alpha_2) \supseteq Q(\alpha_2),$$

applying Lemma 6 and the definition of $Q(\alpha)$. Hence $\boldsymbol{L}\text{-cl}\, Q(\alpha_2) \supseteq P_0(\alpha_1)$, M being closed and increasing, and so

$$\bigcap \{\boldsymbol{L}\text{-cl}\, Q(\alpha);\ \alpha \in \boldsymbol{R}\} \subseteq \bigcap \{P_0(\alpha);\ \alpha \in \boldsymbol{R}\} = \emptyset$$

by Lemma 6.

3) $\bigcup \{R(\alpha);\ \alpha \in \boldsymbol{R}\} = X$. Trivial.

4) $R(\alpha) = \bigcup \{R(\beta);\ \beta > \alpha\}$. Let $x \in R(\alpha)$, then $x \in \boldsymbol{L}\text{-cl}\, Q(\beta)$ for some $\beta > \alpha$. Choose β' such that $\beta > \beta' > \alpha$; then $x \in R(\beta')$. Hence $R(\alpha) \subseteq \bigcup \{R(\beta);\ \beta > \alpha\}$. The reverse inclusion is trivial.

5) $\alpha_1 < \alpha_2$ implies $R(\alpha_1) \supseteq \mathrm{cl}\, R(\alpha_2)$. Observe first that $R(\alpha) \subseteq \boldsymbol{L}\text{-cl}\, Q(\alpha)$ for all $\alpha \in \boldsymbol{R}$ (if $x \in R(\alpha)$, $x \in \boldsymbol{L}\text{-cl}\, Q(\beta)$ for some $\beta > \alpha$ and so $x \in \boldsymbol{L}\text{-cl}\, Q(\alpha)$ since $\alpha < \beta$ implies $Q(\alpha) \supseteq Q(\beta)$). Now

$$\mathrm{cl}\, R(\alpha_2) \subseteq \mathrm{cl}\, \boldsymbol{L}\text{-cl}\, Q(\alpha_2) = \boldsymbol{L}\text{-cl}\, Q(\alpha_2) \subseteq \bigcup \{\boldsymbol{L}\text{-cl}\, Q(\beta);\ \beta > \alpha_1\} = R(\alpha_1).$$

By Lemma 6, the function $g\colon X \to \boldsymbol{R}$ defined by $gx := \sup\{\alpha \in \boldsymbol{R};\ x \in R(\alpha)\}$ is continuous increasing and $R(\alpha) = P_g(\alpha)$ for all $\alpha \in \boldsymbol{R}$. We now prove that $g = \sup\{f_i\}$ in FL.

Consider $P_i(\alpha)$ for any fixed $i \in I$ and $\alpha \in \boldsymbol{R}$: By Lemma 6,

$$\begin{aligned} P_i(\alpha) &= \bigcup \{P_i(\beta);\ \beta > \alpha\} \subseteq \bigcup \{Q(\beta);\ \beta > \alpha\} \\ &\subseteq \bigcup \{\boldsymbol{L}\text{-cl}\, Q(\beta);\ \beta > \alpha\} = R(\alpha) = P_g(\alpha). \end{aligned}$$

Thus $g \geqslant f_i$ by Lemma 7.

Conversely, assume that $h \geqslant f_i$ for all $i \in I$, $h \in FL$. Hence, by Lemma 7, $P_i(\alpha) \subseteq P_h(\alpha)$ for all $i \in I$, $\alpha \in \boldsymbol{R}$, and so $Q(\alpha) \subseteq P_h(\alpha)$ for all $\alpha \in \boldsymbol{R}$. By Lemma 6,

$P_h(\alpha) \supseteq \boldsymbol{L}\text{-cl}\, P_h(\beta)$ whenever $\beta > \alpha$. Clearly, $Q(\beta) \subseteq P_h(\beta) \subseteq \boldsymbol{L}\text{-cl}\, P_h(\beta)$. Also $\boldsymbol{L}\text{-cl}\, Q(\beta) \subseteq \boldsymbol{L}\text{-cl}\, P_h(\beta) \subseteq P_h(\alpha)$ whenever $\alpha < \beta$. Thus $R(\alpha) = \bigcup \{\boldsymbol{L}\text{-cl}\, Q(\beta);\ \beta > \alpha\} \subseteq P_h(\alpha)$ and by Lemma 7, $g \leqslant h$.

Meet completeness of FL is established by the dual argument, so that if part is proved.

II. Only if part. Assume that $FL = CI(PL)$ is conditionally complete. Let $V \subseteq PL$ be any open increasing set. In view of Theorem 3, it suffices to show that $\boldsymbol{L}\text{-cl}\, V$ is open (the dual argument will take care of $\boldsymbol{U}\text{-cl}\, M$ for any open decreasing subset $M \subseteq PL$). By Lemma 1, $V = \bigcup \{C_\lambda \in \boldsymbol{CU}(PL);\ C_\lambda \subseteq V\}$. Let χ_λ be the characteristic functions of the C_λ. Clearly, $\chi_\lambda \in FL$ and $0 \leqslant \chi_\lambda \leqslant 1$. Hence $g = \sup\{\chi_\lambda\}$ exists in FL. $g \equiv 1$ on V; moreover $g^{-1}(1)$ is closed and increasing, whence $g \equiv 1$ on $\boldsymbol{L}\text{-cl}\, V$. Again by Lemma 1, $\boldsymbol{L}\text{-cl}\, V = \bigcap \{C_\mu \in \boldsymbol{CU}(PL);\ \boldsymbol{L}\text{-cl}\, V \subseteq C_\mu\}$. Consider a point x_0 not in $\boldsymbol{L}\text{-cl}\, V$. There exists, consequently, $C_0 \in \boldsymbol{CU}$ such that $x_0 \bar{\in} C_0$ and $C_0 \supseteq \boldsymbol{L}\text{-cl}\, V$. Denote by χ_0 the characteristic function of C_0. $\chi_0 \equiv 1$ on $\boldsymbol{L}\text{-cl}\, V$; hence $\chi_0 \geqslant \chi_\lambda$ for all λ, and so $\chi_0 \geqslant g = \sup\{\chi_\lambda\}$. But $\chi_0(x_0) = 0$, thus $gx_0 = 0$. But x_0 was arbitrary in $PL \setminus \boldsymbol{L}\text{-cl}\, V$, so we have $g \equiv 1$ on $\boldsymbol{L}\text{-cl}\, V$ and $g \equiv 0$ on $PL \setminus \boldsymbol{L}\text{-cl}\, V$. Hence $\boldsymbol{L}\text{-cl}\, V = g^{-1}(\frac{1}{2}, \to)$ is open as a preimage of an open set under a continuous function. This completes the proof of Theorem 8. ■

5. DuBois–Reymond separability. A Boolean algebra is said to be *DuBois–Reymond separable* iff $a_1 < a_2 < a_3 < \ldots < b_3 < b_2 < b_1$ (a_i, b_j elements of the algebra) implies the existence of an element c in the algebra satisfying

$$a_1 < a_2 < a_3 < \ldots < c < \ldots < b_3 < b_2 < b_1 .$$

See Walker [10] for motivation and related topics. Of course, this definition makes sense in any partially ordered set. The following theorem generalizes a key part of Seever's paper [8].

THEOREM 9. *FL is Dubois–Reymond separable if and only if so is L.*

The proof falls into three parts. First, we need the concept of a Q-space. This is obtained by modifying one of the several definitions of an F-space in the setting of ordered spaces. More precisely, we have

DEFINITION. Let X be any ordered space. We call X a *Q-space* provided the following condition is satisfied in X:

Let U, V be open and disjoint,

$$U = \bigcup \{K_i,\ i \in \omega \text{ and } K_i \text{ closed decreasing}\},$$
$$V = \bigcup \{M_j,\ j \in \omega \text{ and } M_j \text{ closed increasing}\}.$$

Then there exist F, G closed disjoint, F decreasing, G increasing such that $U \subseteq F$ and $V \subseteq G$.

We will prove:

A) If L — equivalently, $\boldsymbol{CU}(PL)$ — is DuBois–Reymond separable, then PL is a Q-space,

B) if PL is a Q-space, then $FL = CI(PL)$ is DuBois–Reymond separable and

C) FL DuBois–Reymond separable implies the same for L.

A) Assume $\boldsymbol{CU}(PL)$ is DuBois–Reymond separable. By a prime $'$ we will mark set complements with respect to PL. Let U, V be given as in the definition of a Q-space. Consider any pair K_i, M_j. Applying Lemma 1 and the compactness of PL, we may find $C_i \in \boldsymbol{CL}(PL)$, $D_j \in \boldsymbol{CU}(PL)$ such that $K_i \subseteq C_i$, $M_j \subseteq D_j$, $C_i \cap \text{cl}\, V = \emptyset$, $D_j \cap \text{cl}\, U = \emptyset$. Put $C_i^* = C_i \setminus D_j$, $D_j^* = D_j \setminus C_i \cdot C_i^*$, D_j^* have the properties just mentioned for C_i, D_j; additionally, they are disjoint. This gets the induction started. Assume now that we have constructed, for $1 \leqslant k \leqslant n$, sets $C_k^* \in \boldsymbol{CL}$, $D_k^* \in \boldsymbol{CU}$ such that $C_k^* \subseteq C_{k+1}^*$, $D_k^* \subseteq D_{k+1}^*$ (for $k \leqslant n-1$), $C_k^* \cap D_k^* = \emptyset$, $C_k^* \cap \text{cl}\, V = \emptyset$, $D_k^* \cap \text{cl}\, U = \emptyset$, $K_1 \cup \ldots \cup K_k \subseteq C_k^*$, $M_1 \cup \ldots \cup M_k \subseteq D_k^*$. There exist $C \in \boldsymbol{CL}$, $D \in \boldsymbol{CU}$ such that $C \cap \text{cl}\, V = \emptyset$, $D \cap \text{cl}\, U = \emptyset$, $C \supseteq K_1 \cup \ldots \cup K_{n+1}$, $D \supseteq M_1 \cup \ldots \cup M_{n+1}$. Put

$$C_{n+1}^* = C_n^* \cup \big(C \setminus (D \cup D_n^*)\big), \qquad D_{n+1}^* = D_n^* \cup \big(D \setminus (C \cup C_n^*)\big).$$

C_{n+1}^*, D_{n+1}^* have the required properties. Moreover,

$$D_1^* \subseteq D_2^* \subseteq D_3^* \subseteq \ldots \subseteq C_3^{*\prime} \subseteq C_2^{*\prime} \subseteq C_1^{*\prime},$$

so we may find, $\boldsymbol{CU}$ being DuBois–Reymond separable, $D \in \boldsymbol{CU}$ such that

$$D_1^* \subseteq D_2^* \subseteq D_3^* \subseteq \ldots \subseteq D \subseteq \ldots \subseteq C_3^{*\prime} \subseteq C_2^{*\prime} \subseteq C_1^{*\prime}.$$

Obviously, $V \subseteq D$ and $U \subseteq D'$, so PL is a Q-space.

B) Let $X \in \boldsymbol{TOD}$, X a Q-space. We introduce the following notation: If A, B are upper ends in X, we write $A \ll B$ iff $\boldsymbol{L}\text{-cl}\, A \subseteq \boldsymbol{U}\text{-int}\, B$. To establish DuBois–Reymond separability, we imitate Seever's proof ([8], pp. 269–270). This adaptation rests on the following

LEMMA 10. *Let $X \in \boldsymbol{TOD}$ and $A, B \subseteq X$, $A \ll B$. Then there exists $W \in \boldsymbol{U}(X)$, $W = \bigcup \{F_i;\ i \in \omega$ and $F_i' \in \boldsymbol{L}(X)\}$ such that $A \ll W \ll B$.*

Proof. Let $G = \boldsymbol{L}\text{-cl}\, A$, $U = \boldsymbol{U}\text{-int}\, B$, $F = U'$. As in the proof of A), we may find $C \in \boldsymbol{CL}(X)$, $D \in \boldsymbol{CU}(X)$ such that $C \cap D = \emptyset$, $F \subseteq C$, $G \subseteq D$. Denote by χ_C (χ_D, resp.) the characteristic function of C (D, resp.). It is not hard to see that $f = \frac{1}{2}(1 - \chi_C + \chi_D) \in CI(X)$, $f \equiv 1$ on G and $f \equiv 0$ on F. Let $W = P_f(\frac{1}{2}) \in \boldsymbol{U}(X)$, and let $D \subseteq \boldsymbol{R}$ be any countable dense subset. Then $P_f(\frac{1}{2}) = \bigcup \{f^{-1}[\alpha, \to);\ \alpha \in D$ and $\alpha > \frac{1}{2}\}$; hence numbering $\{\alpha \in D,\ \alpha > \frac{1}{2}\}$ in some suitable way, we may put $F_i = f^{-1}[\alpha_i, \to)$. Obviously, the F_i are closed upper ends. Moreover, $A \subseteq G \subseteq W \subseteq f^{-1}[\frac{1}{2}, \to) =: M \subseteq U \subseteq B$. G and M are closed increasing, W and U open increasing, so $\boldsymbol{L}\text{-cl}\, A \subseteq \boldsymbol{U}\text{-int}\, W = W$ and $\boldsymbol{L}\text{-cl}\, W \subseteq U = \boldsymbol{U}\text{-int}\, B$. This completes the proof of Lemma 10.

Returning to the proof of B), assume that f_n, $g_m \in CI(X)$ satisfy $g_1 \leqslant g_2 \leqslant g_3 \leqslant \ldots \leqslant f_3 \leqslant f_2 \leqslant f_1$. Moreover, we may assume that $0 < g_1 x \leqslant f_1 x < 1$ for all $x \in X$. Let D be the set of dyadic rationals. Define $U(\alpha) = \bigcup \{P_{g_m}(\alpha);\ m \in \omega\}$, $V(\alpha) = \{x \in X;\ f_n x < \alpha$ for some $n \in \omega\}$ $(\alpha \in D)$. Since $P_{g_m}(\alpha) = \bigcup \{g_m^{-1}[\beta, \to);$

$\beta \in D$ and $\beta > \alpha\}$, $U(\alpha)$ is open and a countable union of closed upper ends. Similarly, $V(\alpha)$ is open and a countable union of closed lower ends. Moreover, $U(\alpha) \cap V(\alpha) = \emptyset$. X being a Q-space, we find F closed decreasing, and G closed increasing such that $V(\alpha) \subseteq F$, $U(\alpha) \subseteq G$ and $F \cap G = \emptyset$. Accordingly, $U(\alpha) \subseteq G \subseteq F' \subseteq V(\alpha)'$, whence $U(\alpha) \ll V(\alpha)'$. With the aid of Lemma 10, it is now possible to construct a family $\{W(\alpha);\ \alpha \in D\}$ of open upper ends satisfying

1) $U(\alpha) \ll W(\alpha) \ll V(\alpha)'$ for all $\alpha \in D$,

2) $W(\alpha) \ll W(\beta)$ whenever $\alpha, \beta \in D$ and $\alpha > \beta$,

3) $\bigcap \{W(\alpha);\ \alpha \in D\} = \emptyset$, $\bigcup \{W(\alpha);\ \alpha \in D\} = X$

in exactly the same way as outlined in Seever [8], pp. 269–270. The family of sets $\{W(\alpha)\}$ produces by Lemma 6 a function $f \in CI(X)$ such that

$$g_1 \leqslant g_2 \leqslant g_3 \leqslant \ldots \leqslant f \leqslant \ldots \leqslant f_3 \leqslant f_2 \leqslant f_1 ;$$

so X is DuBois-Reymond separable.

C) Let FL be DuBois–Reymond separable and assume $C_i, D_j \in \boldsymbol{CU}(PL)$ are given such that $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots \subseteq D_3 \subseteq D_2 \subseteq D_1$. Denote by φ_i the characteristic functions of C_i, and by ψ_j those of D_j. Then $\varphi_i, \psi_j \in CI(PL)$ and

$$\varphi_1 \leqslant \varphi_2 \leqslant \varphi_3 \leqslant \ldots \leqslant \psi_3 \leqslant \psi_2 \leqslant \psi_1 .$$

Accordingly, there exists $f \in CI(PL)$ separating the two sequences. Put $F = f^{-1}(1)$, $G = f^{-1}(0)$. Then $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots \subseteq F \subseteq G' \subseteq \ldots \subseteq D_3 \subseteq D_2 \subseteq D_1$. Applying Lemma 1, $G' = \bigcup \{C_\mu \in \boldsymbol{CU};\ C_\mu \subseteq G'\}$. F is compact, hence $F \subseteq C_{\mu_1} \cup \ldots \cup C_{\mu_n} =: C_0 \subseteq G'$, $C_0 \in \boldsymbol{CU}$. Hence finally $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots \subseteq C_0 \subseteq \ldots \subseteq D_3 \subseteq D_2 \subseteq D_1$, and Theorem 9 is proved ∎.

Q-spaces are the natural generalization to ordered spaces of one of the several equivalent characterizations of F-spaces, see [2]. It would be interesting to examine whether there are, correspondingly, equivalent descriptions of Q-spaces, especially one related to the definition of an F-space as a space whose ring of continuous real-valued functions has the property that every finitely generated ideal is principal.

Another problem remaining open is the generalization of Theorem 9 to cardinality-dependent versions of DuBois–Reymond separability: Say that L has property $I(m)$ — m any infinite cardinal — iff $\{a_\lambda\}$, $\{b_\mu\}$ are two subsets of L of cardinalities strictly less than m, $\{a_\lambda\}$ directed upwards and $\{b_\mu\}$ directed downwards, and $a_\lambda \leqslant b_\mu$ for each pair (λ, μ), then there exists $c \in L$ such that $a_\lambda \leqslant c \leqslant b_\mu$ for all λ, μ. Of course, DuBois–Reymond separability is equivalent to $I(\aleph_0^+)$ for any lattice. Question: Is $L \models I(m) \Leftrightarrow FL \models I(m)$ true for any m?

6. $CI(X)$ vs. $C(X)$. As mentioned in Section 2, $CI(X)$ is a sublattice of $C(X)$ for any ordered space X. In general, the corresponding embedding need not be regular, that is, the sup of a collection $\{f_i;\ i \in I\} \subseteq CI(X)$ taken within $C(X)$ may be strictly smaller than that taken within $CI(X)$, provided that both suprema exist, and similarly for infima. Two questions arise naturally: 1) For which ordered spaces X is $CI(X)$ a regular sublattice of $C(X)$ and 2) what conditions on L ensure that PL has this property? The two theorems in this section provide partial answers.

First, we recall that an ordered space X is an *I-space* (see Priestley [7]) provided that for each open $V \subseteq X$ the smallest upper end and the smallest lower end containing V are open sets themselves. The next lemma provides an alternative formulation.

LEMMA 11. *X is an I-space iff* $\operatorname{cl} A$ *is increasing for every increasing* $A \subseteq X$ *and* $\operatorname{cl} B$ *is decreasing for every decreasing* $B \subseteq X$. ∎

Proof. One half is essentially contained in the proof of Lemma 2(ii) in [7], the other half is taken care of by the dual argument. ∎

THEOREM 12. *Let X be a compact I-space. Then $CI(X)$ is a regular sublattice of $C(X)$.*

Proof. Let $f \in C(X)$ and define

$$f^+ : X \to \boldsymbol{R} \quad \text{by} \quad f^+x = \inf\{fy;\ y \geqslant x\},$$
$$f^- : X \to \boldsymbol{R} \quad \text{by} \quad f^-x = \sup\{fy;\ y \leqslant x\}.$$

Evidently, f^+ is increasing and $f^+ \leqslant f$, f^- is increasing and $f \leqslant f^-$. We claim that the following statements hold for any $\alpha \in \boldsymbol{R}$:

(i) $(f^+)^{-1}[\alpha, \to)$ = largest upper end contained in $f^{-1}[\alpha, \to)$;

(ii) $(f^+)^{-1}(\leftarrow, \alpha]$ = smallest lower end containing $f^{-1}(\leftarrow, \alpha]$;

(iii) $(f^-)^{-1}[\alpha, \to)$ = smallest upper end containing $f^{-1}[\alpha, \to)$;

(iv) $(f^-)^{-1}(\leftarrow, \alpha]$ = largest lower end contained in $f^{-1}(\leftarrow, \alpha]$.

Ad (i). Clearly, $(f^+)^{-1}[\alpha, \to)$ is increasing and contained in $f^{-1}[\alpha, \to)$. Now let $S \subseteq f^{-1}[\alpha, \to)$, S increasing, and consider $x_0 \in S$. For all $y \in X$, $y \geqslant x_0$ implies $fy \geqslant \alpha$, since $y \in S \subseteq f^{-1}[\alpha, \to)$. Hence $\inf\{fy;\ y \geqslant x_0\} \geqslant \alpha$, that is, $f^+x_0 \geqslant \alpha$ and $x_0 \in (f^+)^{-1}[\alpha, \to)$.

Ad (ii). $x \in (f^+)^{-1}(\leftarrow, \alpha] \Leftrightarrow f^+x \leqslant \alpha \Leftrightarrow (\exists y)(x \leqslant y \ \&\ fy \leqslant \alpha) \Leftrightarrow x \leqslant y$ for some $y \in f^{-1}(\leftarrow, \alpha]$.

(iii) and (iv) are proved similarly. The point now is that the sets described in (i)–(iv) are all closed under our assumptions: For (i), $\operatorname{cl}(f^+)^{-1}[\alpha, \to)$ is increasing by Lemma 11 and contained in $f^{-1}[\alpha, \to)$, hence $\operatorname{cl}(f^+)^{-1}[\alpha, \to) = (f^+)^{-1}[\alpha, \to)$. For (ii) and (iii) observe that, by Lemma 1 of [7], X is a so-called compact ordered space in the sense of Nachbin [4], whence ([7], pp. 508–509) it follows that the upper and the lower end spanned by a closed set are closed themselves. For (iv), use the other half of Lemma 11. Summing up, our assumptions imply that f^+ and f^- are continuous.

Now let $\{f_i;\ i \in I\} \subseteq CI(X)$ and assume $f_0 \in CI(X)$ is the sup of $\{f_i\}$ taken within $CI(X)$, $g \in C(X)$ the corresponding sup taken within $C(X)$. Consequently, $g \leqslant f_0$. Consider $g^+ \# g^+$ is continuous, increasing and $g^+ \leqslant g$. Moreover, $g^+ \geqslant f_i$ for all $i \in I$: Otherwise, there exist $x_0 \in X$ and $j \in I$ such that $f_j x_0 > g^+x_0 \# g^+x_0 = \inf\{gy;\ y \geqslant x_0\}$, so there exists $y_0 \geqslant x_0$ such that $gy_0 < f_j x_0$. But $f_j y_0 \geqslant f_j x_0$, hence $f_j y_0 > gy_0$, contradicting $g = \sup\{f_i\}$ in $C(X)$.

Accordingly, $g^+ \geqslant f_0 \geqslant g \geqslant g^+$, that is, $g = g^+ = f_0$. Dually for infima, so $CI(X)$ is a regular sublattice of $C(X)$. ■

We recall from [7] that the minimal Boolean extension $B(L)$ of any $L \in D_{01}$ may be obtained from PL by "forgetting the order": more precisely, $B(L)$ is isomorphic to the algebra of *all* clopen subsets of PL. Minimal Boolean extensions are connected with the second question raised above in the following way:

THEOREM 13. *If the embedding of L into its minimal Boolean extension $B(L)$ is regular, then $CI(PL)$ is a regular sublattice of $C(PL)$.*

Proof. The assumption is equivalent to PL being an I-space, see Proposition 17 of [7]. ■

Theorems 12 and 13 are not fully satisfactory since they provide only sufficient conditions. The problem of giving exact characterizations of the spaces and lattices in question remains open.

We conclude by remarking that it is possible to generalize our key theorems (8 and 9) to the case where L lacks universal bounds. One would then consider the lattices $CI(P(L_{01}))$, where L_{01} is obtained from L by adjoining a zero and a unit regardless of the fact that L already may — but need not — have such elements. Much of the theory developed would remain valid, but we feel that the generality gained by such a procedure does not compensate the required technical clumsiness.

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On CE-images of the Hilbert cube and characterization of Q-manifolds

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Abstract. It is shown that a locally compact ANR, X, admitting arbitrarily small (i.e. close to id_X) maps $f, g\colon X \to X$ with $f(X) \cap g(X) = \emptyset$ is a Q-manifold. This is applied to show that if $\mathcal{A}$ is a semicontinuous decomposition of Q such that each $A \in \mathcal{A}$ has trivial shape then $Q/\mathcal{A} \cong Q$ (resp. $Q/\mathcal{A} \times [0, 1] \cong Q$) provided $Q/\mathcal{A} \in$ AR and the union of non-degenerate elements of $\mathcal{A}$ is contained in a countable union of Z-sets in Q (resp. in a countable union of finite-dimensional compacta). A short proof of the Curtis-Schori hyperspace theorem is included in the Appendix.

In 1975, R. D. Edwards established the following profound result (see [11] and [8], § 43):

EDWARDS' THEOREM. *If M is a manifold modelled on the Hilbert cube Q and $\pi\colon M \to X$ is a proper* CE*-map of M onto a locally compact* ANR, *then*

$$\pi \times \mathrm{id}_Q\colon M \times Q \to X \times Q$$

is a limit of homeomorphisms and, in particular, $X \times Q$ is a Q-manifold.

However, it is of interest to know under what additional conditions on π the space X is itself a Q-manifold. Specifically, for the case $M = Q$, the following problems were posed in [1]:

(a) Suppose that the union $S(\pi)$ of non-degenerate point inverses of π is contained in a countable union of Z-sets. Is then $X \cong Q$ (1)?

(b) Under what conditions on π is $X \times [0, 1] \cong Q$?

In connection with (a) it follows from a theorem of J. E. West that $X \cong Q$ if $S(\pi)$ is contained in a single Z-set of Q; see [19] and [8], § 42. In connection with (b) it was shown by J. L. Bryant and by T. A. Chapman that $X \times [0, 1] \cong Q$ and $X \times X \cong Q$ if π has only one non-degenerate point inverse A which is an arc. This was subsequently generalized by Z. Čerin [9] to the case $A \cong [0, 1]^n$, $n < \infty$. In [1] it is mentioned that R. D. Edwards has proved that $Q/A \times [0, 1] \cong Q$ for any finite-dimensional compactum A in Q of trivial shape (unpublished).

In this note we solve (a) in affirmative and we also show that if $S(\pi)$ is a countable union of finite-dimensional compacta, then $X \times [0, 1] \cong Q$ (see § 4). In fact we

(1) We write $X \cong Y$ to denote that X and Y are homeomorphic.